Chapter 1
An Introduction to Asymptotic Homogenization

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1.1 Introduction

Real world physical systems are usually *multiscale* in nature. They are characterized by strong heterogeneities, geometrical complexity, and different constituents which can interplay among several hierarchical levels of organization. Typical examples include, but are not limited to, fluid flow through geometrically complex and porous structure (encountered, for instance, when dealing with oil and gas recovery problems or physiological fluid flow through biological tissues and organs), as well as mechanical and chemical interactions among the various constituents of composite materials (such as, for example, soil or biological hard tissue, e.g. bone and tendons). From a modeling viewpoint, it is necessary to have a comprehensive understanding of the real world phenomena formulating qualitative and quantitative predictions (via analytical and numerical tools) to pursue validation against appropriate experimental data. As a matter of fact, this is basically a two-fold issue as (a) it is in general nontrivial and, especially for three-dimensional real problems, practically impossible to fully resolve *microscale* material and geometrical complexity and (b) experimental measurements usually provide average information on a *macroscale*, i.e. where the difference between different constituents cannot be easily detected. These arguments motivated the development of specific mathematical techniques

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designed to provide computationally feasible *macroscopic* mathematical models which, however, encode the crucial role of the *microstructure* (in terms of material heterogeneities, geometry, fine scale physical coupling, etc.). Although there exist various averaging techniques to obtain a macroscopic description of multiphase physical systems, such as the mixture theory (see, e.g., [4] and [5] for porous media), most of them lead to macroscale description of the system where information on the role of the microstructure is partially or entirely lost.

The *asymptotic homogenization* technique exploits the sharp length scale separation that exists in multiscale systems and a power series representation of the fields to provide macroscale systems of partial differential equations (PDEs) that satisfy both (a) and (b), as the derived models encode the role of the microstructure in their coefficients (hydraulic conductivities, diffusivities, elastic stiffness, etc.). As a drawback, the actual computation of the coefficients for multidimensional problems is only possible assuming appropriate regularity assumptions for the fields involved in the mathematical description (such as *local periodicity*). Furthermore, the microscopic description should be based on linearized balance equations (although nonlinear contributions can arise on a macroscale level) in order to decouple microscale and macroscale spatial variations of the fields.

Here, we introduce the technique via a very simple set of basic examples. We follow a direct approach widely explored in the literature (see, e.g. [2, 3, 13, 14, 16, 22]) which is well suited to introduce asymptotic homogenization to undergraduate/graduate students or scientists coming across this topic for the first time. Issues related to the theoretical foundation of the technique in terms of existence and uniqueness of the homogenized problems are not discussed here and we therefore refer the reader to the pioneering works [15][1] and [1] concerning *H-convergence* and *two-scale convergence*, respectively.

This book chapter is organized as follows:

- In Sect. 1.2, we start from the one-dimensional diffusion problem highlighting the concept of multiscale (spatial) variations and the basic assumptions that are needed to provide a macroscopic description of the problem via asymptotic homogenization. These include spatial variations decoupling, power series expansion, and local boundedness. We derive the diffusion-type homogenized problem and present the analytic form of the effective diffusion coefficient, which also holds for non-periodic microscale variations.
- In Sect. 1.3, we extend the one-dimensional formulation to the multi-dimensional diffusion problem and introduce the assumption of local periodicity, which is in this case necessary to compute the coefficients of the homogenized model. We show that the microscale information is encoded in the homogenized diffusion tensor, which can be computed solving a diffusion-type problem on a single periodic cell.

[1] An English translation can be found in [8], chapter 3.

- In Sect. 1.4, we present the asymptotic homogenization of the Stokes' problem, which leads to Darcy's law for porous media. In this case, the length scale separation is purely geometrical and is captured via an explicit non-dimensionalization process. The effective hydraulic conductivity is to be computed solving a Stokes'-type periodic cell problem.
- In Sect. 1.5, we present concluding remarks.

1.2 One Dimensional Diffusion Problem

We consider the one-dimensional diffusion-type boundary value problem (BVP)

$$\frac{d}{d\tilde{x}}\left(D(\tilde{x})\frac{du(\tilde{x})}{d\tilde{x}}\right) = f(\tilde{x}); \quad 0 < \tilde{x} < 1, \tag{1.1}$$

$$u(0) = a; \quad u(1) = b; \quad a, b \in \mathbb{R}, \tag{1.2}$$

where (1.2) represent non-homogeneous Dirichlet boundary conditions. Here, $f(\tilde{x})$ represents a known, spatially varying volume source, $D(\tilde{x})$ is the smooth, strictly positive, spatially varying diffusion coefficient, and $u(\tilde{x})$ the unknown scalar field. We assume that $f(\tilde{x})$ is regular enough such that a unique solution of (1.1–1.2) exists. BVPs of the type (1.1–1.2) are often encountered in the literature to model various physical phenomena, for example, the linear elastic displacement of an elastic rope, the temperature distribution for heat conduction, diffusion of pollutants, etc. Next we introduce the idea of multiscale spatial variations and formalize it via a basic set of assumptions.

1.2.1 Basic Set of Assumptions

We are interested in investigating the behavior of the problem solution $u(\tilde{x})$ when the diffusion coefficient $D(\tilde{x})$ exhibits *multiscale* spatial variations, i.e., when it displays a different behavior depending on the spatial resolution that we take into account. This is clearly highlighted in Fig. 1.1, where a representative solution of the problem (1.1–1.2) is plotted in the full domain (i.e. the unit length segment), and against a very small portion of it (zoomed in), where spatial variations on such a small scale can be clearly seen.

Next, we highlight the rigorous steps to deduce (a) the macroscopic profile of the solution of the one-dimensional diffusion problem and (b) how microscopic variations of the diffusion coefficient affect the macroscale behavior of the solution. At this stage, it is helpful to understand what *spatial scale separation* means in mathematical terms. Let us first introduce an informal, yet instructive argument. The problem (1.1–1.2) holds on the full unit segment, and our spatial coordinate $\tilde{x}$ spans

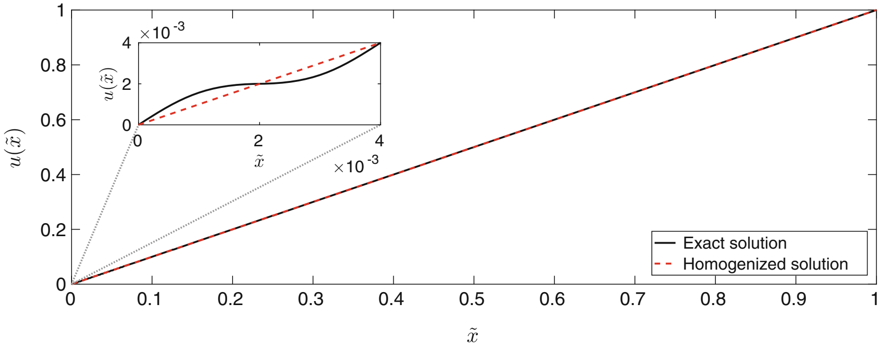

Fig. 1.1 The exact solution $u(\tilde{x}) = \frac{\tilde{x}+c\epsilon\sin(\tilde{x}/\epsilon)}{1+c\epsilon\sin(1/\epsilon)}$ of the BVP (1.1–1.2) for $a = 0$, $b = 1$, $D(\tilde{x}) = 1/(1 + c\cos(\tilde{x}/\epsilon))$, $\epsilon = 2/\pi \cdot 10^{-3}$, $c = 0.9$, $f = 0$ is plotted in black vs. the red-dashed *homogenized* solution $u(\tilde{x}) = \tilde{x}$ for $\tilde{x} \in (0, 1)$ (full plot) and $\tilde{x} \in (0, 4 \cdot 10^{-3})$ (inlay, zoomed in)

the full domain and represents the *physical* mapping for our problem. However, we aim at separating spatial macroscopic variations (see Fig. 1.1 in the domain $(0, 1)$), and microscale spatial variations that are detectable when “zooming in” (see Fig. 1.1 in $(0, 4\cdot 10^{-3})$). To do this, we introduce a characteristic macroscopic length L (that is 1 for our specific problem), and another, much smaller characteristic microscopic length d (that is, for example, $4 \cdot 10^{-3}$ in the particular case shown in Fig. 1.1). We non-dimensionalize our physical spatial coordinate $\tilde{x}$ with respect to both the *microscale* d and the *macroscale* L, i.e.

$$\tilde{x} = Lx_M = dx_m. \tag{1.3}$$

Here x_M represents a non-dimensional coarse scale spatial mapping, as it is non-dimensionalized with respect to the macroscale L, whereas x_m represents a fine scale mapping, as it maps spatial variations resolved on the fine scale d. The two spatial coordinates are related by Eq. (1.3), i.e.

$$x_m = x_M/\epsilon, \tag{1.4}$$

where we define

$$\epsilon = \frac{d}{L}. \tag{1.5}$$

The small parameter ϵ measures the spatial scale separation between the microscale d and the macroscale L.

Remark 1.1 We remark that, even though a non-dimensional analysis is not always performed in the multiscale asymptotics literature, it is important to understand the relationship between macro and micro spatial variations of the fields. The macroscale and microscale variables (also referred to as *slow* and *fast* scales, respectively) are typically denoted by x and $y = x/\epsilon$, respectively, and, using a commonly adopted abuse of notation, the physical variable is usually also denoted by x. The latter identification is rigorous when spatial scales decoupling is carried out after a non-dimensional analysis, as the physical spatial variable is already non-dimensionalized with respect to the macroscale L in that case. Such an analysis is highly recommended when dealing with multiphysics problems that typically comprise several parameters exhibiting different asymptotic behaviors with respect to ϵ, see, e.g. [19, 23]. However, the same, correct results are also obtained by avoiding an explicit non-dimensional analysis (as the local scale y will consistently stay dimensional and account for finer spatial variation the smaller ϵ is), provided that the correct asymptotic behavior of any variable and parameter involved in the differential problem is consistently taken into account. Here, we deal with a very simple problem (which is already in non-dimensional form, with macroscale length $L = 1$), so we just point out the nature of different spatial variables once and for all in this introductory section and avoid complicating the notation for the multidimensional problems illustrated in the following sections.

We are now ready to state the first crucial assumption

Assumption I (Length Scale Separation) *We assume that there exist two distinct spatial scales, referred to as the* microscale d *and the* macroscale L*, such that their ratio*

$$\epsilon = \frac{d}{L} \ll 1\,. \tag{1.6}$$

Our BVP (1.1–1.2) is currently stated in terms of the physical spatial scale $\tilde{x}$, which encodes both macroscale and microscale spatial variations. We aim to transform a single scale problem into a *multiscale* problem, and this leads us to the following assumption:

Assumption II (Spatial Variations Decoupling) *We assume that the unknown field u and the diffusion coefficient D that appear in the BVP (1.1–1.2) are functions of two formally independent spatial variables $x = \tilde{x}$, referred to as the* macroscale *and*

$$y = \frac{\tilde{x}}{\epsilon}, \tag{1.7}$$

(continued)

Assumption II (continued)
referred to as the microscale *variable. In particular, we may write*

$$u = u(x, y), \quad D = D(x, y), \tag{1.8}$$

where

$$x \in (0, 1), \quad y \in (0, +\infty). \tag{1.9}$$

As a direct consequence of Assumption II, derivatives involving the physical spatial scale are now to be understood as *total* (*material*), that is

$$\frac{d(\bullet)}{d\tilde{x}} = \frac{\partial(\bullet)}{\partial x} + \frac{dy}{d\tilde{x}}\frac{\partial(\bullet)}{\partial y} = \frac{\partial(\bullet)}{\partial x} + \frac{1}{\epsilon}\frac{\partial(\bullet)}{\partial y}. \tag{1.10}$$

We are interested in determining the macroscale behavior of differential problems of the type (1.1) in the presence of a sharp length scale separation between the macroscale and the microscale. Hence, it is convenient to consider a regular multiscale expansion for our unknown variable, as follows

Assumption III (Power Series Expansion) *We assume that the multiscale unknown $u(x, y)$ can be formally represented by a regular expansion in power series of ϵ, i.e.*

$$u(x, y) \equiv u^{\epsilon}(x, y) = \sum_{l=0}^{\infty} u^{(l)}(x, y)\epsilon^{l}. \tag{1.11}$$

The reader interested in rigorous issues related to the power series representation (1.11), which is appropriate under suitable regularity assumptions (even weaker than those assumed here), can refer to [10].

Accounting for Assumptions II and III, it seems that we have greatly complicated our problem (1.1), as we are now dealing with one more spatial variable and with infinitely many unknowns $u^{(l)}$. However, our aim is to determine the macroscale behavior of the problem solution whenever the length scale separation that characterizes the problem is sufficiently sharp, that is, for $\epsilon \to 0$. Thus, we will exploit our assumptions and the properties of the various coefficients to derive a macroscale differential problem for the leading order term of the multiscale power series expansion, i.e. $u^{(0)}$.

In order to prevent our multiscale functions from forming singularities with respect to the newly introduced microscale variable y, we also need a number of regularity requirements.

Assumption IV (Local Boundedness and Regularity) *We assume that*

- *Every field $u^{(l)}$, the external source f, and the coefficient D retain, with respect to the macroscale variable x, the same smoothness that characterizes the fields $u(\tilde{x})$, $f(\tilde{x})$, and $D(\tilde{x})$ appearing in (1.1) with respect to the variable $\tilde{x}$.*
- *Any function $u^{(l)}(x, y)$ that appears in (1.11) is* locally bounded*, i.e.*

$$\lim_{y \to +\infty} |u^{(l)}(x, y)| < +\infty \quad \forall x \in (0, 1) \text{ and } \forall l \in \mathbb{N}. \tag{1.12}$$

- *The multiscale diffusion coefficient $D(x, y)$ is strictly positive, locally bounded in the sense of (1.12), and there exist two strictly positive smooth functions $D_m(x)$ and $D_M(x)$ satisfying, for every $y \in (0, +\infty)$ and for every $x \in (0, 1)$*

$$D_m(x) \le D(x, y) \le D_M(x). \tag{1.13}$$

- *The volume source f is y-constant for the sake of simplicity, i.e.*

$$f = f(x)\,. \tag{1.14}$$

We are now ready to apply the asymptotic homogenization technique to the problem (1.1). We intend to obtain a macroscale differential problem for the leading, zero-th order term $u^{(0)}$ that appears in the power series representation (1.11) of $u(x, y)$.

1.2.2 The Homogenized Problem

Let us enforce Assumptions I to IV. The multiscale problem associated to (1.1) then reads, by means of (1.10), as follows:

$$\epsilon^2 \frac{\partial}{\partial x}\left(D(x, y)\frac{\partial u^\epsilon}{\partial x}(x, y)\right) + \epsilon \frac{\partial}{\partial x}\left(D(x, y)\frac{\partial u^\epsilon}{\partial y}(x, y)\right) + \epsilon \frac{\partial}{\partial y}\left(D(x, y)\frac{\partial u^\epsilon}{\partial x}(x, y)\right) + \frac{\partial}{\partial y}\left(D(x, y)\frac{\partial u^\epsilon}{\partial y}(x, y)\right) = \epsilon^2 f(x), \tag{1.15}$$

where we have multiplied both the right and the left hand side by ϵ^2 and u^ϵ denotes the power series representation (1.11). We then formally equate the same powers of ϵ in ascending order, starting from ϵ^0, in (1.15) until we obtain all the necessary conditions to derived a closed macroscale differential problem for the zero-th order component $u^{(0)}$. The derived *homogenized* problem will describe the one-dimensional diffusion process for well separated microscale and macroscale spatial variations, that is, for $\epsilon \to 0$.

$\boxed{\epsilon^0}$

Equating the coefficients of power ϵ^0 in (1.15) yields

$$\frac{\partial}{\partial y}\left(D(x,y)\frac{\partial u^{(0)}}{\partial y}(x,y)\right) = 0. \tag{1.16}$$

We integrate over the microscale y and divide by $D(x,y)$ to obtain[2]

$$u^{(0)} = c_0(x)\int_0^y \frac{1}{D(x,s)}\,\mathrm{d}s + c_1(x), \tag{1.17}$$

where c_0 and c_1 are y-constant functions to be determined. We now enforce Assumption IV (in particular relationship (1.13) concerning the existence of $D_M(x)$), to deduce

$$\int_0^y \frac{1}{D(x,s)}\,\mathrm{d}s \geq \int_0^y \frac{1}{D_M(x)}\,\mathrm{d}s = \frac{y}{D_M(x)}. \tag{1.18}$$

Hence, $u^{(0)}$ is not a bounded function of y unless $c_0(x) = 0$, that is

$$u^{(0)} = c_1(x) \tag{1.19}$$

only depends on the macroscale x. From now on, we thus simplify the notation and write $u^{(0)}(x)$.

$\boxed{\epsilon^1}$

We equate the coefficients of power ϵ^1 in (1.15) and we account for the macroscale character of the leading order solution $u^{(0)}(x)$ to obtain:

$$\frac{\partial}{\partial y}\left(D(x,y)\frac{\partial u^{(0)}}{\partial x}(x)\right) + \frac{\partial}{\partial y}\left(D(x,y)\frac{\partial u^{(1)}}{\partial y}(x,y)\right) = 0. \tag{1.20}$$

[2]Note that the lower integration point is set to zero without loss of generality as the results are unchanged by integrating formally from any y_0 to y. Indeed, the homogenized coefficient that appears in (1.26) is invariant with respect to translations as $y - y_0$ is still approaching infinity as y approaches infinity.

We integrate over the microscale y to obtain:

$$D(x,y)\frac{\partial u^{(0)}}{\partial x}(x) + D(x,y)\frac{\partial u^{(1)}}{\partial y}(x,y) = b_0(x), \tag{1.21}$$

where b_0 is a y-constant function. Dividing by $D(x,y)$ and further integrating over the microscale y yields

$$u^{(1)}(x,y) = b_0(x)\int_0^y \frac{1}{D(x,s)}\,\mathrm{d}s - \frac{\partial u^{(0)}}{\partial x}y + b_1(x), \tag{1.22}$$

where b_1 is another y-constant function. We have to ensure that $u^{(1)}$ stays locally bounded as $y \to +\infty$ (cf. Assumption IV). We first notice that, applying (1.13), the integral function $\int_0^y \frac{1}{D(x,s)}\,\mathrm{d}s$ is bounded from below and above, i.e.

$$\frac{y}{D_M(x)} \leq \int_0^y \frac{1}{D(x,s)}\,\mathrm{d}s \leq \frac{y}{D_m(x)}, \tag{1.23}$$

and it becomes unbounded as y approaches $+\infty$. However, the term $\frac{\partial u^{(0)}}{\partial x}y$ is also unbounded as $y \to +\infty$ and both these contributions are $O(y)$. Therefore, we impose that they balance each other as y approaches $+\infty$, i.e.

$$\lim_{y\to+\infty}\left(b_0(x)\int_0^y \frac{1}{D(x,s)}\,\mathrm{d}s - \frac{\partial u^{(0)}(x)}{\partial x}y\right) = 0, \tag{1.24}$$

where the right hand side of (1.24) can be set to zero without loss of generality as any y-constant contribution could be previously taken into account redefining, for instance, the function b_1. The relationship (1.24) can be rewritten as

$$\frac{1}{\langle D^{-1}\rangle_\infty}\frac{\partial u^{(0)}(x)}{\partial x} = b_0(x), \tag{1.25}$$

where we define

$$\langle D^{-1}\rangle_\infty := \lim_{y\to+\infty}\frac{1}{y}\int_0^y \frac{1}{D(x,s)}\,\mathrm{d}s. \tag{1.26}$$

We differentiate relationship (1.25) with respect to the macroscale x to obtain

$$\frac{\partial}{\partial x}\left(\frac{1}{\langle D^{-1}\rangle_\infty}\frac{\partial u^{(0)}(x)}{\partial x}\right) = \frac{\partial b_0(x)}{\partial x}. \tag{1.27}$$

The differential problem (1.27) actually holds on the macroscale only and it describes the behavior of the leading order field $u^{(0)}$. Hence, this problem represents precisely our mathematical goal, provided that we are able to close it via a relationship for the right hand side $\frac{\partial b_0}{\partial x}$ in terms of known quantities. As we shall see below, we can obtain such a condition exploiting the local boundedness properties of the second order term $u^{(2)}$.

$\boldsymbol{\epsilon^2}$

We now equate the coefficients of power ϵ^2 in (1.15) to obtain:

$$\frac{\partial}{\partial x}\left(D(x,y)\frac{\partial u^{(0)}(x)}{\partial x}\right)+\frac{\partial}{\partial x}\left(D(x,y)\frac{u^{(1)}(x,y)}{\partial y}\right)+$$
$$\frac{\partial}{\partial y}\left(D(x,y)\frac{\partial u^{(1)}(x,y)}{\partial x}\right)+\frac{\partial}{\partial y}\left(D(x,y)\frac{u^{(2)}(x,y)}{\partial y}\right)=f(x). \tag{1.28}$$

Since from (1.21) we have

$$\frac{\partial u^{(1)}(x,y)}{\partial y}=\frac{b_0(x)}{D(x,y)}-\frac{\partial u^{(0)}}{\partial x} \tag{1.29}$$

then

$$\frac{\partial}{\partial x}\left(D(x,y)\frac{\partial u^{(1)}(x,y)}{\partial y}\right)=\frac{\partial}{\partial x}\left(b_0(x)-D(x,y)\frac{\partial u^{(0)}(x)}{\partial x}\right). \tag{1.30}$$

We substitute (1.30) in (1.28) to obtain, rearranging terms:

$$\frac{\partial}{\partial y}\left(D(x,y)\frac{u^{(2)}(x,y)}{\partial y}\right)=f(x)-\frac{\partial b_0(x)}{\partial x}-\frac{\partial}{\partial y}\left(D(x,y)\frac{\partial u^{(1)}(x,y)}{\partial x}\right). \tag{1.31}$$

We integrate over the microscale y, divide by $D(x,y)$ and further integrate over y to finally obtain the following expression for $u^{(2)}(x,y)$:

$$u^{(2)}(x,y)=(f(x)-\frac{\partial b_0(x)}{\partial x})\int_0^y\frac{s}{D(x,s)}\,\mathrm{d}s-\int_0^y\frac{\partial u^{(1)}(x,y)}{\partial x}\,\mathrm{d}s$$
$$+\,d_0(x)\int_0^y\frac{1}{D(x,s)}\,\mathrm{d}s+d_1(x), \tag{1.32}$$

where $d_0(x)$, $d_1(x)$ are y-constant functions. We now notice that, by Assumption IV, $D(x,y)$ and $u^{(1)}(x,y)$ are bounded functions of y. Hence, the first term of the right hand side of (1.32) is $O(y^2)$, while the second and third terms are $O(y)$ in the limit

$y \to +\infty$. In particular, it is necessary to require that the $O(y^2)$ (which cannot be balanced by other terms for $y \to +\infty$) identically vanishes, which implies[3]

$$f(x) = \frac{\partial b_0(x)}{\partial x}. \tag{1.33}$$

We exploit relationship (1.33) to close the differential problem (1.27), such that, accounting also for the appropriate boundary conditions dictated by our original problem (1.1–1.2), the *homogenized* BVP for the leading order coefficient $u^{(0)}(x)$ reads:

$$\frac{d}{dx}\left(\bar{D}(x)\frac{du^{(0)}(x)}{dx}\right) = f(x); \quad 0 < x < 1, \tag{1.34}$$

$$u^{(0)}(0) = a; \quad u^{(0)}(1) = b; \quad a, b \in \mathbb{R}, \tag{1.35}$$

where $\bar{D}(x)$ is the *homogenized* diffusion coefficient defined by

$$\bar{D}(x) = \left\langle D^{-1}\right\rangle_{\infty}^{-1}(x) = \left(\lim_{y\to+\infty} \frac{1}{y}\int_0^y \frac{1}{D(x,s)}\,\mathrm{d}s\right)^{-1}. \tag{1.36}$$

The BVP (1.34–1.35) formally resembles the original one (1.1–1.2). However, microscale variations are now smoothed out and the homogenized coefficient $\bar{D}(x)$, as defined by (1.36), is the *harmonic* mean of the original $D(x, y)$ and not the simple arithmetic average defined by $D_{\text{AVG}}(x) = \lim_{y\to+\infty} \frac{1}{y}\int_0^y D(x,s)\,\mathrm{d}s$. In fact, the harmonic average is more representative of the trend of the coefficient throughout the whole domain, i.e. high amplitude oscillations that are present in a subset which is much smaller than the whole domain do not greatly contribute to the final result. The convergence, for decreasing value of ϵ, of the exact solution of (1.1–1.2) to the homogenized one (obtained by solving the homogenized problem (1.34) using the appropriate coefficient $\bar{D}$) versus the "averaged solution" (obtained by solving the problem (1.34) using the arithmetic average D_{AVG}) is shown for a particular choice of boundary conditions in Fig. 1.2.

We would like to conclude this section remarking that the technique is applicable to nonperiodic local variations of the fields, as shown in Fig. 1.3. The reader can replicate the examples shown in Figs. 1.2 and 1.3 computing the homogenized solution analytically and compare it to the actual solution of the BVP (1.1–1.2). The latter is to be computed numerically for the nonperiodic example shown in Fig. 1.3.

[3]In order for $u^{(2)}(x, y)$ to be a bounded function of y, it is also necessary to require that the $O(y)$ terms compensate each other. However, as long as we focus on the leading order approximation $u^{(0)}$, it is sufficient to exploit local boundedness with respect to the $O(y^2)$ term only, as the latter involves the macroscale function $b_0(x)$ (see Eq. (1.27)) that can eventually close the macroscale problem for $u^{(0)}$.

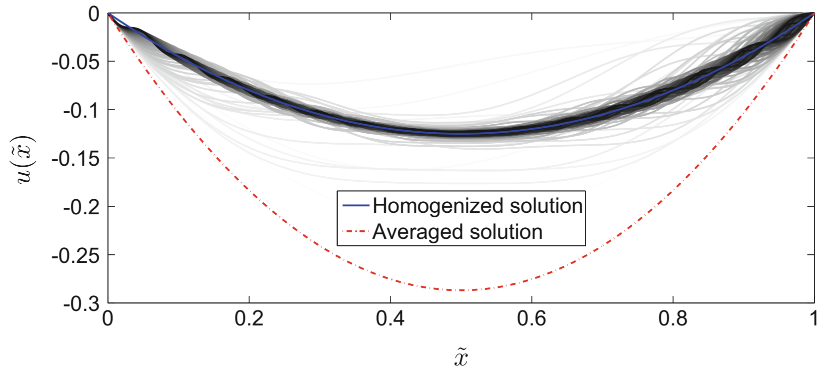

Fig. 1.2 The exact solution of the BVP (1.1–1.2) for $a = 0$, $b = 0$, $f = 1$, $D(\tilde{x}) = 1/(1 + c\cos(\tilde{x}/\epsilon))$, $c = 0.9$, and $0.01 < \epsilon < 1$, is shown in grey scale and it gets darker and darker the smaller ϵ becomes. The solution of the real problem is converging to the *homogenized* solution (shown in blue) and not to the averaged one, represented by the dash-dot line in red

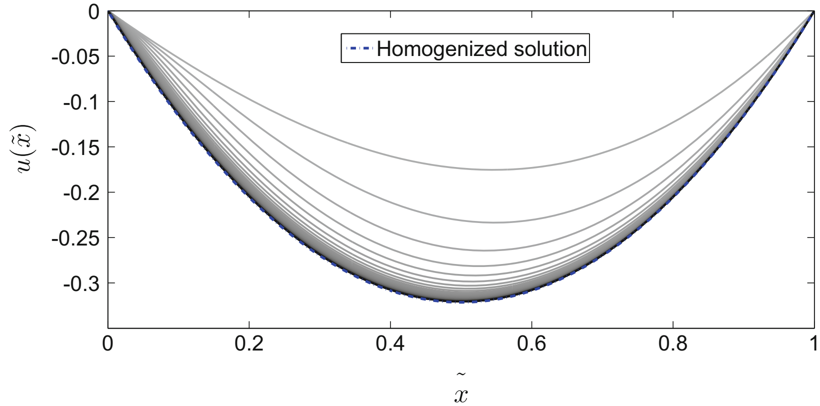

Fig. 1.3 The exact solution of the BVP (1.1–1.2) for $a = 0$, $b = 0$, $f = 1$, $D = (\arctan(x/\epsilon) + 1)^{-1}$, and $0.02 < \epsilon < 1$, is shown in grey scale and it gets darker and darker the smaller ϵ becomes. The solution of the real problem is converging to the *homogenized* solution represented by the dash-dot line in blue

1.3 Multidimensional Diffusion Problem

We aim to generalize the diffusion problem introduced in the previous section to dimension $n \in 1, 2, 3$. We then consider the following classical diffusion problem for a scalar field u in an open connected set $\Omega \subset R^n$ with smooth boundary $\partial\Omega$, i.e.

$$\nabla \cdot (D(\mathbf{x})\nabla u(\mathbf{x})) = f(\mathbf{x}),\ \mathbf{x} \in \Omega \tag{1.37}$$

equipped, for example, with non-homogeneous Dirichlet boundary conditions

$$u(\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} \in \partial\Omega. \tag{1.38}$$

Here, $D(\mathbf{x})$ is the strictly positive and smooth spatially varying diffusion coefficient, $f(\mathbf{x})$ a known volume source and $g(\mathbf{x})$ a known function dictating the behavior of the solution at the boundary. The functions f and g are assumed sufficiently regular such that a solution of the classical diffusion problem (1.37–1.38) exists. At this stage, we assume length scale separation, spatial variable decoupling (that implies transformation of differential operators) and power series representation, so that we can readily generalize assumptions (I–III), together with relationship (1.10). The multiscale, multidimensional problem associated to the n-dimensional diffusion problem (1.37) then formally reads

$$\begin{aligned} &\epsilon^2\nabla_{\mathbf{x}} \cdot (D(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}} u^{\epsilon}(\mathbf{x},\mathbf{y})) + \epsilon\nabla_{\mathbf{x}} \cdot \big(D(\mathbf{x},\mathbf{y})\nabla_{\mathbf{y}} u^{\epsilon}(\mathbf{x},\mathbf{y})\big) + \\ &\epsilon\nabla_{\mathbf{y}} \cdot (D(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}} u^{\epsilon}(\mathbf{x},\mathbf{y})) + \nabla_{\mathbf{y}} \cdot \big(D(\mathbf{x},\mathbf{y})\nabla_{\mathbf{y}} u^{\epsilon}(\mathbf{x},\mathbf{y})\big) = \epsilon^2 f(\mathbf{x},\mathbf{y}), \end{aligned} \tag{1.39}$$

where we have included microscale variation of the volume source for the sake of generality. Here, $\nabla_{\mathbf{x}}$ and $\nabla_{\mathbf{y}}$ represent the gradient with respect to the macroscale and microscale variables $\mathbf{x}$ and $\mathbf{y}$, respectively. We are using the same symbol $\mathbf{x}$ for the macroscale and the physical spatial variables for the sake of simplicity of notation. Since we introduced the additional microscale spatial variable $\mathbf{y}$, we then need to state appropriate regularity assumptions for every multiscale quantity that appears in (1.39). We could indeed generalize Assumption IV and apply the asymptotic homogenization technique to obtain a well-posed macroscale problem. However, whenever $n > 1$, we cannot in general obtain single closed form expressions for the homogenized coefficients (see, e.g., [13]). In general, the latter are to be computed solving microscale differential problems that in principle hold on the whole microscale domain (that extends up to infinity in the limit $\epsilon \to 0$). One of the most important goals for this asymptotic homogenization technique is to determine an *effective* differential problem that describes the macroscale behavior without resolving the full details of the microscale, thus enhancing computational feasibility. At the same time, the macroscale problem should retain information on the microscale encoded in the homogenized coefficient that should be readily accessible, as is the case for the integral (1.36) which defines the one-dimensional homogenized diffusion coefficient. As is widely enforced in the asymptotic homogenization literature, we can focus on a smaller portion of the microscale by assuming $\mathbf{y}$-*periodicity* of the fields involved in (1.39). We state the periodicity assumption in three dimensions, as it can be readily restricted for $n = 1, 2$.

Assumption V (Local Periodicity) *There exists a family of vectors*

$$\mathbf{R}(\eta,\, \kappa,\, \upsilon) := \eta \mathbf{I}_1 + \kappa \mathbf{I}_2 + \upsilon \mathbf{I}_3,\;\; \eta,\, \kappa,\, \upsilon \in \mathbb{Z} \tag{1.40}$$

with fixed vectors $\mathbf{I}_1,\ \mathbf{I}_2,\ \mathbf{I}_3 \in \mathbb{R}^3$ *that constitute a basis of* $\mathbb{R}^3$*, such that, for every field that appears in (1.39), collectively denoted by* ψ*, we have*

$$\psi(\mathbf{x}, \mathbf{y}) = \psi(\mathbf{x}, \mathbf{y} + \mathbf{R}(\eta,\, \kappa,\, \upsilon)),\;\; \forall \eta,\, \kappa,\, \upsilon \in \mathbb{Z}. \tag{1.41}$$

Note that Assumption V is stated for arbitrarily shaped periodic cells and that rectangular (cuboid in three dimensions) periodic cells are simply obtained assuming $\mathbf{I}_n \propto \mathbf{e}_n$ for every n. We therefore account for Assumption V (instead of generalizing local boundedness stated in (1.12)), appropriately extended to the external source $f(x, y)$, while we retain the remaining points of Assumption IV concerning the regularity of the diffusion coefficient and all the fields with respect to the macroscale variable $\mathbf{x}$. Thus, microscopic variations of multiscale fields can now be studied on a single periodic cell defined by the vectors $\mathbf{I}, .., \mathbf{I}_n$. A simple cartoon representing a two-dimensional rectangular cell is shown in Fig. 1.4.

We now proceed by equating the same powers of ϵ in ascending order, starting from ϵ^0, as we have done in the one dimensional case. It is important to bear in mind that we are assuming local periodicity and that the arising differential conditions, though retaining a parametric dependence in terms of the macroscale $\mathbf{x}$, hold on the periodic cell (which we also refer to as Ω to avoid complicating the notation) spanned by the microscale variable $\mathbf{y}$.

$\boxed{\epsilon^0}$

Equating the same powers of ϵ^0 in (1.39) yields

$$\nabla_{\mathbf{y}} \cdot \big(D(\mathbf{x}, \mathbf{y}) \nabla_{\mathbf{y}} u^{(0)}(\mathbf{x}, \mathbf{y})\big) = 0 \text{ in } \Omega. \tag{1.42}$$

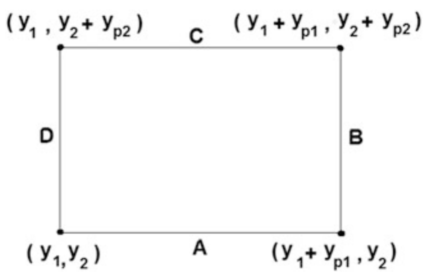

Fig. 1.4 A representation of a two-dimensional rectangular cell defined by the vectors $\mathbf{I}_1 = y_{p1}\mathbf{e}_1$ and $\mathbf{I}_2 = y_{p2}\mathbf{e}_2$, where $y_{p1}, y_{p2} \in \mathbb{R}^+$

Relationship (1.42), equipped with periodicity conditions on $\partial\Omega$, constitutes a standard diffusion-type cell problem that admits a unique solution up to a $\mathbf{y}$-constant function. In particular, any constant is also periodic and solves (1.42), thus we deduce that $u^{(0)}$ is independent of y, i.e.

$$u^{(0)} = u^{(0)}(\mathbf{x}). \tag{1.43}$$

$\boxed{\epsilon^1}$

We equate the same powers of ϵ^1 in (1.39) and account for (1.43) to obtain

$$\nabla_{\mathbf{y}} \cdot \big(D(\mathbf{x},\mathbf{y})\nabla_{\mathbf{y}} u^{(1)}(\mathbf{x},\mathbf{y})\big) = -\nabla_{\mathbf{y}} \cdot \big(D(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}} u^{(0)}(\mathbf{x})\big) \text{ in } \Omega, \tag{1.44}$$

The above problem reads as a periodic cell problem for the first order coefficient $u^{(1)}(\mathbf{x},\mathbf{y})$ and it is once again a classical diffusion-type problem, equipped with a volume load on the right-hand side and periodic boundary conditions on $\partial\Omega$. It admits a unique solution up to an arbitrary $\mathbf{y}$-constant function $c(\mathbf{x})$. Since the problem is linear and the vector function $\nabla_{\mathbf{x}} u^{(0)}$ is $\mathbf{y}$-constant, we state the following solution *ansatz*

$$u^{(1)}(\mathbf{x},\mathbf{y}) = \mathbf{a}(\mathbf{x},\mathbf{y}) \cdot \nabla_{\mathbf{x}} u^{(0)} + c(\mathbf{x}). \tag{1.45}$$

Relationship (1.45) is indeed the solution of the problem (1.44) provided that the vector $\mathbf{a}$ solves the following cell problem

$$\nabla_{\mathbf{y}} \cdot \big(D(\mathbf{x},\mathbf{y})\nabla_{\mathbf{y}} \mathbf{a}\big) = -\nabla_{\mathbf{y}} D(\mathbf{x},\mathbf{y}) \text{ in } \Omega, \tag{1.46}$$

where $\mathbf{a}$ is $\mathbf{y}$-periodic and a further condition is needed to achieve uniqueness, for example by fixing the integral average of $\mathbf{a}$ over the periodic cell Ω. As for the one-dimensional case, we need one last step to obtain a closed macroscale problem for the leading order coefficient $u^{(0)}$.

$\boxed{\epsilon^2}$

We are now ready to conclude the multiscale procedure by equating the same powers of ϵ^2 in (1.39), that yields

$$\begin{aligned}&\nabla_{\mathbf{x}} \cdot \big(D\nabla_{\mathbf{x}} u^{(0)}\big) + \nabla_{\mathbf{x}} \cdot \big(D\nabla_{\mathbf{y}} u^{(1)}\big) + \\ &\nabla_{\mathbf{y}} \cdot \big(D\nabla_{\mathbf{x}} u^{(1)}\big) + \nabla_{\mathbf{y}} \cdot \big(D\nabla_{\mathbf{y}} u^{(2)}\big) = f\end{aligned} \tag{1.47}$$

We now average relationship (1.47) over the periodic cell, i.e. we apply the following cell average operator:

$$\langle (\bullet) \rangle_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} (\bullet) \mathrm{d}\mathbf{y}, \tag{1.48}$$

where $|\Omega|$ is the volume (or area, in two dimensions) of the periodic cell. Application of (1.48) to (1.47) yields

$$\nabla_{\mathbf{x}} \cdot \left(\langle D \rangle_{\Omega} \nabla_{\mathbf{x}} u^{(0)} \right) + \nabla_{\mathbf{x}} \cdot \left\langle D \nabla_{\mathbf{y}} u^{(1)} \right\rangle_{\Omega} + \\ \frac{1}{|\Omega|} \int_{\partial\Omega} D \nabla_{\mathbf{x}} u^{(1)} \cdot \mathbf{n} \mathrm{dS} + \frac{1}{|\Omega|} \int_{\partial\Omega} D \nabla_{\mathbf{y}} u^{(2)} \cdot \mathbf{n} \mathrm{dS} = \langle f \rangle_{\Omega}, \tag{1.49}$$

where the surface integrals arise after applying the divergence theorem with respect to $\mathbf{y}$. Both the surface contributions actually read as integrals of the scalar product between a periodic function and the unit vector $\mathbf{n}$ normal to $\partial\Omega$ over their corresponding periodic cell. Accounting for periodicity, these terms identically reduce to zero, as every contribution on a single face of the periodic cell (edge in two-dimensions), is exactly canceled by the contribution on its corresponding parallel face considering the change in sign of the unit outward normal vector $\mathbf{n}$ (for example, in Fig. 1.4, the contribution over side A exactly cancels the one over side C, and the same holds for sides B and D).

Equation (1.49) can be further rearranged accounting for ansatz (1.45) obtaining:

$$\nabla_{\mathbf{x}} \cdot \left(\langle D \rangle_{\Omega} \nabla_{\mathbf{x}} u^{(0)} \right) + \nabla_{\mathbf{x}} \cdot \left(\left\langle D (\nabla_{\mathbf{y}} \mathbf{a})^{\mathsf{T}} \right\rangle_{\Omega} \nabla_{\mathbf{x}} u^{(0)} \right) = \langle f \rangle_{\Omega}. \tag{1.50}$$

Finally, (1.50) can be rewritten as a macroscale diffusion problem for $u^{(0)}$ as follows:

$$\nabla_{\mathbf{x}} \cdot \left(\mathsf{D}(\mathbf{x}) \nabla_{\mathbf{x}} u^{(0)} \right) = \hat{f}(\mathbf{x}), \tag{1.51}$$

equipped with appropriate macroscale boundary conditions, for example of the type (1.38). The homogenized diffusion tensor D and the macroscale volume source $\hat{f}$ are defined as

$$\mathsf{D}(\mathbf{x}) = \langle D \rangle_{\Omega} \mathsf{I} + \left\langle D (\nabla_{\mathbf{y}} \mathbf{a})^{\mathsf{T}} \right\rangle_{\Omega}, \tag{1.52}$$

or, componentwise

$$D_{ij}(\mathbf{x}) = \langle D \rangle_{\Omega} \delta_{ij} + \left\langle D \frac{\partial a_j}{\partial y_i} \right\rangle_{\Omega} \tag{1.53}$$

and

$$\hat{f}(\mathbf{x}) = \langle f \rangle_{\Omega}, \tag{1.54}$$

where I is the identity tensor. The homogenized problem (1.51) is to be solved on the macroscale only, and microscale information is encoded in the components of the effective diffusivity tensor (1.52), which can be computed solving the diffusion-type cell problems given by (1.46) and exploiting (1.52). A few remarks and exercises now follow.

Remark 1.2 (Anisotropy) The homogenized problem reads as an anisotropic diffusion problem in the limit $\epsilon \to 0$. Hence, the microscale inhomogeneities characterizing the physical diffusion coefficient $D(\mathbf{x}, \mathbf{y})$ translate into anisotropy on the macroscale. In particular, the degree of anisotropy is in general related to the specific form of the coefficient $D(\mathbf{x}, \mathbf{y})$, that dictates the shape and relative dimension of the periodic cell where the vector $\mathbf{a}$, and, in turn the components of the tensor D are to be computed.

Remark 1.3 (Computational Feasibility) Let us consider a diffusion coefficient of the type $D(\mathbf{y})$. Then, the cell problem (1.46) solely depends on the microscale variable $\mathbf{y}$ and can be solved once, independently from the macroscale $\mathbf{x}$. In this case, replacing the original problem (1.37) with the homogenized problem (1.51) greatly reduces computational complexity. Given the solution to (1.46), it is straightforward to compute the effective diffusivity tensor (1.52) and finally solve the classical, homogeneous diffusion problem (1.51) on a coarse grid which captures macroscopic variations of the fields only. Whenever the coefficient D retains a macroscopic variation, then it is in principle necessary to solve one cell problem for every macroscale point $\mathbf{x}$. However, since the macroscale domain is supposed to be represented by a coarse grid, computing a limited number of diffusion-type prescribed cell problems is, in most cases, still more advantageous than resolving the full microscale variations embedded in the original inhomogeneous diffusion problem.

Remark 1.4 (On the Role of Periodicity) We have carried out the asymptotic homogenization steps for the n-dimensional diffusion problem assuming periodicity of the microscale (cf. Assumption V) instead of local boundedness. As we remarked at the beginning of this section, this choice is primarily motivated by practical reasons, as the periodicity assumption enabled us to reconstruct microscopic information focusing on a limited portion of the microstructure, namely, the periodic cell. However, we would like to remark that this assumption is not necessary to derive the homogenized problem, and the analytic form of the microstructural problem as such, as everything could have been carried out assuming local boundedness only. In this case, the asymptotic homogenization technique serves as a powerful tool to derive reliable macroscale problems that can be used to model appropriate physical scenarios of interest, without computing the coefficients themselves. In this case, the latter are supposed to be obtained via other sources, for example experimental measurements. The reader could, as an exercise, derive the effective governing equations for the n-dimensional diffusion problem assuming local boundedness only, as done for the derivation of the equation of poroelasticity in [7].

Remark 1.5 (Macroscopic Uniformity) We implicitly assumed the so-called *macroscopic uniformity*, i.e. the periodic cell is independent of the macroscale. This assumption, which is often assumed implicitly in the asymptotic homogenization literature, allowed us to derive (1.49) assuming

$$\langle \nabla_{\mathbf{x}} \cdot (\bullet) \rangle_{\Omega} = \nabla_{\mathbf{x}} \cdot \langle (\bullet) \rangle_{\Omega} . \tag{1.55}$$

Whenever $\Omega = \Omega(\mathbf{x})$, relationship (1.55) does not hold, and proper application of the generalized Reynold's transport theorem is to be enforced to obtain additional macroscale volume sources that modify the homogenized diffusion problem (see, e.g. [13, 19, 20] and alternative approaches concerning multiscale definition of the unit normal vector for non macroscopically uniform domains, such as [6, 11, 12]). Furthermore, whenever the periodic cell retains a parametric dependence on the macroscale variable $\mathbf{x}$, the problem requires the solution of a periodic cell problem for each macroscale point $\mathbf{x}$, as we observed for the case of macroscopically varying diffusion coefficients D.

We conclude this section proposing the following exercises.

Exercise 1.1 Assume $n = 1$ and $D(x, y) = D(x, y + y_p)$. Solve the cell problem analytically in such a particular case and prove that

$$\bar{D}(x) = \left(\frac{1}{y_p} \int_0^{y_p} \frac{1}{D(x, s)} \, \mathrm{d}s \right)^{-1} , \tag{1.56}$$

that is exactly the periodic counterpart of the relationship (1.36) derived in Sect. 1.2.

Exercise 1.2 Assume $n = 2$, $\mathbf{y} = (y_1, y_2)$ and $D(\mathbf{x}, \mathbf{y}) = D_0(\mathbf{x}) D_A(y_1) D_B(y_2)$, with $D_A(y_1) = D_A(y_1 + a)$ and $D_B(y_2) = D_B(y_2 + b)$. Solve the cell problem analytically and prove the following relationships for the components of the resulting homogenized diffusivity tensor D

$$D_{12} = D_{21} = 0, \tag{1.57}$$

$$D_{11} = \left(\frac{1}{a} \int_0^{a} \frac{1}{D_A(s)} \, \mathrm{d}s \right)^{-1} \frac{1}{b} \int_0^{b} D_B(s) \, \mathrm{d}s, \tag{1.58}$$

$$D_{22} = \left(\frac{1}{b} \int_0^{b} \frac{1}{D_B(s)} \, \mathrm{d}s \right)^{-1} \frac{1}{a} \int_0^{a} D_A(s) \, \mathrm{d}s. \tag{1.59}$$

The reader is invited to explore several possible variations of the diffusion coefficient $D(\mathbf{x}, \mathbf{y})$ and investigate how these affects the resulting components of the homogenized tensor D.

1.4 Porous Media Flow: Homogenization of the Stokes' Problem

The last introductory example we present concerns fluid flow in porous media. These materials are typically involved when dealing with several physical scenarios of practical interest, such as fluid flow through sand and rocks, and interstitial flow through biological tissues, for example bone, cell aggregates, organs and tumors. Here we analyze a simple, yet paradigmatic case, that is, the interaction between a solid rigid phase and an incompressible Newtonian fluid slowly flowing through the pores. We identify the whole physical domain with the open set $\Omega \subset \mathbb{R}^3$, $\bar{\Omega} = \bar{\Omega}_f \cup \bar{\Omega}_s$, where Ω_f and Ω_s are the fluid and solid regions, respectively. The fluid flow through the pores is then governed by the Stokes' problem, i.e.

$$\mu \nabla^2_{\mathbf{x}} \mathbf{v} = \nabla p, \;\; \mathbf{x} \in \Omega_f \tag{1.60}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{v} = 0, \;\; \mathbf{x} \in \Omega_f \tag{1.61}$$

$$\mathbf{v} = \mathbf{0}, \;\; \text{on } \Gamma, \tag{1.62}$$

where $\mathbf{v}$ is the fluid velocity, p the fluid pressure, and $\Gamma = \partial\Omega_f \cap \partial\Omega_s$ represents the interface between the two phases. Equations (1.60–1.62) represent the fluid stress balance, the incompressibility constraint and the no slip conditions for a low Reynolds number Newtonian incompressible fluid, respectively. We aim, once again, at obtaining a macroscale representation for such a problem, which is in this case particularly well suited, as the three dimensional porous structure could be, in general, extremely complex and the problem (1.60–1.62) practically impossible to solve also with numerical techniques. In particular, the sharp length scale separation in such a system relies on its geometry, rather than the analytic form of rapidly varying coefficients. In fact we can identify our microscale d with the pore *radius* (or an equivalent, average linear measure for non-cylindrical pores), and our macroscale with the average (linear) size of the whole domain, or, equivalently, with the average length of the pores. A sketch of the porous microstructure is provided in Fig. 1.5.

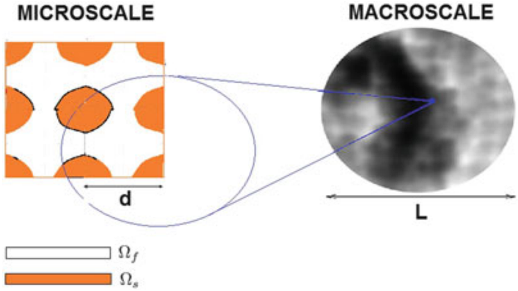

Fig. 1.5 The pore microstructure (shown on the left) against the macrostructure, where the geometrical variations are smoothed out (shown on the right)

In this case, the multiscale nature of the problem is clearly dictated by the geometry itself, and it is necessary to perform an explicit non-dimensionalization process to fully account for the scale separation that characterizes the system.

1.4.1 Non-Dimensionalisation

We rescale our relevant fields as follows

$$\mathbf{x} = L\mathbf{x}', \quad \mathbf{v} = \frac{Cd^2}{\mu}\mathbf{v}', \quad p = CLp' \tag{1.63}$$

where C denotes the magnitude of a characteristic pressure gradient. Here, we scale the spatial coordinate by the characteristic size of the domain (pore length) L, whereas the characteristic velocity V is suggested by the parabolic profile of a viscous fluid flowing in a straight cylindrical channel of radius d, i.e.

$$V \propto \frac{Cd^2}{\mu}, \tag{1.64}$$

see classic fluid-dynamics textbooks, such as [9]. Since differential operators transform as

$$\nabla^2_{\mathbf{x}} = \frac{1}{L^2}\nabla_{\mathbf{x}'} \tag{1.65}$$

and

$$\nabla_{\mathbf{x}} = \frac{1}{L}\nabla_{\mathbf{x}'}, \tag{1.66}$$

the non-dimensional Stokes' problem reads, in terms of the non-dimensional quantities (1.63) and neglecting the primes for the sake of simplicity of notation:

$$\epsilon^2\nabla^2_{\mathbf{x}}\mathbf{v} = \nabla p, \ \mathbf{x} \in \Omega_f \tag{1.67}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{v} = 0, \ \mathbf{x} \in \Omega_f \tag{1.68}$$

$$\mathbf{v} = \mathbf{0}, \ \text{ on } \Gamma, \tag{1.69}$$

where we recall that $\epsilon = d/L$.

1.4.2 The Homogenized Problem

We are dealing with a porous medium, and therefore assume that the average pore radius d is much smaller than the average size of the domain L, such that $\epsilon \ll 1$. We then apply the asymptotic homogenization Assumptions I to III, together with the local periodicity and regularity Assumptions IV and V (exploited in the same way as we have done for the n-dimensional diffusion problem) to the non-dimensional Stokes' problem. We further multiply both the right and the left hand sides of (1.67–1.69) by suitable powers of ϵ to obtain

$$\epsilon^3 \nabla_{\mathbf{x}}^2 \mathbf{v}^\epsilon + \epsilon^2 \nabla_{\mathbf{x}} \cdot \left(\nabla_{\mathbf{y}} \mathbf{v}^\epsilon\right) + \epsilon^2 \nabla_{\mathbf{y}} \cdot (\nabla_{\mathbf{x}} \mathbf{v}^\epsilon) + \epsilon \nabla_{\mathbf{y}}^2 \mathbf{v}^\epsilon = \nabla_{\mathbf{y}} p^\epsilon + \epsilon \nabla_{\mathbf{x}} p^\epsilon, \tag{1.70}$$

$$\nabla_{\mathbf{y}} \mathbf{v}^\epsilon + \epsilon \nabla_{\mathbf{x}} \mathbf{v}^\epsilon = 0 \tag{1.71}$$

in Ω_f and

$$\mathbf{v}^\epsilon = \mathbf{0} \tag{1.72}$$

on Γ.

We now equate the same powers of ϵ in ascending order from ϵ^0 in each of the Stokes' problem equations (1.70–1.72). Since we are in a periodic setting, we identify Ω_f and Ω_s with the corresponding fluid and solid phase within the periodic cell, which we call Ω.

$\boxed{\boldsymbol{\epsilon^0}}$

Equating the same powers of ϵ^0 in the stress balance equation (1.70) yields

$$\nabla_{\mathbf{y}} p^{(0)}(\mathbf{x}, \mathbf{y}) = 0 \Rightarrow p^{(0)} = p^{(0)}(\mathbf{x}), \tag{1.73}$$

that is, the leading order pressure depends only on the macroscale $\mathbf{x}$. The ϵ^0 conditions arising from the incompressibility constraint (1.71) and the no slip condition (1.72) read

$$\nabla_{\mathbf{y}} \cdot \mathbf{v}^{(0)} = 0, \quad \text{in } \Omega_f \tag{1.74}$$

and

$$\mathbf{v}^{(0)} = \mathbf{0}, \quad \text{on } \Gamma \tag{1.75}$$

respectively.

$\boxed{\epsilon^1}$

Equating the same powers of ϵ in (1.70–1.72) leads to the following conditions

$$\nabla_{\mathbf{y}}^2 \mathbf{v}^{(0)} = \nabla_{\mathbf{y}} p^{(1)} + \nabla_{\mathbf{x}} p^{(0)} \quad \text{in } \Omega_f, \tag{1.76}$$

$$\nabla_{\mathbf{y}} \cdot \mathbf{v}^{(1)} + \nabla_{\mathbf{x}} \cdot \mathbf{v}^{(0)} = \mathbf{0} \quad \text{in } \Omega_f, \tag{1.77}$$

and

$$\mathbf{v}^{(1)} = 0, \quad \text{on } \Gamma. \tag{1.78}$$

We now exploit the conditions obtained by equating the same powers of ϵ to close the macroscale problem for the leading order fields $\mathbf{v}^{(0)}$ and $p^{(0)}$.

We collect conditions (1.76), (1.74) and (1.75) together to obtain the following auxiliary Stokes' problem for the fields $(\mathbf{v}^{(0)}, p^{(1)})$

$$\nabla_{\mathbf{y}}^2 \mathbf{v}^{(0)} = \nabla_{\mathbf{y}} p^{(1)} + \nabla_{\mathbf{x}} p^{(0)} \quad \text{in } \Omega_f, \tag{1.79}$$

$$\nabla_{\mathbf{y}} \cdot \mathbf{v}^{(0)} = 0, \quad \text{in } \Omega_f \tag{1.80}$$

$$\mathbf{v}^{(0)} = \mathbf{0}, \quad \text{on } \Gamma, \tag{1.81}$$

supplemented by $\mathbf{y}$-periodicity on the external boundary of the cell $\partial\Omega_f \setminus \Gamma$. We now exploit linearity of the system (1.79–1.81) and the fact that, according to (1.73), the leading order pressure $p^{(0)}$ depends on the macroscale only, to formulate the following ansatz for the solution

$$\mathbf{v}^{(0)} = -\mathsf{W} \nabla_{\mathbf{x}} p^{(0)}, \tag{1.82}$$

$$p^{(1)} = -\mathbf{P} \cdot \nabla_{\mathbf{x}} p^{(0)} + \bar{p}(\mathbf{x}). \tag{1.83}$$

The above expressions represent the unique (up to a $\mathbf{y}$-constant arbitrary function $\bar{p}(\mathbf{x})$) solution of the auxiliary Stokes' problem (1.79–1.81), provided that the auxiliary second rank tensor W and vector $\mathbf{P}$ solve the following Stokes'-type periodic cell problem

$$\nabla_{\mathbf{y}}^2 \mathsf{W}^{\mathsf{T}} = \nabla_{\mathbf{y}} \mathbf{P} - \mathsf{I} \quad \text{in } \Omega_f, \tag{1.84}$$

$$\nabla_{\mathbf{y}} \cdot \mathsf{W} = 0 \quad \text{in } \Omega_f, \tag{1.85}$$

$$\mathsf{W} = 0, \quad \text{on } \Gamma, \tag{1.86}$$

The differential problem (1.84–1.86) is closed by $\mathbf{y}$-periodic conditions on $\partial\Omega_f \setminus \Gamma$ and a further condition on the auxiliary vector $\mathbf{P}$ to ensure the solution uniqueness, for example

$$\langle \mathbf{P} \rangle_{\Omega_f} = 0. \tag{1.87}$$

The problem (1.84–1.86) explicitly reads, by components

$$\frac{\partial W_{ij}}{\partial y_k \partial y_k} = \frac{\partial P_i}{\partial y_j} - \delta_{ij} \quad \text{in } \Omega_f, \tag{1.88}$$

$$\frac{\partial W_{ij}}{\partial y_j} = 0 \quad \text{in } \Omega_f, \tag{1.89}$$

$$W_{ij} = 0, \quad \text{on } \Gamma, \tag{1.90}$$

where $i, j, k = 1, 2, 3$ and sum over repeated indices is understood. Thus, the auxiliary Stokes-type problem (1.84–1.86) requires the solution of three standard Stokes' problem for every fixed $i = 1, 2, 3$. The three Stokes' periodic cell problems differ in the volume load, that is $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, for $i = 1, 2, 3$, respectively. Integral average of the solution ansatz (1.82) over the fluid domain leads to the macroscale governing equation relating the leading order velocity and pressure, namely

$$\left\langle \mathbf{v}^{(0)} \right\rangle_{\Omega_f} = -\left\langle \mathsf{W} \right\rangle_{\Omega_f} \nabla_{\mathbf{x}} p^{(0)}, \tag{1.91}$$

i.e., the fluid flow is described by the Darcy's law on the macroscale domain. Hence, since the average leading order velocity can be computed via (1.91), we just need one more scalar equation for the leading pressure. We consider the integral average of relationship (1.77) and apply the divergence theorem with respect to the microscale variable $\mathbf{y}$ to obtain:

$$\frac{1}{|\Omega_f|} \int_{\partial\Omega_f/\Gamma} \mathbf{v}^{(1)} \cdot \mathbf{n}_{\Omega_f} \, \mathrm{dS} + \frac{1}{|\Omega_f|} \int_{\Gamma} \mathbf{v}^{(1)} \cdot \mathbf{n}_{\Gamma} \, \mathrm{dS} + \nabla_{\mathbf{x}} \cdot \left\langle \mathbf{v}^{0} \right\rangle_{\Omega_f} = 0, \tag{1.92}$$

where $\mathbf{n}_\Gamma$ and $\mathbf{n}_{\partial\Omega_f}$ are the unit outward vectors normal to Γ and $\partial\Omega_f \setminus \Gamma$, respectively. Since the contributions over the external boundary of Ω_f cancel out because of $\mathbf{y}$-periodicity and Eq. (1.78) holds on Γ, no surface contribution remains. The partial differential equation for the leading order pressure $p^{(0)}$ then reads as an effective divergence-free constraint for the average fluid velocity, that is:

$$\nabla_{\mathbf{x}} \cdot \left\langle v^{(0)} \right\rangle_{\Omega_f} = -\nabla_{\mathbf{x}} \cdot \left(\left\langle W \right\rangle_{\Omega_f} \nabla_{\mathbf{x}} p^{(0)} \right) = 0. \tag{1.93}$$

Therefore, asymptotic homogenization of the Stokes' problem for porous media flow leads to the incompressible Darcy's law for the average fluid velocity. The effective, non-dimensional hydraulic conductivity is given by the tensor $\left\langle \mathsf{W} \right\rangle_{\Omega_f}$ which can be computed by solving the Stokes'-type periodic cell problem (1.84–1.86) on the periodic cell Ω. In this case, as long as macroscopic uniformity is assumed (i.e. $\Omega = \Omega(\mathbf{y})$), the effective hydraulic conductivity is homogeneous and solely depends on the geometry of the cell, which is in turn representative of the porous medium structure.

Remark 1.6 It is well-known that a Darcy's law of the type (1.91) can be experimentally verified also for heterogeneous, nonperiodic, porous microstructure. In fact, the Darcy's law for porous media flow can be derived by mixture theory (see, e.g., [21]), where no periodicity is assumed, and also in the context of asymptotic homogenization, assuming local boundedness only. However, local periodicity enables us to derive *computationally feasible* microscale problems that can be solved in practice on a small portion of the microstructure, as done for example in [17] in the context of tumor blood transport. This way, Darcy's law does not play merely the role of the effective set of governing equations for the fluid flow, but also encodes precise information concerning the geometry of the porous structure.

Remark 1.7 (Geometric Homogenization) Note that this example shows that the asymptotic homogenization technique can be carried out also for physical systems that are not characterized by fine scale variations of the coefficients. We have started from the Stokes' problem at constant viscosity and have finally obtained Darcy's law exploiting the sharp length scale separation that exists in the *geometry* itself, which is captured via an explicit non-dimensionalization analysis. In the most general case, physical systems can exhibit both fine scale variations of the coefficients and geometric heterogeneities. For example, when dealing with elastic composite materials (see, e.g. [14, 22] and recently developed computational analysis such as [18]), both oscillations of the elastic coefficients within a single elastic phase and the difference between different phases may be observed on the fine scale, and the two contributions lead in general to distinct contributions that appear in the corresponding cell problems that are to be computed to determine the effective elasticity tensor.

1.5 Concluding Remarks

We have presented a brief introduction to the asymptotic homogenization technique. The material is intended to serve as a first step to foster the curiosity of students and scientists approaching the topic for the first time. We have applied the technique to simple examples, such as the diffusion problem and the Stokes' problem for porous media flow. These are only partially representative of the whole realm of multiscale, multiphysics problems and have been chosen to drive the reader's attention towards the fundamental significance of spatial scale decoupling and non-dimensionalization, and the importance of appropriate regularity assumption (local boundedness, local periodicity) in deriving appropriate homogenized PDEs. We have deliberately ignored advanced, cutting edge applications, as this book chapter solely serves as a simple, basic introduction to the topic. However, we believe that this introductory work may help the interested readers to understand fundamental issues concerning the technique and to raise their awareness when facing complex multiscale problems involving the interplay among several physical phenomena.

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