On the Aggregation State of Simple Materials

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1. Introduction

In standard continuum mechanics, a material body is termed *simple* if its local stress response depends on the history of the deformation gradient $\mathbf{F} := \nabla f$ (cf. [4], Sect. 28). This constitutive notion singles out a large class of material bodies, irrespectively of their *aggregation state*: to distinguish solids from fluids (let alone to sort the latter into liquids and gases [2]) further specifications are needed.

According to a criterion proposed by W. Noll, in *simple solids* the stress state accompanying any given deformation history should be insensitive to all previous rotations in a given subgroup of the orthogonal group; in *simple fluids*, to all previous volume-preserving mixing deformations. These additional constitutive requirements translate into well-known algebraic conditions, whose role, when combined with the restrictions following from appropriate insensitivity to observer changes, is to allow for different representations of the general response laws for solids and fluids.

No such criterion for sorting different aggregation types, and hence no such representations for the response mappings, is at our disposal for *complex* material bodies, those not being simple. The aim of this note is to introduce notions of simplicity and aggregation state slightly different from the standard ones, by way of a couple of significant examples.

In my view, the main merit of these new notions is that their format is easier to generalize so as to cover the complex case, in that they both are expressed in terms of internal power. I here demonstrate their effectiveness by showing that, if the Piola stress depends on the history of the deformation gradient in the following way:

$$\mathbf{S} = \widehat{\mathbf{S}}(\mathbf{F}, \dot{\mathbf{F}}) = \widehat{\mathbf{S}}^{eq}(\mathbf{F}) + \widehat{\mathbf{S}}^{vs}(\mathbf{F}, \dot{\mathbf{F}}), \tag{1.1}$$

with

$$\widehat{\mathbf{S}}^{eq}(\mathbf{F}) := \widehat{\mathbf{S}}(\mathbf{F}, \mathbf{0}), \quad \widehat{\mathbf{S}}^{vs}(\mathbf{F}, \dot{\mathbf{F}}) := \widehat{\mathbf{S}}^{vs}(\mathbf{F}, \dot{\mathbf{F}}) - \widehat{\mathbf{S}}^{vs}(\mathbf{F}, \mathbf{0}), \tag{1.2}$$

then,

$$\begin{aligned} &\widehat{\mathbf{S}}^{eq}(\mathbf{F}) = -\pi \mathbf{F}^{-T},\ \pi \in \mathbb{R}, && \text{for a fluid;} \\ &\mathbf{F}^T \widehat{\mathbf{S}}^{eq}(\mathbf{F}) = \big(\mathbf{F}^T \widehat{\mathbf{S}}^{eq}(\mathbf{F})\big)^T && \text{for an isotropic solid.} \end{aligned} \tag{1.3}$$

2. Simple material bodies

I propose to call a material body $\mathcal{B}$ *simple* when, for whatever subbody $\mathcal{P}$, the basic duality between kinematic and dynamic state variables is declared by laying down the *internal-power form*

$$P(\mathcal{P},\mathbf{v}) := \int_{\mathcal{P}} \mathbf{Z}\cdot D\mathbf{v}, \quad \mathbf{v}\in\mathcal{V}. \tag{2.1}$$

Here, (i) subbodies are pointwise identified with regions of an euclidean point space, used as observational background; (ii) $\mathcal{V}$ is a collection of vector-valued test fields over $\mathcal{B}$, including all *realizable velocity* fields $\mathbf{v} = \dot{f}$ and closed under the operation $\mathcal{O}$ of observer change, in the sense that, if $\mathbf{v}\in\mathcal{V}$, then $\mathbf{v}^+ = \mathcal{O}(\mathbf{v}) = \mathbf{t}+\mathbf{Q}\mathbf{v}\in\mathcal{V}$ for all translations $\mathbf{t}$ and all orthogonal tensors $\mathbf{Q}$; (iii) D is the gradient operator with respect to positional changes in the background space; (iv) $\mathbf{Z}$, the *stress* field dual to the *strain-rate* field $D\mathbf{v}$, is required to be appropriately *indifferent* to observer changes, so as to let the internal power be *invariant* under observer changes, in the sense that $P^+ = P$; (v) the stress response to deformations depends on the history of the first deformation gradient only.[1]

For the background space a *referential copy* of a given euclidean point space, and for $D\equiv\nabla$, the *referential gradient* operator, $\mathbf{Z}$ is interpreted as the *Piola stress* $\mathbf{S}$, a stress measure indifferent to observer changes, in the sense that $\mathbf{S}^+ = \mathbf{Q}\mathbf{S}$ for all orthogonal tensors $\mathbf{Q}$; the internal power takes the form

$$P(\mathcal{P},\mathbf{v}) = \int_{\mathcal{P}_R} \mathbf{S}\cdot\nabla\mathbf{v}, \tag{2.2}$$

where $\mathcal{P}_R$ is the referential region occupied by the subbody $\mathcal{P}$. If the *current copy* of the given euclidean space is the chosen background space, and hence $D\equiv \mathrm{grad}$, the *current gradient* operator, then the appropriate indifferent stress measure is the *Cauchy stress* $\mathbf{T}$, for which $\mathbf{T}^+ = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$; one writes the internal

[1] A thermodynamic argument could be given to prove that (2.1) implies that higher gradients cannot influence the stress state, but I refrain from introducing the necessary machinery, given the declared scope of the present note. Likewise, instead of a generic constitutive dependence of stress on deformation history, I shall assume the simplest dependence $(1.1)_1$.

power as follows:

$$P(\mathcal{P},\mathbf{v})=\int_{\mathcal{P}}\mathbf{T}\cdot\operatorname{grad}\mathbf{v},\tag{2.3}$$

with $\mathcal{P}$ denoting both the subbody $\mathcal{P}$ and the region of the current space it occupies. Given that

$$\nabla\mathbf{v}=(\operatorname{grad}\mathbf{v})\mathbf{F},\quad \mathbf{F}=\nabla f,$$

one finds that

$$\mathbf{T}:=(\det\mathbf{F})^{-1}\mathbf{S}\mathbf{F}^{T},\tag{2.4}$$

where $\det\mathbf{F}$ is the jacobian of the deformation f of the reference into the current space.

In continuum mechanics, it is common practice to use the Piola stress and the internal-power formulation (2.2) when dealing with solids, the Cauchy stress and the internal-power formulation (2.3) when dealing with fluids. The reasons for this are that, for a solid, both referential and current observations are routinely recorded and compared, whereas a fluid body is generally observed only in its current shape: in fact, not only a careful and persistent identification of fluid elements is practically difficult but also tracing their paths is in most cases irrelevant to evaluate the stress field accompanying a given motion process. Given that my intention is to make use of certain invariance properties of the internal power to distinguish the stress response of solids from that of fluids, I should not risk to beg the question. Therefore, in the following I choose for the internal power the referential formulation (2.2). Of course, the implicit understanding is that material elements are presumed identifiable, no matter how difficult the identification might be due to their aggregation state.[2]

3. Simple fluids and simple solids, "à la" Noll and not

The standard doctrine, where distinctions in aggregation state are based on Noll's criterion, is best illustrated in the case of simple *elastic* materials,

[2]This issue was first raised explicitly by Hellinger, in 1914; see [1]).

whose response depends only on the current value of the deformation gradient. Given a map $\mathbf{F} \mapsto \hat{\sigma}(\mathbf{F})$ delivering the stored energy per unit volume of $\mathcal{B}_R$, the associated stress-response map is

$$\mathbf{F} \mapsto \hat{\mathbf{S}}(\mathbf{F}) := \partial\hat{\sigma}(\mathbf{F}),$$

and the associated *material symmetry group* $\mathcal{G}$ consists of all those second-order tensors $\mathbf{H} \in \mathcal{U} := \{\mathbf{H} \,|\, \det \mathbf{H} = 1\}$ that satisfy

$$\hat{\sigma}(\mathbf{FH}) = \hat{\sigma}(\mathbf{F}) \tag{3.1}$$

for all $\mathbf{F}$ in the domain of $\hat{\sigma}$. *Simple elastic fluids* compose the material class for which the group $\mathcal{U}$ is a subgroup of $\mathcal{G}$; as is well-known, this definition reflects the physical expectation that, for a fluid, *volume-preserving deformations* (be they obtained by mixing or shaking) are not energy- or stress-detectable by any further deformation. For *simple elastic solids*, $\mathcal{G}$ is a subgroup of $\mathcal{R} := \{\mathbf{R} \in \mathcal{U} \,|\, \mathbf{R}^{-1} = \mathbf{R}^T\}$, the maximal subgroup of $\mathcal{U}$ consisting of all rotations; this time, the physical expectation is that, for a solid, some (and possibly all) *rigid deformations* are not detectable by any further deformation.

3.1. Fluids

As a consequence of the standard notion of simple fluid, one can show that another physical expectation is satisfied, namely, that *for all elastic fluids the stress has the form of a pressure, depending only on volume changes*. In fact, as is well-known, with the use of the multiplicative decomposition

$$\mathbf{F} = \varphi\widetilde{\mathbf{F}}, \quad \varphi := (\det \mathbf{F})^{1/3}, \quad \widetilde{\mathbf{F}} \in \mathcal{U},$$

it follows from (3.1) that, whatever $\hat{\sigma}$, there is a function $\bar{\sigma}(\det \mathbf{F}) := \hat{\sigma}(\varphi\mathbf{1})$, that is *invariant* under observer changes and induces the *indifferent* stress-response map

$$\hat{\mathbf{S}}(\mathbf{F}) = -\hat{\pi}(\det\mathbf{F})\mathbf{F}^{-T}, \quad \hat{\pi}(\det\mathbf{F}) := (\det \mathbf{F})\,\bar{\sigma}'(\det\mathbf{F}).[3]$$

[3] With the use of (2.4), this result can be stated as follows:

$$\hat{\mathbf{T}}(\det\mathbf{F}) = -\bar{\sigma}'(\det\mathbf{F})\mathbf{1}.$$

Here is a line of reasoning that yields the same result, when a notion of simplicity based on internal power is accepted.

Suppose, for simplicity, that the constitutive dependence of the Piola stress on the deformation history is specified by (1.1)-(1.2), so that, just as the stress, the internal power density is split additively into *equilibrium* and *viscous* parts:

$$\mathbf{S}\cdot\dot{\mathbf{F}}=\mathbf{S}^{eq}\cdot\dot{\mathbf{F}}+\mathbf{S}^{vs}\cdot\dot{\mathbf{F}}.$$

Call a simple material a *fluid* if, given an admissible process $t\mapsto\mathbf{F}(t)$, the equilibrium internal power is partwise invariant for whatever choice of a concomitant time-dependent *volume-preserving* change of reference placement $t\mapsto\mathbf{H}(t)\in\mathcal{U}$:

$$\widehat{\mathbf{S}}^{eq}(\mathbf{F})\cdot\dot{\mathbf{F}}=\widehat{\mathbf{S}}^{eq}(\mathbf{F}\mathbf{H})\cdot(\mathbf{F}\mathbf{H})^{\cdot},$$

i.e., if

$$(\widehat{\mathbf{S}}^{eq}(\mathbf{F})-\widehat{\mathbf{S}}^{eq}(\mathbf{F}\mathbf{H})\mathbf{H}^{T})\cdot\dot{\mathbf{F}}=\mathbf{F}^{T}\widehat{\mathbf{S}}^{eq}(\mathbf{F}\mathbf{H})\cdot\dot{\mathbf{H}}. \tag{3.2}$$

Since $\dot{\mathbf{H}}$ can be chosen in any manner consistent with

$$\dot{\mathbf{H}}\cdot\mathbf{H}^{-T}=0, \tag{3.3}$$

we have from (3.2) that

$$\mathbf{F}^{T}\widehat{\mathbf{S}}^{eq}(\mathbf{F}\mathbf{H})=-\pi\mathbf{H}^{-T},\quad \pi\in\mathbb{R}. \tag{3.4}$$

With this, (3.2) reduces to

$$(\widehat{\mathbf{S}}^{eq}(\mathbf{F})+\pi\mathbf{F}^{-T})\cdot\dot{\mathbf{F}}=0, \tag{3.5}$$

an invariant statement that can also be written in the following form:

$$\widehat{\mathbf{S}}^{eq}(\mathbf{F})\cdot\dot{\mathbf{F}}=-\pi\,\mathrm{div}\,\mathbf{v},\quad \mathbf{v}=\dot{f};$$

Granted mass conservation:

$$\rho_R=(\det\mathbf{F})\rho$$

(here ρ_R and ρ are the referential and current mass densities), the customary constitutive law

$$\widetilde{\mathbf{T}}(\rho)=-\tilde{\pi}(\rho)\mathbf{1}$$

is arrived at.

thus, for fluids, the equilibrium power expenditure in a test process depends only on volume changes, because it is unaffected by whatever previous mixing.

To proceed further, it is convenient to assume that the geometric structure of the state space is as rich as possible, by way of a postulate of *state accessibility* and *free process continuation* that, at times, is unduly left tacit when dealing with the constitutive theory of continuous media. In the present contex, such postulate boils down to assuming that, for each given pair $(\mathbf{F}_0, \dot{\mathbf{F}}_0)$, there is an admissible process such that $\mathbf{F}(t) = \mathbf{F}_0,\ \dot{\mathbf{F}}(t) = \dot{\mathbf{F}}_0$.

Due to the implicit quantification following from the postulate in question, relation (3.5) implies that

$$\widehat{\mathbf{S}}^{eq}(\mathbf{F}) = -\pi \mathbf{F}^{-T}, \quad \pi \in \mathbb{R}.$$

Thus, *for all simple fluids the equilibrium stress is a pressure*. Note that this result is arrived at without presuming that the fluid is elastic, the case when the pressure is constitutively specified in terms of volume (or density) changes.

3.2. Solids

I exemplify further my internal-power approach to sorting aggregation states and stress responses to deformation processes by characterizing a material class of simple solids that includes standard elastic isotropic solids as a subclass.

Once again, assume that (1.1)-(1.2) hold. Call a simple material an *isotropic solid* if its equilibrium internal power is partwise invariant whatever the continuation $\dot{\mathbf{F}}$ of a given admissible process $t \mapsto \mathbf{F}(t)$ and for whatever choice of a concomitant time-dependent *rigid* change of reference placement $t \mapsto \mathbf{R}(t) \in \mathcal{R}$:

$$\widehat{\mathbf{S}}^{eq}(\mathbf{F}) \cdot \dot{\mathbf{F}} = \widehat{\mathbf{S}}^{eq}(\mathbf{F}\mathbf{R}) \cdot (\mathbf{F}\mathbf{R})^{\cdot},$$

a condition that can be written as

$$(\widehat{\mathbf{S}}^{eq}(\mathbf{F}) - \widehat{\mathbf{S}}^{eq}(\mathbf{F}\mathbf{R})\mathbf{R}^T) \cdot \dot{\mathbf{F}} = \mathbf{F}^T \widehat{\mathbf{S}}^{eq}(\mathbf{F}\mathbf{R}) \cdot \dot{\mathbf{R}}. \tag{3.6}$$

Now, $\dot{\mathbf{R}}$ can be chosen in any manner consistent with

$$\dot{\mathbf{R}} \cdot \mathbf{A}\mathbf{R} = 0 \quad \text{for all } \mathbf{A} \in \mathrm{Sym},$$

where Sym denotes the collection of all symmetric second-order tensors. Therefore, for all $\mathbf{A} \in \mathrm{Sym}$, we have from (3.6) that

$$\mathbf{F}^T \widehat{\mathbf{S}}^{eq}(\mathbf{F}\mathbf{R}) = \mathbf{A}\mathbf{R}, \tag{3.7}$$

and hence, in addition, that

$$(\widehat{\mathbf{S}}^{eq}(\mathbf{F}) - \mathbf{F}^{-T}\mathbf{A}) \cdot \dot{\mathbf{F}} = 0.$$

Finally, provided that $\dot{\mathbf{F}}$ is arbitrarily choosable, we arrive at the following representation result for the equilibrium-stress response mapping of isotropic solids:

$$\mathbf{F}^T \widehat{\mathbf{S}}^{eq}(\mathbf{F}) \in \mathrm{Sym}. \tag{3.8}$$

It is easy to check that any *hyperelastic* isotropic response mapping

$$\mathbf{S} = \partial_{\mathbf{F}}\sigma(\mathbf{F}\cdot\mathbf{F}, \mathbf{F}^*\cdot\mathbf{F}^*, \det\mathbf{F})$$

delivers a stress that satisfies (3.8). More generally, relation (3.8) may be interpreted as a statement of the fact that the *equilibrium Eshelby tensor* $\mathbf{E}^{eq} := \psi\mathbf{1} - \mathbf{F}^T\mathbf{S}^{eq}$ is symmetric (cf. [3], p. 78; here ψ denotes the free energy per unit referential volume).

4. Final comments

A salient feature of the here proposed notion of aggregation state is that it discriminates solids from fluids solely on the basis of their equilibrium response. In fact, as I plan to show in a future paper, characterizations of the viscous response of solids and fluids are also possible, that are especially relevant when complex material bodies are dealt with.

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