Virtual Powers on Diffused Subbodies and Normal Traces of Tensor-Valued Measures

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1. Introduction

In the last years there has been a considerable effort in the weakening of the concept of *balance law* in Continuum Mechanics through many directions, from the point of view of the stress fields up to the irregularity of the boundary of the (sub)bodies. After a first important step made by Gurtin, Williams and Ziemer [11], who introduced the class of sets with finite perimeter as somehow "optimal" concept for the treating of balance laws (and of the related concept of flux through a surface), the papers by Šilhavý [17, 18] introduced the notion of *almost every subbody* and treated some classes of stresses more irregular than before, such as L^p-tensor fields with divergence in L^p. Subsequently, our paper [6] pointed out how it was possible to obtain results for L^1_{loc}-tensor fields with divergence measure on normalized subsets with finite perimeter by requiring the balance property to hold true on a very simple class of subbodies (almost all cubes), thus showing that those simple subsets are enough to qualify Cauchy's tensor field and the validity of the balance law. On the other hand, from [10, 8, 9] we were becoming more and more convinced that the balance of power could provide a more general and nice setting of the whole problem, since it contains the possibility of treating, in a natural way, higher grade materials and it is more intrinsic from an analytical point of view, especially when working in a non-Euclidean framework. In papers [14, 15, 2], some of the authors extended what was known in the case of fluxes to the more distributional setting related to powers, and in the paper [7] we studied materials of grade 2 and the corresponding (weak) form of the balance law. At the same time, Šilhavý [20, 21] generalized many of the properties of a flux to objects which are extendable to the case of subbodies having boundary with infinite Hausdorff measure, as the case of fractals, which are some flat chains in the sense of Whitney, and Harrison [12] developed a theory of still more irregular objects, called chainlets, to which some properties of fluxes and powers may extend. Also, in the same years, Chen and Frid [4, 5] showed some applications of this area of interest to conservation laws.

We start here from the belief that a *body* could (or should) be in general modeled with a function, which may tend or not to the characteristic function of a set in the simplest cases (see the notion of *presence* in [3]). This has, from one side, no logical complication like fuzzy sets, since the subbody is simply a function, and on the other side this seems to be the most common and natural case when several scales are taken into account. In fact, a quite general setting of the concept of power expended on a velocity field is possible and natural, and it also gives rise to an interesting definition of what a *contact power* is. Moreover, it is also a powerful analytical tool, since it allows us to state sufficient conditions to pass to the limit to a characteristic function, *i.e.* to localized subbodies.

Once a sufficiently general representation for the power is granted, the next step is the following: in the case of first order contact powers, the volume integrals in the representation formula can be seen as a definition of a particular distribution of order one, but in some cases they may define a distribution of order zero, supported by the topological boundary of the set. We investigate here some sufficient conditions in order to ensure that this distribution is indeed of order zero, that is, a measure, and therefore we find natural extensions of Gauss-Green formula on arbitrary open sets and, in particular, a generalized notion of normal trace for measures with divergence measure. Results in the same direction have been obtained in [4, 5, 20, 21]. Finally, we turn our attention to a very special case of subbodies, namely the rectangles in the plane, and apply our results to obtain a more explicit expression of what the trace of a measure on a side or at a vertex may look like. This is not merely an exercise, since there are many different ways to state a trace result, depending on what is known on the fields (for instance, if the trace is calculated from inside or from both sides of the rectangle). It is however clear that, with some technicality, these results hold also for locally regular sets. The real application we show is the famous Flamant solution for the stress in an elastic body with a vector-valued Dirac measure exerted at the boundary (see Podio-Guidugli [16] for an extensive treatment of the topic): again, this is not at all linear elasticity, as it may appear at a first glance. Flamant's solution is a stress solution and therefore it is not restricted to whatever elastic material: it's pure balance. We then recover the trace of the stress solution in an anlytical way, using the notion of (generalized) vector potential.

2. Virtual powers on diffused subbodies

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and denote with $\mathfrak{M}(\Omega)$ the set of positive Borel measures finite on compact subsets of Ω. Given an integer $N \geqslant 1$, we introduce the finite dimensional linear spaces $\mathrm{Sym}_0 := \mathbb{R}^N$ and

$$\mathrm{Sym}_j := \{\boldsymbol{f} : (\mathbb{R}^n)^j \to \mathbb{R}^N : \quad \boldsymbol{f} \text{ is } j\text{-linear and symmetric}\}$$

for $j \geqslant 1$. In particular,

$$\forall \boldsymbol{v} \in C^\infty\big(\Omega; \mathbb{R}^N\big), \forall x \in \Omega : \quad \nabla^{(j)}\boldsymbol{v}(x) \in \mathrm{Sym}_j\,,$$

where $\nabla^{(j)}\boldsymbol{v}$ denotes the j-th derivative of $\boldsymbol{v}$. We denote with Sym_j^* the dual space of Sym_j.

We define the collection of *diffused subbodies* of Ω as

$$\Theta(\Omega) = \{\vartheta \in C_0(\Omega) : \quad 0 \leqslant \vartheta \leqslant 1 \text{ on } \Omega\}\,.$$

Definition 2.1. *A power of order $k \in \mathbb{N}$ is a function $P : \Theta(\Omega) \times C^\infty\big(\Omega; \mathbb{R}^N\big) \to \mathbb{R}$ such that*

1. *for every $\boldsymbol{v} \in C^\infty\big(\Omega; \mathbb{R}^N\big)$, $P(\vartheta, \boldsymbol{v}) = P(\vartheta_1, \boldsymbol{v}) + P(\vartheta_2, \boldsymbol{v})$ whenever $\vartheta, \vartheta_1, \vartheta_2 \in \Theta(\Omega)$ satisfy $\vartheta = \vartheta_1 + \vartheta_2$;*

2. *for every $\vartheta \in \Theta(\Omega)$, $P(\vartheta, \cdot)$ is linear;*

3. *for every compact set $K \subseteq \Omega$ there exists $c_K \geqslant 0$ such that for every $\vartheta \in \Theta(\Omega)$ with $\mathrm{supt}\,\vartheta \subseteq K$ and for every $\boldsymbol{v} \in C^\infty\big(\Omega; \mathbb{R}^N\big)$,*

$$|P(\vartheta, \boldsymbol{v})| \leqslant c_K \sum_{j=0}^{k} \|\nabla^{(j)}\boldsymbol{v}\|_{\infty, \mathrm{supt}\,\vartheta}\,,$$

where $\|\nabla^{(j)}\boldsymbol{v}\|_{\infty,S} := \sup\{|\nabla^{(j)}\boldsymbol{v}(x)| : x \in S\}$.

The set $C^\infty\big(\Omega; \mathbb{R}^N\big)$ is the space of *test velocities*. In the standard framework, $N = n$ in Continuum Mechanics and $n = 1$ in Thermodynamics, but indeed N can be arbitrary, for instance in the presence of hidden parameters.

2.1. Integral representation

We will prove now an integral representation for P. For a fixed ϑ, the linear functional $P(\vartheta,\cdot)$ is a distribution on Ω of order k and such a representation is a standard result in the theory of distributions. The point here is that we find measures which are independent of ϑ.

Theorem 2.1. *For every power P of order k there exist $k+1$ measures $\mu_j \in \mathfrak{M}(\Omega)$ and $k+1$ Borel maps $T_j : \Omega \to \mathrm{Sym}_j^*$ such that $|T_j| = 1$ μ_j-a.e. and*

$$\forall \vartheta \in \Theta(\Omega),\, \forall \boldsymbol{v} \in C^\infty\big(\Omega;\mathbb{R}^N\big): \quad P(\vartheta,\boldsymbol{v}) = \sum_{j=0}^{k} \int_\Omega \vartheta \langle T_j, \nabla^{(j)}\boldsymbol{v}\rangle \, d\mu_j \,. \tag{2.1}$$

Moreover, the tensor-valued measures $T_j d\mu_j$ are uniquely determined.

Proof – Let us first treat the case $N=1$.

In the first step, we extend the definition of P to every $\vartheta \in C_0(\Omega)$, obtaining a bi-linear functional. Let $\vartheta \in C_0(\Omega)$, $\vartheta \geqslant 0$. Given $h \in \mathbb{N}$ such that $2^{-h}\vartheta(x) \leqslant 1$ for all $x \in \Omega$, we set

$$\widehat{P}(\vartheta, v) := 2^h P(2^{-h}\vartheta, v).$$

This definition is easily seen to be independent of h. In this way $\widehat{P}(\cdot, v)$ is an additive function defined on every $\vartheta \geqslant 0$. Moreover, for every compact set $K \subseteq \Omega$ and every $\vartheta \in \Theta(\Omega) \setminus \{0\}$ with $\operatorname{supt}\vartheta \subseteq K$, we can take $h \in \mathbb{Z}$ such that $2^{h-1} < \|\vartheta\|_\infty \leqslant 2^h$, obtaining

$$\begin{aligned}\forall v \in C^\infty(\Omega): \quad |\widehat{P}(\vartheta, v)| &= 2^h |P(2^{-h}\vartheta, v)| \\ &\leqslant 2^h c_K \sum_{j=0}^{k} \|\nabla^{(j)} v\|_{\infty,\operatorname{supt}\vartheta} \\ &\leqslant 2 c_K \|\vartheta\|_\infty \sum_{j=0}^{k} \|\nabla^{(j)} v\|_{\infty,\operatorname{supt}\vartheta} \,.\end{aligned} \tag{2.2}$$

Now we define $\widetilde{P} : C_0(\Omega) \times C^\infty(\Omega) \to \mathbb{R}$ by setting

$$\widetilde{P}(\vartheta, v) := \widehat{P}(\vartheta^+, v) - \widehat{P}(\vartheta^-, v)\,.$$

In view of (2.2), $\widetilde{P}$ satisfies

$$|\widehat{P}(\vartheta, v)| \leqslant 4c_K \|\vartheta\|_\infty \sum_{j=0}^{k} \|\nabla^{(j)} v\|_{\infty, \operatorname{supt} \vartheta}\,, \tag{2.3}$$

for every $\vartheta \in C_0(\Omega)$ with $\operatorname{supt} \vartheta \subseteq K$ and for every $v \in C^\infty(\Omega)$.

Let us denote again with P such an extension and write c_K instead of $4c_K$. We claim that P is linear in ϑ. Indeed, to prove additivity notice that

$$(\vartheta_1 + \vartheta_2)^+ + (\vartheta_1^- + \vartheta_2^-) = (\vartheta_1 + \vartheta_2)^- + (\vartheta_1^+ + \vartheta_2^+)$$

and that all terms in parentheses are positive. By adding and subtracting the term $P(\vartheta_1^- + \vartheta_2^-, v)$, from the above identity it follows

$$\begin{aligned} P(\vartheta_1 + \vartheta_2, v) &= P((\vartheta_1 + \vartheta_2)^+, v) - P((\vartheta_1 + \vartheta_2)^-, v) \\ &= P(\vartheta_1^+ + \vartheta_2^+, v) - P(\vartheta_1^- + \vartheta_2^-, v) \\ &= P(\vartheta_1^+, v) + P(\vartheta_2^+, v) - P(\vartheta_1^-, v) - P(\vartheta_2^-, v) \\ &= P(\vartheta_1, v) + P(\vartheta_2, v). \end{aligned} \tag{2.4}$$

It remains to prove that P is homogeneous of degree 1 in ϑ. By additivity, it is easily seen that $P(q\vartheta, v) = qP(\vartheta, v)$ for every $q \in \mathbb{Q}$. For $\lambda \in \mathbb{R}$, let (q_h) be a sequence of rational numbers such that $q_h \to \lambda$. Then

$$|P(\lambda\vartheta, v) - \lambda P(\vartheta, v)| \leqslant |P((\lambda - q_h)\vartheta, v)| + |\lambda - q_h| P(\vartheta, v).$$

In view of (2.3), the right-hand side vanishes for $h \to \infty$, hence P is bi-linear on $C_0(\Omega) \times C^\infty(\Omega)$.

In the second step, define $m_k = \operatorname{card}\{\alpha \in \mathbb{N}^n : |\alpha| \leq k\} = \binom{n+k}{k}$ and

$$\eta_\alpha(x) = \frac{x^\alpha}{\alpha!}\,.$$

It is clear that

$$D^\beta \eta_\alpha = \begin{cases} \dfrac{x^{\alpha-\beta}}{(\alpha-\beta)!} & \text{if } \beta \leqslant \alpha \\ 0 & \text{if } \beta_i > \alpha_i \text{ for some } 1 \leqslant i \leqslant n\,. \end{cases}$$

In particular, $D^\alpha\eta_\alpha = 1$ and the $m_k \times m_k$ matrix $[D^\beta\eta_\alpha]$ is non singular (indeed $\det[D^\beta\eta_\alpha] = 1$). This uniquely defines m_k linear maps $\Lambda_\alpha : C_0(\Omega;\mathbb{R}^{m_k}) \to C_0(\Omega)$ such that for every $w \in C_0(\Omega;\mathbb{R}^{m_k})$ one has $\operatorname{supt}\Lambda_\alpha(w) \subseteq \operatorname{supt} w$ and

$$w_\beta = \sum_{|\alpha|\le k} \Lambda_\alpha(w) D^\beta\eta_\alpha\,,$$

where w_β denotes the β-th component of w. Then we can define the linear functional

$$\varphi : C_0(\Omega;\mathbb{R}^{m_k}) \to \mathbb{R}\,,\quad \varphi(w) := \sum_{|\alpha|\le k} P(\Lambda_\alpha(w),\eta_\alpha).$$

Let $c_\alpha > 0$ such that $\|\Lambda_\alpha(w)\|_\infty \leqslant c_\alpha\|w\|_\infty$. Taking a compact set $K \subseteq \Omega$ and $\operatorname{supt} w \subseteq K$, by the properties of P we find

$$\begin{aligned}|\varphi(w)| &\leqslant c_K \sum_{|\alpha|\le k} \|\Lambda_\alpha(w)\|_\infty \sum_{j=0}^{k} \|\nabla^{(j)}\eta_\alpha\|_{\infty,K}\\ &= \|w\|_\infty \left(c_K \sum_{|\alpha|\le k}\sum_{j=0}^{k} c_\alpha \|\nabla^{(j)}\eta_\alpha\|_{\infty,K}\right)\end{aligned}$$

which shows that φ is a distribution of order zero. In particular, there exist m_k measures $\nu_\alpha \in \mathfrak{M}(\Omega)$ and m_k bounded Borel functions $p_\alpha : \Omega \to \mathbb{R}$ such that

$$\forall w \in C_0(\Omega;\mathbb{R}^{m_k}) : \quad \varphi(w) = \sum_{|\alpha|\leqslant k} \int_\Omega w_\alpha p_\alpha \, d\nu_\alpha. \tag{2.5}$$

Now fix $\gamma \in \mathbb{N}^n$ with $|\gamma| \leqslant k$ and $\vartheta \in C_0(\Omega)$, and take the special case $\widehat{w}_\beta = \vartheta D^\beta\eta_\gamma$. Since

$$\sum_{|\alpha|\le k} \Lambda_\alpha(\widehat{w}) D^\beta\eta_\alpha = \widehat{w}_\beta = \vartheta D^\beta\eta_\gamma\,,$$

it follows that

$$\Lambda_\gamma(\widehat{w}) = \vartheta\,, \qquad \Lambda_\alpha(w) = 0 \quad \text{for } \alpha \ne \gamma,$$

so that (2.5) yields

$$P(\vartheta,\eta_\gamma) = \sum_{|\alpha|\le k} \int_\Omega \vartheta D^\alpha\eta_\gamma p_\alpha \, d\nu_\alpha\,,$$

which is the claim in the special case $v = \eta_\gamma = \frac{x^\gamma}{\gamma!}$. By linearity, the representation formula holds for all polynomials with degree at most k and, switching to the tensorial notation,

$$P(\vartheta, v) = \sum_{j=0}^{k} \int_\Omega \vartheta \langle T_j, \nabla^{(j)} v\rangle \, d\mu_j \tag{2.6}$$

for every polynomial v with degree at most k.

Now we want to prove that the formula (2.6) extends to all of $C^\infty(\Omega)$. Let us fix $v \in C^\infty(\Omega)$. By (2.3), $P(\cdot, v)$ is a distribution of order 0 on Ω, hence there exist $\mu_v \in \mathfrak{M}(\Omega)$ and a Borel function $p_v : \Omega \to \mathbb{R}$ such that $|p_v| = 1$ μ_v-a.e. and

$$\forall \vartheta \in C_0(\Omega) : \qquad P(\vartheta, v) = \int_\Omega \vartheta p_v \, d\mu_v \,.$$

Now fix also $\vartheta \in C_0(\Omega)$. For $h \in \mathbb{N}$ consider a finite collection $\{I_{h,m}\}_m$ of disjoint open n-intervals in Ω such that for every $h \in \mathbb{N}$, $\{I_{h+1,m}\}_m$ is a refinement of $\{I_{h,m}\}_m$,

$$\operatorname{supt} \vartheta \subseteq \bigcup_m \overline{I_{h,m}} \subseteq \Omega \,, \qquad \max_m \{\operatorname{diam}(I_{h,m})\} < 2^{-h}$$

and every boundary $\partial I_{h,m}$ is negligible for the measures $\mu_1, \dots, \mu_k, \mu_v$. Then for every $h, m \in \mathbb{N}$ consider two functions $\varphi_{h,m}, \psi_{h,m} \in C_0^\infty(I_{h,m})$ such that $0 \leqslant \varphi_{h,m}, \psi_{h,m} \leqslant 1$,

$$(\mu_1 + \cdots + \mu_k + \mu_v)\Big(\Big\{x \in \Omega : \quad \sum_m \varphi_{h,m} < 1\Big\}\Big) < 2^{-h},$$

and $\psi_{h,m} = 1$ in a neighborhood of $\operatorname{supt} \varphi_{h,m}$. Finally, denote with $T_{h,m}$ the Taylor approximation of v of order k around the center of $I_{h,m}$ and recall that

$$\sum_{j=0}^{k} \Big(\max_m \|\nabla^{(j)}(v - T_{h,m})\|_{\infty, I_{h,m}}\Big) \to 0 \qquad \text{as } h \to \infty \,. \tag{2.7}$$

Then one has

$$
\begin{aligned}
P\Big(\vartheta\sum_m \varphi_{h,m}, v\Big) &= \sum_m P\Big(\vartheta\varphi_{h,m}, T_{h,m}\Big) \\
&\qquad + \sum_m P\Big(\vartheta\varphi_{h,m}, (v - T_{h,m})\psi_{h,m}\Big) \\
&= \sum_m P(\vartheta\varphi_{h,m}, T_{h,m}) \\
&\qquad + P\Big(\vartheta\sum_m \varphi_{h,m}, \sum_m (v - T_{h,m})\psi_{h,m}\Big)\,.
\end{aligned}
\tag{2.8}
$$

Now set $K_h = \operatorname{supt}(\vartheta\sum_m \varphi_{h,m})$ and $K = \operatorname{supt}\vartheta$. By applying (2.3) on the last term of the right-hand side of (2.8) one gets

$$
\begin{aligned}
&\left|P\Big(\vartheta\sum_m \varphi_{h,m}, \sum_m (v - T_{h,m})\psi_{h,m}\Big)\right| \\
&\qquad\leqslant c_K \Big\|\vartheta\sum_m \varphi_{h,m}\Big\|_\infty \sum_{j=1}^k \Big\|\nabla^{(j)}\Big[\sum_m (v - T_{h,m})\psi_{h,m}\Big]\Big\|_{\infty,K_h} \\
&\qquad\leqslant c_K \|\vartheta\|_\infty \sum_{j=1}^k \Big(\max_m \|\nabla^{(j)}(v - T_{h,m})\|_{\infty,I_{h,m}}\Big)\,.
\end{aligned}
$$

Taking into account (2.8), (2.7) and the representation (2.6), it follows

$$
\begin{aligned}
P(\vartheta, v) &= \lim_{h\to\infty} P\Big(\vartheta\sum_m \varphi_{h,m}, v\Big) = \lim_{h\to\infty} \sum_m P(\vartheta\varphi_{h,m}, T_{h,m}) \\
&= \sum_{j=0}^k \int_\Omega \vartheta\langle T_j, \nabla^{(j)} v\rangle\, d\mu_j\,,
\end{aligned}
$$

which ends up the proof in the case $N = 1$.

In the general case, for every $i = 1, \ldots, N$, we can define a power $P_i : \Theta(\Omega)\times C^\infty(\Omega) \to \mathbb{R}$ of order k by $P_i(\vartheta, v) = P(\vartheta, v\boldsymbol{e_i})$, where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N$ is the canonical basis in $\mathbb{R}^N$. Since

$$
P(\vartheta, \boldsymbol{v}) = \sum_{i=1}^N P_i(\vartheta, v_i)\,, \qquad \boldsymbol{v} = (v_1, \ldots, v_N)\,,
$$

the assertion follows from the previous case. □

Remark 2.1 - One can ask if a similar representation holds if assumption *1* is weakened as

4. for every $\boldsymbol{v} \in C^\infty\big(\Omega;\mathbb{R}^N\big)$ and every $\vartheta_1, \vartheta_2 \in \Theta(\Omega)$ with $\operatorname{supt}\vartheta_1 \cap \operatorname{supt}\vartheta_2 = \varnothing$,

$$P(\vartheta_1 + \vartheta_2, \boldsymbol{v}) = P(\vartheta_1, \boldsymbol{v}) + P(\vartheta_2, \boldsymbol{v})$$

namely, if the additivity holds only for "well-separated" bodies. However, such an assumption is too weak, since for instance the functional

$$P(\vartheta, \boldsymbol{v}) := \sum_{j=0}^{k} \int_\Omega \vartheta^2 \nabla^{(j)} \boldsymbol{v} \, d\mathscr{L}^n$$

satisfies *4*, *2*, *3* but cannot be represented by (2.1).

Remark 2.2 - If a power of order k satisfies the property

5. there exist $\lambda_0, \dots, \lambda_k \in \mathfrak{M}(\Omega)$ such that

$$|P(\vartheta, \boldsymbol{v})| \leqslant \sum_{j=0}^{k} \int_\Omega \vartheta |\nabla^{(j)} \boldsymbol{v}| \, d\lambda_j$$

which is stronger than *3* of Definition 2.1, then it can be proved that each μ_j of Theorem 2.1 is indeed absolutely continuous with respect to λ_j. Hence there exist $k+1$ bounded Borel functions $T_j : \Omega \to \mathrm{Sym}_j^*$ uniquely determined λ_j-a.e. such that

$$P(\vartheta, \boldsymbol{v}) = \sum_{j=0}^{k} \int_\Omega \vartheta \langle T_j, \nabla^{(j)} \boldsymbol{v} \rangle \, d\lambda_j \, .$$

2.2. Weak balance and contact powers

Definition 2.2. *A power P is said to be* weakly balanced *if for every compact set $K \subseteq \Omega$ there exists $c_K \geqslant 0$ such that for every $\vartheta \in \Theta(\Omega)$ with $\operatorname{supt}\vartheta \subseteq K$ and for every $\boldsymbol{v} \in C^\infty\big(\Omega;\mathbb{R}^N\big)$ with $(1-\vartheta)\boldsymbol{v} = \boldsymbol{0}$ on Ω, one has*

$$|P(\vartheta, \boldsymbol{v})| \leqslant c_K \|\boldsymbol{v}\|_{\infty,K} \, .$$

In particular, P is said to be a contact power *if*

$$P(\vartheta, \boldsymbol{v}) = 0 \quad \textit{whenever } (1-\vartheta)\boldsymbol{v} = \mathbf{0} \textit{ on } \Omega.$$

A power is said to be a distance power *if it has order zero.*

Of course, contact and distance powers are automatically weakly balanced. On the other hand, a decomposition theorem holds.

Theorem 2.2 (Distance-contact decomposition). *A weakly balanced power P of order k can be decomposed in a unique way as a sum of a distance power $P^{(d)}$ and a contact power $P^{(c)}$.*

Proof – By Theorem 2.1 one has

$$P(\vartheta, \boldsymbol{v}) = \sum_{j=0}^{k} \int_{\Omega} \vartheta \langle T_j, \nabla^{(j)} \boldsymbol{v} \rangle \, d\mu_j$$

for every $\vartheta \in \Theta(\Omega)$ and $\boldsymbol{v} \in C^{\infty}(\Omega; \mathbb{R}^N)$.

Given a compact set K in Ω, let $\vartheta \in \Theta(\Omega)$ with $\vartheta = 1$ on K. If $\boldsymbol{v} \in C_0^{\infty}(\Omega; \mathbb{R}^N)$ has support in K, the weak balance yields

$$|P(\vartheta, \boldsymbol{v})| = \left| \sum_{j=0}^{k} \int_{\Omega} \langle T_j, \nabla^{(j)} \boldsymbol{v} \rangle \, d\mu_j \right| \leqslant \widehat{c}_K \|\boldsymbol{v}\|_{\infty,\Omega} \,,$$

hence the left-hand side is a distribution of order 0,

$$\text{(2.9)} \qquad \forall \boldsymbol{v} \in C_0^{\infty}(\Omega; \mathbb{R}^N) : \quad \sum_{j=0}^{k} \int_{\Omega} \langle T_j, \nabla^{(j)} \boldsymbol{v} \rangle \, d\mu_j = \int_{\Omega} \langle B, \boldsymbol{v} \rangle \, d\nu \,,$$

where $\nu \in \mathfrak{M}(\Omega)$ and $B : \Omega \to (\mathbb{R}^N)^*$ is a bounded Borel function. We set by definition

$$P^{(d)}(\vartheta, \boldsymbol{v}) = \int_{\Omega} \vartheta \langle B, \boldsymbol{v} \rangle \, d\nu \,, \qquad P^{(c)}(\vartheta, \boldsymbol{v}) = P(\vartheta, \boldsymbol{v}) - P^{(d)}(\vartheta, \boldsymbol{v}) \,.$$

The first is clearly a distance power. Moreover, if we take ϑ and $\boldsymbol{v}$ such that

$(1-\vartheta)\boldsymbol{v}=\mathbf{0}$, then

$$P^{(c)}(\vartheta,\boldsymbol{v}) = \sum_{j=0}^{k}\int_{\Omega}\vartheta\langle T_j,\nabla^{(j)}\boldsymbol{v}\rangle\,d\mu_j - \int_{\Omega}\vartheta\langle B,\boldsymbol{v}\rangle\,d\nu$$
$$= \sum_{j=0}^{k}\int_{\Omega}\langle T_j,\nabla^{(j)}\boldsymbol{v}\rangle\,d\mu_j - \int_{\Omega}\langle B,\boldsymbol{v}\rangle\,d\nu = 0\,.$$

Suppose now that

$$P = P^{(d)} + P^{(c)} = \widehat{P}^{(d)} + \widehat{P}^{(c)}.$$

For every $\boldsymbol{v}\in C_0^\infty\big(\Omega;\mathbb{R}^N\big)$, taking $\vartheta\in\Theta(\Omega)$ such that $\vartheta=1$ on $\operatorname{supt}\boldsymbol{v}$, one has in particular that $(1-\vartheta)\boldsymbol{v}=\mathbf{0}$, hence, with obvious notation,

$$\int_{\Omega}\langle B,\boldsymbol{v}\rangle\,d\nu = P^{(d)}(\vartheta,\boldsymbol{v}) = \widehat{P}^{(d)}(\vartheta,\boldsymbol{v}) = \int_{\Omega}\langle\widehat{B},\boldsymbol{v}\rangle\,d\widehat{\nu}\,.$$

Therefore $B\,d\nu$ and $\widehat{B}\,d\widehat{\nu}$ denote the same vector measure and $P^{(d)}=\widehat{P}^{(d)}$. By difference, the identity of $P^{(c)}$ and $\widehat{P}^{(c)}$ is also proved. □

Up to here, a diffused subbody ϑ was merely a continuous function. If one wants to say more, a possibility is to suppose ϑ more regular. Let's therefore suppose $\vartheta\in\Theta(\Omega)\cap C^k(\Omega)$.

Theorem 2.3. *Let P be a weakly balanced power of order k. Then for every $\vartheta\in\Theta(\Omega)\cap C^k(\Omega)$ and every $\boldsymbol{v}\in C^\infty\big(\Omega;\mathbb{R}^N\big)$ one has*

$$P^{(d)}(\vartheta,\boldsymbol{v}) = \sum_{j=0}^{k}\int_{\Omega}\langle T_j,\nabla^{(j)}(\vartheta\boldsymbol{v})\rangle\,d\mu_j\,, \tag{2.10}$$

$$P^{(c)}(\vartheta,\boldsymbol{v}) = \sum_{j=1}^{k}\int_{\Omega}\langle T_j,\vartheta\nabla^{(j)}\boldsymbol{v}-\nabla^{(j)}(\vartheta\boldsymbol{v})\rangle\,d\mu_j\,. \tag{2.11}$$

Proof – If we replace $\boldsymbol{v}$ with $\vartheta\boldsymbol{v}$ in (2.9), we get

$$\int_{\Omega}\langle B,\vartheta\boldsymbol{v}\rangle\,d\nu = \sum_{j=0}^{k}\int_{\Omega}\langle T_j,\nabla^{(j)}(\vartheta\boldsymbol{v})\rangle\,d\mu_j$$

and the proof is straightforward. □

It is worth noting that the expression $\vartheta\nabla^{(j)}\boldsymbol{v} - \nabla^{(j)}(\vartheta\boldsymbol{v})$ vanishes whenever $\vartheta = 0$ or $\vartheta = 1$ on an open set; it means that the contact part of a power is supported by (the closure of) the region where $0 < \vartheta < 1$, which can be interpreted as the "boundary" of the subbody.

In particular, in the case $k = 1$ from (2.11) and (2.9) we have

$$\begin{aligned} &\forall\vartheta \in \Theta(\Omega) \cap C^1(\Omega),\ \forall\boldsymbol{v} \in C^\infty\big(\Omega;\mathbb{R}^N\big): \\ &\qquad P^{(c)}(\vartheta,\boldsymbol{v}) = -\int_\Omega \langle T_1, \boldsymbol{v}\otimes\nabla\vartheta\rangle\, d\mu_1\,, \end{aligned} \tag{2.12}$$

$$\forall\boldsymbol{v} \in C_0^\infty\big(\Omega;\mathbb{R}^N\big): \quad \int_\Omega \langle T_0, \boldsymbol{v}\rangle\, d\mu_0 + \int_\Omega \langle T_1, \nabla\boldsymbol{v}\rangle\, d\mu_1 = 0\,, \tag{2.13}$$

where the tensor product is defined in the usual way

$$(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{c} := (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}\,.$$

In the case $k = 2$ we find

$$\begin{aligned} P^{(c)}(\vartheta,\boldsymbol{v}) &= \int_\Omega \langle T_1, \vartheta\nabla\boldsymbol{v} - \nabla(\vartheta\boldsymbol{v})\rangle\, d\mu_1 \\ &\qquad + \int_\Omega \langle T_2, \vartheta\nabla\nabla\boldsymbol{v} - \nabla\nabla(\vartheta\boldsymbol{v})\rangle\, d\mu_2 \\ &= -\int_\Omega \langle T_1, \boldsymbol{v}\otimes\nabla\vartheta\rangle\, d\mu_1 \\ &\qquad - \int_\Omega \langle T_2, \boldsymbol{v}\otimes\nabla\nabla\vartheta + \nabla\boldsymbol{v}\otimes\nabla\vartheta\rangle\, d\mu_2 \end{aligned} \tag{2.14}$$

where for $S \in \mathrm{Lin}(\mathbb{R}^n;\mathbb{R}^n)$, $T \in \mathrm{Lin}(\mathbb{R}^n;\mathbb{R}^N)$, $\boldsymbol{a} \in \mathbb{R}^N$ and $\boldsymbol{u} \in \mathbb{R}^n$ we set

$$(\boldsymbol{a}\otimes S)_{ijk} := a_i S_{jk}\,, \quad (T\otimes\boldsymbol{u})_{ijk} := T_{ki}u_j + T_{ij}u_k\,.$$

2.3. The case of localized subbodies

After proving the existence and uniqueness of the tensor-valued measures $T_j d\mu_j$, the standard theory of localized subbodies, *i.e.* subbodies which are subsets of the body, can be recovered by setting

$$P(M,\boldsymbol{v}) = \sum_{j=0}^{k} \int_M \langle T_j, \nabla^{(j)}\boldsymbol{v}\rangle\, d\mu_j \tag{2.15}$$

for every Borel subset M with compact closure in Ω and every $\boldsymbol{v} \in C^\infty(\Omega; \mathbb{R}^N)$. Moreover, the following approximation result can be easily deduced by the Dominated Convergence Theorem.

Theorem 2.4. *Let M be a Borel subset with compact closure in Ω such that $\mu_j(\partial_* M) = 0$[1] for $j = 0, \ldots, k$. Let (ϑ_h) be a sequence in $\Theta(\Omega)$ such that $\vartheta_h \to \chi_M$ as $h \to \infty$ pointwise μ_j-a.e. for $j = 0, \ldots, k$. Then*

$$\forall \boldsymbol{v} \in C^\infty(\Omega; \mathbb{R}^N) : \quad \lim_{h \to \infty} P(\vartheta_h, \boldsymbol{v}) = P(M, \boldsymbol{v}) .$$

Such a sequence (ϑ_h) can be obtained, for instance, regularizing by convolution the characteristic function χ_M .

In particular, in [7], under suitable conditions on the set M and the measures μ_j, a representation formula involving the boundary of the subbody is given for the cases $k = 1$ and $k = 2$.

3. Normal traces of measures with divergence measure

The representation of a contact power as a boundary integral (by means of some generalized Gauss-Green Theorem) is one of the key links between the theory of powers and the usual approach by forces and stresses in Continuum Mechanics. Let us consider the classical situation of a power P of order one. Given an open subset M with compact closure in Ω, (2.15) yields the integral representation

$$P(M, \boldsymbol{v}) = \int_M \langle T_0, \boldsymbol{v} \rangle \, d\mu_0 + \int_M \langle T_1, \nabla \boldsymbol{v} \rangle \, d\mu_1$$

for any $\boldsymbol{v} \in C^\infty(\Omega; \mathbb{R}^N)$. If P is a contact power, it follows from (2.13) that $T_1 \mu_1$ has divergence measure, indeed

$$\operatorname{div}(T_1 \mu_1) = T_0 \mu_0 \quad \text{in the sense of distributions.}$$

For simplicity, we drop the subscript from T_1 and μ_1, meaning that T is a Borel $(N \times n)$-tensor field with $|T| \leqslant 1$ and $\mu \in \mathfrak{M}(\Omega)$. Moreover, sometimes we will

[1] The symbol $\partial_* \Omega$ denotes the *measure-theoretic boundary* of Ω, see for instance [6].

denote with T the resulting tensor-valued measure $T\mu$. Hence we can write

$$(3.1)\quad P(M,\boldsymbol{v}) = \int_M \boldsymbol{v}\cdot \operatorname{div}(T\mu) + \int_M T\cdot\nabla\boldsymbol{v}\,d\mu = \int_M \boldsymbol{v}\cdot\operatorname{div}\mathsf{T} + \int_M \nabla\boldsymbol{v}\cdot d\mathsf{T}\,.$$

In this section we focus on a fixed open set M with compact closure in Ω. For this reason, we denote M by Ω and investigate some conditions under which a general tensor-valued measure with divergence measure T on Ω admits a trace of its normal component on the boundary.

More precisely, we assume that Ω is a bounded open set in $\mathbb{R}^n$ and $\mathsf{T} = T\mu$ is a tensor-valued measure on Ω whose distributional divergence in Ω is still a measure. Moreover, both T and $\operatorname{div}\mathsf{T}$ are assumed to have bounded variation in Ω, *i.e.*

$$|\mathsf{T}|(\Omega) = \int_\Omega |T|\,d\mu < +\infty\,, \qquad |\operatorname{div}\mathsf{T}|(\Omega) < +\infty\,,$$

but we do not require any smoothness on $\partial\Omega$.

In the following, we will denote with $\mathrm{Lip}(\overline{\Omega};\mathbb{R}^N)$ the space of all Lipschitz functions on the closure of Ω.

We can now introduce $\mathsf{T}\boldsymbol{n}|_{\partial\Omega}$, the *distributional normal trace* of T to $\partial\Omega$.

Definition 3.1. *We denote with the symbol* $\mathsf{T}\boldsymbol{n}|_{\partial\Omega}$ *the (vector-valued) distribution on* $\mathbb{R}^n$ *defined as*

$$\forall\boldsymbol{v}\in C_0^\infty\left(\mathbb{R}^n;\mathbb{R}^N\right):\quad \langle\mathsf{T}\boldsymbol{n}|_{\partial\Omega},\boldsymbol{v}\rangle = \int_\Omega T\cdot\nabla\boldsymbol{v}\,d\mu + \int_\Omega \boldsymbol{v}\cdot\operatorname{div}(T\mu)\,.$$

It is clear that $\mathsf{T}\boldsymbol{n}|_{\partial\Omega}$ is a distribution of order at most one with support in $\partial\Omega$. We denote with the same symbol also the natural extension to any $\boldsymbol{v}\in\mathrm{Lip}(\overline{\Omega};\mathbb{R}^N)\cap C^1(\Omega;\mathbb{R}^N)$.

Definition 3.2. *We say that a function* $\vartheta\in\mathrm{Lip}(\overline{\Omega})\cap C^1(\Omega)$ *is* relative *to the open set* Ω, *if*

$$\begin{cases} 0<\vartheta(x)\leqslant 1 & \text{for } x\in\Omega\,,\\ \vartheta(x)=0 & \text{for } x\in\partial\Omega\,. \end{cases}$$

For every open set Ω, without further assumptions, such a function does exist. Indeed, it is a classical result of differential topology that there exists

$\vartheta \in C^\infty(\mathbb{R}^n)$ which is strictly positive in Ω and vanishes in $\mathbb{R}^n \setminus \Omega$ (see for instance [13, Exercise 2.2.1]).

In the next result we show that the distributional normal trace $\mathsf{T}\boldsymbol{n}|_{\partial\Omega}$ can be characterized also by a limiting procedure. The assertion should be compared with [19, Theorem 8.1(i) and Proposition 7.4], where the question is treated in the setting of flat measures, which allows to drop the assumption that $\boldsymbol{v} \in C^1(\Omega;\mathbb{R}^N)$.

Theorem 3.1. *Let Ω be a bounded open set and let ϑ be relative to Ω. Then for every $\boldsymbol{v} \in \mathrm{Lip}(\overline{\Omega};\mathbb{R}^N) \cap C^1(\Omega;\mathbb{R}^N)$ one has*

$$\langle \mathsf{T}\boldsymbol{n}|_{\partial\Omega}, \boldsymbol{v}\rangle = -\lim_{r\to 0^+} \frac{1}{r}\int_{\Omega\cap\{0<\vartheta<r\}} \boldsymbol{v}\cdot(T\nabla\vartheta)\,d\mu\,.$$

Moreover, for every $\boldsymbol{v} \in \mathrm{Lip}(\overline{\Omega};\mathbb{R}^N) \cap C^1(\Omega;\mathbb{R}^N)$ it also holds

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{on } \partial\Omega \quad \Longrightarrow \quad \int_\Omega T\cdot\nabla\boldsymbol{v}\,d\mu + \int_\Omega \boldsymbol{v}\cdot\mathrm{div}(T\mu) = 0\,.$$

Proof – For every $r > 0$, let $\delta_r \in]0,1[$ be such that

$$\mu\Big(\Big\{x\in\Omega:\,0<\frac{\vartheta(x)}{r}<\delta_r\Big\}\Big) < r^2\,,$$
$$\mu\Big(\Big\{x\in\Omega:\,1-\delta_r<\frac{\vartheta(x)}{r}<1\Big\}\Big) < r^2\,.$$

Then consider a function $g_r \in C^\infty(\mathbb{R})$ such that $0 \leqslant g \leqslant 1$, $g(x) = 1$ for every $x \in [\delta_r, 1-\delta_r]$ and $g(x) = 0$ for every $x \leqslant \delta_r/2$ and $x \geqslant 1$. Let G_r be the primitive of g_r satisfying $G_r(0) = 0$, and define

$$\vartheta_r(x) = G_r\Big(\frac{\vartheta(x)}{r}\Big)\,.$$

In particular,

$$\nabla\vartheta_r(x) = \frac{1}{r}g_r\Big(\frac{\vartheta(x)}{r}\Big)\nabla\vartheta(x)$$

and $\vartheta_r \in C^1_0(\Omega;\mathbb{R})$ with $\mathrm{supt}\,\vartheta_r \subseteq \{\vartheta \geqslant \delta_r/2\}$.

Now, since $\vartheta_r\boldsymbol{v}$ is a C^1-function, from the definition of distributional divergence one gets

$$\int_\Omega \vartheta_r\nabla\boldsymbol{v}\cdot T\,d\mu + \int_\Omega \vartheta_r\boldsymbol{v}\cdot\mathrm{div}(T\mu) = -\int_\Omega (T\nabla\vartheta_r)\cdot\boldsymbol{v}\,d\mu\,.$$

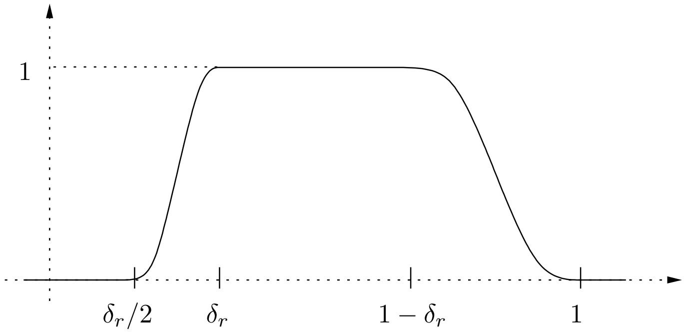

Figure 1 – The function g_r .

By the Dominated Convergence Theorem, as $r \to 0^+$ the left-hand side tends to

$$\int_\Omega \nabla \boldsymbol{v} \cdot T \, d\mu + \int_\Omega \boldsymbol{v} \cdot \mathrm{div}(T\mu) \,,$$

which is the definition of $\langle \mathsf{T}\boldsymbol{n}|_{\partial\Omega}, \boldsymbol{v}\rangle$. The right-hand side writes

$$-\int_\Omega (T\nabla\vartheta_r) \cdot \boldsymbol{v} \, d\mu = -\frac{1}{r}\int_{\Omega\cap\{0<\vartheta<r\}} (T\nabla\vartheta)\cdot \boldsymbol{v}\, d\mu$$
$$+\frac{1}{r}\int_{\Omega\cap(\{0<\frac{\vartheta}{r}<\delta_r\}\cup\{1-\delta_r<\frac{\vartheta}{r}<1\})} \Big(1 - g_r\Big(\frac{\vartheta(x)}{r}\Big)\Big)(T\nabla\vartheta)\cdot \boldsymbol{v}\, d\mu \,.$$

By the choice of δ_r, the last integral vanishes as $r \to 0^+$, and the first formula is proved.

Assume now that $\boldsymbol{v} = \boldsymbol{0}$ on $\partial\Omega$. Arguing by components, we may assume without loss of generality that $N = 1$.

Let us first treat the case in which

$$\{x \in \Omega : v(x) = 0\} \text{ is } \mu\text{-negligible.} \tag{3.2}$$

Let $\varphi : \mathbb{R} \longrightarrow [0,1]$ be a C^∞-function such that $\varphi = 0$ on $[-1,1]$ and $\varphi = 1$ outside $]-2,2[$. For every $k \geqslant 1$, let $\psi_k : \mathbb{R} \longrightarrow \mathbb{R}$ be the smooth function such that $\psi_k(0) = 0$ and $\psi_k'(s) = \varphi(ks)$. Since $\psi_k(v) \in C^1_0(\Omega)$, we have

$$\int_\Omega \psi_k'(v)\, T\cdot\nabla v\, d\mu + \int_\Omega \psi_k(v)\,\mathrm{div}(T\mu) = 0\,.$$

On the other hand, $(\psi_k(v))$ is convergent to v uniformly on Ω and $(\psi_k'(v))$ is convergent to 1 pointwise μ-a.e. in Ω by (3.2). Passing to the limit as $k \to \infty$, we get

$$\int_\Omega T \cdot \nabla v \, d\mu + \int_\Omega v \, \mathrm{div}(T\mu) = 0 \, .$$

In the general case, let $w \in C^\infty(\mathbb{R}^n)$ be such that $w > 0$ in Ω and $w = 0$ in $\mathbb{R}^n \setminus \Omega$. Since the sets

$$\{x \in \Omega : \, v(x) + \varepsilon w(x) = 0\} \, , \qquad \varepsilon > 0$$

are pairwise disjoint, there exists $\varepsilon > 0$ such that $\{x \in \Omega : \, v(x) + \varepsilon w(x) = 0\}$ is μ-negligible. From the previous step, we infer that

$$\int_\Omega T \cdot \nabla (v + \varepsilon w) \, d\mu + \int_\Omega (v + \varepsilon w) \, \mathrm{div}(T\mu) = 0 \, .$$

Again from the previous step we also deduce that

$$\int_\Omega T \cdot \nabla (\varepsilon w) \, d\mu + \int_\Omega (\varepsilon w) \, \mathrm{div}(T\mu) = 0 \, ,$$

as the set $\{x \in \Omega : \, \varepsilon w(x) = 0\}$ is empty. By subtracting the two equations, the assertion follows. □

Remark 3.1 - By the previous result, the distribution $\mathsf{T}\boldsymbol{n}|_{\partial\Omega}$ can be extended to

$$\left\{ \boldsymbol{u} \in C(\partial\Omega; \mathbb{R}^N) : \;\; \boldsymbol{u} = \boldsymbol{v}\big|_{\partial\Omega} \text{ for some } \boldsymbol{v} \in \mathrm{Lip}(\overline{\Omega}; \mathbb{R}^N) \cap C^1(\Omega; \mathbb{R}^N) \right\}$$

by setting

$$\langle \mathsf{T}\boldsymbol{n}|_{\partial\Omega}, \boldsymbol{u} \rangle = \int_\Omega T \cdot \nabla \boldsymbol{v} \, d\mu + \int_\Omega \boldsymbol{v} \cdot \mathrm{div}(T\mu) \, .$$

The previous theorem ensures that the definition is independent of the choice of $\boldsymbol{v}$.

It is of great interest the case in which the distribution $\mathsf{T}\boldsymbol{n}|_{\partial\Omega}$ is a measure, *i.e.* it is of order zero as a distribution in $\mathbb{R}^n$. In the following theorem we give a sufficient condition, already considered in [19, Theorem 8.1(ii)].

Theorem 3.2. *Let Ω be a bounded open set and let ϑ be relative to Ω. If*

$$\limsup_{r\to 0^+} \frac{1}{r} \int_{\Omega\cap\{0<\vartheta<r\}} |T\nabla\vartheta|\, d\mu < +\infty\,, \tag{3.3}$$

then $\mathsf{T}\boldsymbol{n}|_{\partial\Omega}$ is a distribution of order zero in $\mathbb{R}^n$ with support in $\partial\Omega$, so that the pairing $\langle \mathsf{T}\boldsymbol{n}|_{\partial\Omega}, \boldsymbol{v}\rangle$ has a natural meaning for any $\boldsymbol{v} \in C(\partial\Omega;\mathbb{R}^N)$.

Moreover, for every $\boldsymbol{v} \in C(\overline{\Omega};\mathbb{R}^N)$ one has

$$\langle \mathsf{T}\boldsymbol{n}|_{\partial\Omega}, \boldsymbol{v}\rangle = -\lim_{r\to 0^+} \frac{1}{r} \int_{\Omega\cap\{0<\vartheta<r\}} (T\nabla\vartheta)\cdot \boldsymbol{v}\, d\mu\,.$$

Proof – The proof relies on some standard facts of Measure Theory. Let (γ_r) be the family of measures on $\overline{\Omega}$ defined by

$$\langle \gamma_r, \boldsymbol{v}\rangle := -\frac{1}{r} \int_{\Omega\cap\{0<\vartheta<r\}} (T\nabla\vartheta)\cdot \boldsymbol{v}\, d\mu\,.$$

By Theorem 3.1 we know that

$$\forall \boldsymbol{v} \in \mathrm{Lip}(\overline{\Omega};\mathbb{R}^N)\cap C^1(\Omega;\mathbb{R}^N): \quad \lim_{r\to 0^+} \langle \gamma_r, \boldsymbol{v}\rangle = \langle \mathsf{T}\boldsymbol{n}|_{\partial\Omega}, \boldsymbol{v}\rangle\,.$$

Moreover, by (3.3) the measures γ_r have uniformly bounded total variation, hence there exists a measure γ such that $\gamma_r \overset{*}{\rightharpoonup} \gamma$ up to a subsequence. Since $\mathrm{Lip}(\overline{\Omega};\mathbb{R}^N)\cap C^1(\Omega;\mathbb{R}^N)$ is dense in $C(\overline{\Omega};\mathbb{R}^N)$, it follows that $\gamma = \mathsf{T}\boldsymbol{n}|_{\partial\Omega}$, hence the normal trace is a measure, and the full sequence converges. Then the proof is complete. □

4. The rectangle

In Theorem 3.2 we gave a sufficient condition and a limit formula in order to find the measure normal trace on an open subbody. In the present section we want to study the very particular case of $n = 2$ and $\Omega =]a,b[\times]c,d[$, *i.e.* a two-dimensional rectangle. We will find the expression of the normal trace of a tensor-valued measure T by means of the so-called *Disintegration Theorem*.

Theorem 4.1 (Disintegration). *Let $\mu \in \mathfrak{M}(\Omega)$. Then there exist $\lambda_1 \in \mathfrak{M}(\mathbb{R})$ and, for every $x \in \mathbb{R}$, $\gamma_x \in \mathfrak{M}(\mathbb{R})$ such that $\gamma_x(\mathbb{R}) = 1$ and*

$$\left\{x \mapsto \int_{\mathbb{R}} f(x,y)\, d\gamma_x(y)\right\} \qquad \text{is } \lambda_1\text{-measurable,}$$

$$\int_{\Omega} f(x,y)\, d\mu(x,y) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\, d\gamma_x(y) \right) d\lambda_1(x)\,,$$

for every Borel function $f : \mathbb{R}^2 \to [0, +\infty]$.

A similar assertion holds swapping x and y, yielding measures λ_2 and γ_y.

Proof – See [1]. □

Remark 4.1 - Consider the one-dimensional Lebesgue measure $\mathscr{L}^1$ and the Lebesgue decomposition

$$\lambda_1 = \frac{d\lambda_1}{d\mathscr{L}^1}\, \mathscr{L}^1 + \lambda_1^{(s)}\,,$$

$$\lambda_2 = \frac{d\lambda_2}{d\mathscr{L}^1}\, \mathscr{L}^1 + \lambda_2^{(s)}\,,$$

where $d\lambda/d\mathscr{L}^1$ denotes the Radon-Nikodym derivative of the measure λ with respect to $\mathscr{L}^1$ and $\lambda^{(s)}$ the singular part of λ with respect to $\mathscr{L}^1$. Then one can also write

$$\begin{aligned}
\int_{\Omega} f(x,y)\, d\mu(x,y) &= \int_a^b \left(\frac{d\lambda_1}{d\mathscr{L}^1}(x) \int_{]c,d[} f(x,y)\, d\gamma_x(y) \right) d\mathscr{L}^1(x) \\
&\quad + \int_{]a,b[} \left(\int_{]c,d[} f(x,y)\, d\gamma_x(y) \right) d\lambda_1^{(s)}(x) \\
&= \int_c^d \left(\frac{d\lambda_2}{d\mathscr{L}^1}(y) \int_{]a,b[} f(x,y)\, d\gamma_y(x) \right) d\mathscr{L}^1(y) \\
&\quad + \int_{]c,d[} \left(\int_{]a,b[} f(x,y)\, d\gamma_y(x) \right) d\lambda_2^{(s)}(y)\,.
\end{aligned}$$

We recall here the definition of Lebesgue point for a function.

Definition 4.1. *Let $f : \mathbb{R} \to \mathbb{R}^N$ a Borel function. Then we say that $x \in \mathbb{R}$ is* a right Lebesgue point *for f, if there exists $\ell \in \mathbb{R}$ such that*

$$\lim_{r\to 0^+} \frac{1}{r} \int_{]x,x+r[} |f(\xi) - \ell| \, d\mathscr{L}^1(\xi) = 0\,.$$

We say that ℓ is the right Lebesgue value *of f, and analogously for the* left Lebesgue point *and the* left Lebesgue value.

Given such a Borel function f, it is well-known that $\mathscr{L}^1$-a.e. $x \in \mathbb{R}$ is a right (left) Lebesgue point for f with $\ell = f(x)$.

Theorem 4.2. *Assume that*

i. $\displaystyle \limsup_{r\to 0^+} \frac{1}{r} \int_{]a,a+r[\times]c,d[} |T(x,y)\boldsymbol{e}_1| \, d\mu(x,y) < +\infty\,,$

ii. $\displaystyle \lim_{r\to 0^+} \frac{1}{r} \lambda_1^{(s)}(]a,a+r[) = 0\,,$

iii. for every $\boldsymbol{v} \in C([c,d];\mathbb{R}^N)$, $x = a$ is a right Lebesgue point for

$$x \mapsto \frac{d\lambda_1}{d\mathscr{L}^1}(x) \int_{]c,d[} \boldsymbol{v}(y) \cdot T(x,y)\boldsymbol{e}_1 \, d\gamma_x(y)\,. \tag{4.1}$$

Let $\varphi_{\boldsymbol{v}}$ denote the right Lebesgue value of the function (4.1), *i.e.*

$$\varphi_{\boldsymbol{v}} = \lim_{r\to 0^+} \frac{1}{r} \int_a^{a+r} \left[\frac{d\lambda_1}{d\mathscr{L}^1}(x) \int_{]c,d[} \boldsymbol{v}(y) \cdot T(x,y)\boldsymbol{e}_1 \, d\gamma_x(y) \right] d\mathscr{L}^1(x)\,.$$

Then there exists a finite vector-valued measure $\boldsymbol{\nu}_a$ on $[c,d]$ such that

$$\int_{[c,d]} \boldsymbol{v} \cdot \boldsymbol{\nu}_a = - \lim_{r\to 0^+} \frac{1}{r} \int_{]a,a+r[\times]c,d[} \boldsymbol{v}(y) \cdot T(x,y)\boldsymbol{e}_1 \, d\mu(x,y) = -\varphi_{\boldsymbol{v}} \tag{4.2}$$

for every $\boldsymbol{v} \in C([c,d];\mathbb{R}^N)$.

In particular, the normal trace $\boldsymbol{\nu}_a$ is given by the right Lebesgue value of (4.1), which is strictly related to the measures γ_x and λ_1 that come from the Disintegration Theorem.

With a similar construction we can define $\boldsymbol{\nu}_b$, $\boldsymbol{\nu}_c$ and $\boldsymbol{\nu}_d$, paying attention to taking the *right* Lebesgue value for a, c and the *left* Lebesgue value for b, d.

PROOF – Let $\boldsymbol{v} \in C([c,d];\mathbb{R}^N)$ and let $\varphi_{\boldsymbol{v}}$ as in the theorem. Then by Remark 4.1 one has

$$\int_{]a,a+r[\times]c,d[} \boldsymbol{v}(y)\cdot T(x,y)\boldsymbol{e}_1\,d\mu(x,y)$$

$$= \int_a^{a+r}\left[\frac{d\lambda_1}{d\mathscr{L}^1}(x)\int_{]c,d[}\boldsymbol{v}(y)\cdot T(x,y)\boldsymbol{e}_1\,d\gamma_x(y)\right]d\mathscr{L}^1(x)$$

$$+\int_{]a,a+r[}\left(\int_{]c,d[}\boldsymbol{v}(y)\cdot T(x,y)\boldsymbol{e}_1\,d\gamma_x(y)\right)d\lambda_1^{(s)}(x)\,.$$

Now divide by r and let $r\to 0^+$. The last integral vanishes in view of *ii* and the boundedness of the integrand and the measure γ_x. Hence one gets the second identity in (4.2). The fact that $\varphi_{\boldsymbol{v}}$ is indeed a distribution of order zero follows easily by *i*. □

Remark 4.2 - It can be proved, by an approximation procedure, that condition *iii* can be required only for any $\boldsymbol{v}\in C_0^\infty(\mathbb{R};\mathbb{R}^N)$.

Indeed, suppose that *iii* holds for every $\boldsymbol{v}\in C_0^\infty(\mathbb{R};\mathbb{R}^N)$ and consider $\boldsymbol{w}\in C([c,d];\mathbb{R}^N)$. Hence

$$\frac{1}{r}\int_a^{a+r}\left|\frac{d\lambda_1}{d\mathscr{L}^1}(x)\int_{]c,d[}\boldsymbol{w}(y)\cdot T(x,y)\boldsymbol{e}_1\,d\gamma_x(y)-\varphi_{\boldsymbol{w}}\right|d\mathscr{L}^1(x)$$

$$\leqslant \frac{1}{r}\int_a^{a+r}\left|\frac{d\lambda_1}{d\mathscr{L}^1}(x)\int_{]c,d[}\boldsymbol{v}(y)\cdot T(x,y)\boldsymbol{e}_1\,d\gamma_x(y)-\varphi_{\boldsymbol{v}}\right|d\mathscr{L}^1(x)$$

$$+\|\boldsymbol{w}-\boldsymbol{v}\|_\infty\frac{1}{r}\int_{]a,a+r[\times]c,d[}|T(x,y)\boldsymbol{e}_1|\,d\mu(x,y)+|\varphi_{\boldsymbol{w}}-\varphi_{\boldsymbol{v}}|\,.$$

Since the property holds for $\boldsymbol{v}$ and it can be chosen arbitrarily close to $\boldsymbol{w}$ in the uniform norm, keeping into account *i* the right-hand side can be made arbitrarily small.

The following theorem assures that a normal trace can be found for almost every rectangle (in the sense of one-dimensional Lebesgue measure).

Theorem 4.3. *For $\mathscr{L}^1$-a.e. $a, b, c, d \in \mathbb{R}$ the conditions i, ii, iii of Theorem 4.2 are satisfied and*

$$\boldsymbol{\nu}_a = -\frac{d\lambda_1}{d\mathscr{L}^1}(a) T(a, \cdot)\boldsymbol{e}_1 \, \gamma_a$$

$$\boldsymbol{\nu}_b = \frac{d\lambda_1}{d\mathscr{L}^1}(b) T(b, \cdot)\boldsymbol{e}_1 \, \gamma_b$$

$$\boldsymbol{\nu}_c = -\frac{d\lambda_2}{d\mathscr{L}^1}(c) T(\cdot, c)\boldsymbol{e}_2 \, \gamma_c$$

$$\boldsymbol{\nu}_d = \frac{d\lambda_2}{d\mathscr{L}^1}(d) T(\cdot, d)\boldsymbol{e}_2 \, \gamma_d \, .$$

Proof – We prove the theorem for a, the other cases following in the same way.

Since $\lambda_1^{(s)}$ is singular with respect to the Lebesgue measure, it is clear that *ii* holds for a.e. $a \in \mathbb{R}$.

Let us now study *iii*. Since $C_0(\mathbb{R}; \mathbb{R}^N)$ is separable, it admits a countable and dense subset Y. Let $a \in \mathbb{R}$ be a right Lebesgue point for the function (4.1) and such that the right Lebesgue value is

$$\varphi_{\boldsymbol{v}} = \frac{d\lambda_1}{d\mathscr{L}^1}(a) \int_{]c,d[} \boldsymbol{v}(y) \cdot T(a, y)\boldsymbol{e}_1 \, d\gamma_a(y)$$

for every $\boldsymbol{v} \in Y$ and $c, d \in \mathbb{Q}$. It is well-known that this is true for $\mathscr{L}^1$-a.e. $a \in \mathbb{R}$.

Take $\boldsymbol{w} \in C_0(\mathbb{R}; \mathbb{R}^N)$ and let $\varepsilon > 0$. Then there is $\boldsymbol{v} \in Y$ such that $\|\boldsymbol{w} - \boldsymbol{v}\|_\infty < \varepsilon$ and

$$\frac{1}{r} \int_a^{a+r} \left| \frac{d\lambda_1}{d\mathscr{L}^1}(x) \int_{]c,d[} (\boldsymbol{v} - \boldsymbol{w})(y) \cdot T(x, y)\boldsymbol{e}_1 \, d\gamma_x(y) \right| d\mathscr{L}^1(x)$$

$$\leqslant \|\boldsymbol{v} - \boldsymbol{w}\|_\infty \frac{1}{r} \int_a^{a+r} \frac{d\lambda_1}{d\mathscr{L}^1}(x) \, d\mathscr{L}^1(x)$$

which vanishes as $r \to 0^+$, so that we can replace Y with $C_0(\mathbb{R}; \mathbb{R}^N)$.

Moreover, let $c \in \mathbb{R}$ be a Lebesgue point for $d\lambda_2 / d\mathscr{L}^1$ and such that *ii* holds. For $r > 0$, consider $0 \leqslant \delta_r \leqslant r^2$ be such that $c + \delta_r \in \mathbb{Q}$. Then for every

$\boldsymbol{w} \in C_0(\mathbb{R};\mathbb{R}^N)$ one has

$$\begin{aligned}
\frac{1}{r}\int_a^{a+r}&\left|\frac{d\lambda_1}{d\mathscr{L}^1}(x)\int_{]c,c+\delta_r[}\boldsymbol{v}(y)\cdot T(x,y)\boldsymbol{e}_1\,d\gamma_x(y)\right|d\mathscr{L}^1(x)\\
&\leqslant \frac{1}{r}\|\boldsymbol{w}\|_\infty \mu(]a,a+r[\times]c,c+\delta_r[)\\
&\leqslant \frac{1}{r}\|\boldsymbol{w}\|_\infty \lambda_2(]c,c+\delta_r[)\leqslant \frac{1}{r}\|\boldsymbol{w}\|_\infty \lambda_2(]c,c+r^2[)
\end{aligned}$$

and the right-hand side vanishes for $r\to 0^+$. Hence we can replace $c\in\mathbb{Q}$ with a.e. $c\in\mathbb{R}$, and of course the same holds for $d\in\mathbb{R}$.

The proof for property *i* is similar. □

Now we compare the trace introduced in Theorem 4.2 for a rectangle with respect to the general theory of Section 3. In particular, we will prove that, if a,b,c,d satisfy the assumptions *i*–*iii*, then the trace $\mathsf{T}\boldsymbol{n}|_{\partial M}$ is given by the measures $\boldsymbol{\nu}_a,\boldsymbol{\nu}_b,\boldsymbol{\nu}_c,\boldsymbol{\nu}_d$ and, in particular, the Gauss-Green Theorem holds for such a rectangle.

Hereafter we will assume that $\Omega=]a,b\times]c,d[$ with a,b,c,d satisfying the conditions *i*, *ii*, *iii* of Theorem 4.2.

Theorem 4.4. *The distribution* $\mathsf{T}\boldsymbol{n}|_{\partial\Omega}$ *has order zero.*

Proof – Let $Q=[0,1]\times[0,1]$ and consider the function $f:Q\to\mathbb{R}$ defined by

$$f(x,y):=\frac{\sin(\pi x)\sin(\pi y)}{\sqrt{\sin^2(\pi x)+\sin^2(\pi y)}}.$$

Then $f\in \mathrm{Lip}(Q)\cap C^1(\mathrm{int}\,Q)$ with Lipschitz constant $L=\pi$; moreover, there exist $k_1,k_2>0$ such that

$$\{(x,y)\in Q:\ f(x,y)=r\}\subseteq\{(x,y)\in Q:\ k_1 r<d((x,y),\partial Q)<k_2 r\}$$

(indeed, it is enough to choose $k_1<1/\pi$ and $k_2>\sqrt{2}/\pi$).

Define now

$$\vartheta(x,y)=f\left(\frac{x-a}{b-a},\frac{y-c}{d-c}\right).$$

Then ϑ is relative to the set $\Omega=]a,b\times]c,d[$ and, for suitable constants $L,k>0$,

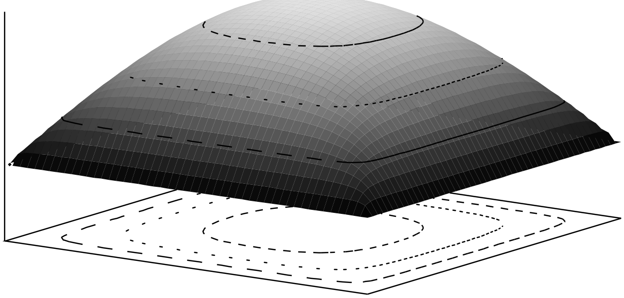

Figure 2 – The function f.

$$\limsup_{r\to 0^+}\frac{1}{r}\int_{\Omega\cap\{0<\vartheta<r\}}|T\nabla\vartheta|\,d\mu$$
$$\leqslant \limsup_{r\to 0^+}\frac{L}{r}\int_{(]a,a+kr[\times]c,d[\,\cup\,]b-kr,b[\times]c,d[)}|T\boldsymbol{e}_1|\,d\mu$$
$$+\limsup_{r\to 0^+}\frac{L}{r}\int_{(]a,b[\times]c,c+kr[\,\cup\,]a,b[\times]d-kr,d[)}|T\boldsymbol{e}_2|\,d\mu<+\infty\,.$$

Hence (3.3) holds and we can apply Theorem 3.2. □

Lemma 4.1. *Let* $\boldsymbol{v}\in \mathrm{Lip}(\overline{\Omega};\mathbb{R}^N)\cap C^1(\Omega;\mathbb{R}^N)$ *be such that* $\boldsymbol{v}(a,y)=\boldsymbol{v}(b,y)=0$ *for every* $y\in[c,d]$. *Then*

$$\langle \mathsf{T}\boldsymbol{n}|_{\partial\Omega},\boldsymbol{v}\rangle=\int_{]a,b[}\boldsymbol{v}(\cdot,c)\cdot\boldsymbol{\nu}_c+\int_{]a,b[}\boldsymbol{v}(\cdot,d)\cdot\boldsymbol{\nu}_d\,.$$

Proof – For every $r>0$, let $\delta_r\in]0,r[$ be such that

$$\mu\Big(\Big\{(x,y)\in\Omega:\,c+r-\delta_r<y<c+r\Big\}\Big)<r^2\,,$$

$$\mu\Big(\Big\{x\in\Omega:\,d-r<y<d-r+\delta_r\Big\}\Big)<r^2\,.$$

Let $f_r\in \mathrm{Lip}([c,d])\cap C^1(]c,d[)$ be such that $f_r(c)=f_r(d)=0$, $f_r(x)=1$ for $c+r\leqslant x\leqslant d-r$ and f_r is linear in $[c,c+r-\delta_r]$ and $[d-r+\delta_r,d]$ (see Figure 4). Let $\boldsymbol{v}\in\mathrm{Lip}(\overline{\Omega};\mathbb{R}^N)\cap C^1(\Omega;\mathbb{R}^N)$. Then the function $f_r(y)\boldsymbol{v}(x,y)$

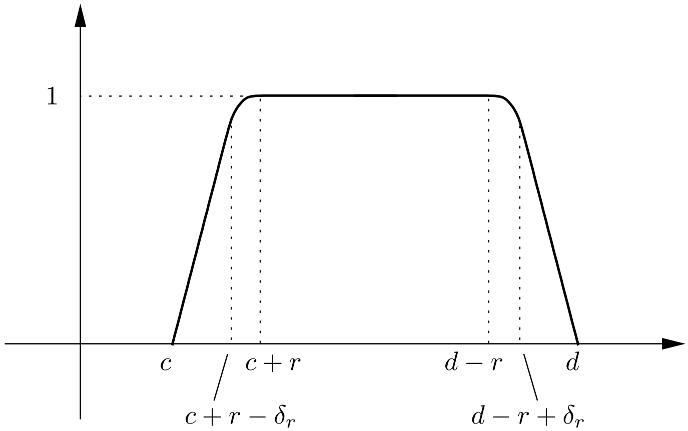

Figure 3 – The function f_r .

vanishes on $\partial\Omega$ and, by Theorem 3.1, one has

$$\int_\Omega f_r \boldsymbol{v} \cdot \operatorname{div}(T\mu) + \int_\Omega f_r \nabla\boldsymbol{v} \cdot T\, d\mu + \int_\Omega (\boldsymbol{v} \otimes \nabla f_r) \cdot T\, d\mu = 0\,.$$

Now, the first two integrals converge to $\langle \mathsf{T}\boldsymbol{n}|_{\partial\Omega}, \boldsymbol{v}\rangle$ as $r \to 0^+$ and last integral writes

$$\int_\Omega (\boldsymbol{v}\otimes\nabla f_r)\cdot T\, d\mu = -\frac{1}{r}\int_{]a,b[\times]c,c+r[} (\boldsymbol{v}\otimes\boldsymbol{e}_2)\cdot T\, d\mu + \frac{1}{r}\int_{]a,b[\times]d-r,d[} (\boldsymbol{v}\otimes\boldsymbol{e}_2)\cdot T\, d\mu + o(r)$$

as $r \to 0^+$. The proof ends up by applying (4.2). □

Theorem 4.5 (Gauss-Green)**.** *Let* $\boldsymbol{v} \in \operatorname{Lip}(\overline{\Omega};\mathbb{R}^N) \cap C^1(\Omega;\mathbb{R}^N)$*. Then*

$$\begin{aligned}\langle \mathsf{T}\boldsymbol{n}|_{\partial\Omega}, \boldsymbol{v}\rangle &= \int_{]a,b[} \boldsymbol{v}(\cdot,c)\cdot\boldsymbol{\nu}_c + \int_{]a,b[} \boldsymbol{v}(\cdot,d)\cdot\boldsymbol{\nu}_d + \int_{[c,d]} \boldsymbol{v}(a,\cdot)\cdot\boldsymbol{\nu}_a + \int_{[c,d]} \boldsymbol{v}(b,\cdot)\cdot\boldsymbol{\nu}_b \\ &= \int_{[a,b]} \boldsymbol{v}(\cdot,c)\cdot\boldsymbol{\nu}_c + \int_{[a,b]} \boldsymbol{v}(\cdot,d)\cdot\boldsymbol{\nu}_d + \int_{]c,d[} \boldsymbol{v}(a,\cdot)\cdot\boldsymbol{\nu}_a + \int_{]c,d[} \boldsymbol{v}(b,\cdot)\cdot\boldsymbol{\nu}_b\,.\end{aligned}$$

In particular, $\boldsymbol{\nu}_c(\{a\}) = \boldsymbol{\nu}_a(\{c\})$ and the same holds for the other vertices.

Proof – Let $\boldsymbol{v} \in \operatorname{Lip}(\overline{\Omega};\mathbb{R}^N) \cap C^1(\Omega;\mathbb{R}^N)$ and consider a function $f_r \in C([a,b]) \cap C^1(]a,b[)$ as in the proof of the above lemma. Then the function

$\boldsymbol{w}_r(x,y) = f_r(x)\boldsymbol{v}(x,y)$ is such that $\boldsymbol{w}_r(a,y) = \boldsymbol{w}_r(b,y) = 0$ for every $y \in [c,d]$. By Lemma 4.1, one has

$$(4.3)\quad \int_\Omega T \cdot \nabla \boldsymbol{w}_r \, d\mu + \int_\Omega \boldsymbol{w}_r \cdot \mathrm{div}(T\mu) = \int_{]a,b[\times\{c\}} \boldsymbol{w}_r \cdot \boldsymbol{\nu}_c + \int_{]a,b[\times\{d\}} \boldsymbol{w}_r \cdot \boldsymbol{\nu}_d \,.$$

We can write the first integral as

$$\begin{aligned}\int_\Omega T \cdot \nabla \boldsymbol{w}_r \, d\mu &= \int_\Omega f_r \nabla \boldsymbol{v} \cdot T \, d\mu + \int_\Omega (\boldsymbol{v} \otimes \nabla f_r) \cdot T \, d\mu \\ &= \int_\Omega f_r \nabla \boldsymbol{v} \cdot T \, d\mu + \frac{1}{r} \int_{]a,a+r[\times]c,d[} (\boldsymbol{v} \otimes \boldsymbol{e}_1) \cdot T \, d\mu \\ &\qquad - \frac{1}{r} \int_{]b-r,b[\times]c,d[} (\boldsymbol{v} \otimes \boldsymbol{e}_1) \cdot T \, d\mu + o(r) \,.\end{aligned}$$

Taking into account (4.2) and the Dominated Convergence Theorem, letting $r \to 0^+$ in (4.3) one gets

$$\langle \mathsf{T}\boldsymbol{n}|_{\partial\Omega}, \boldsymbol{v}\rangle = \int_{]a,b[} \boldsymbol{v}(\cdot,c)\cdot\boldsymbol{\nu}_c + \int_{]a,b[} \boldsymbol{v}(\cdot,d)\cdot\boldsymbol{\nu}_d + \int_{[c,d]} \boldsymbol{v}(a,\cdot)\cdot\boldsymbol{\nu}_a + \int_{[c,d]} \boldsymbol{v}(b,\cdot)\cdot\boldsymbol{\nu}_b$$

which proves the first formula. The second one can be proved by the same procedure. □

5. Example: the Flamant tensor field

In this last section we apply the above theory to the so called *Flamant tensor field*, which is the stress tensor field in an elastic half-plane with a concentrated load applied perpendicularly to its boundary. Our interest in this example started with [15] and was further motivated by [16]. Let us also mention [19, Example 9.2], where a case with similar features was considered, namely that of a Newtonian force.

We have $n = N = 2$. If the half-plane is the set $H = \{(x,y) \in \mathbb{R}^2 : x > 0\}$ and the load $\boldsymbol{f} = f\boldsymbol{e}_1$ is applied at the origin O of the frame of reference, then the stress distribution turns out to be the 2×2 tensor field

$$T(x,y) = -\frac{2f}{\pi} \frac{x}{(x^2+y^2)^2} \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}$$

or, in polar coordinates,

$$T(\rho, \vartheta) = -\frac{2f}{\pi} \frac{\cos \vartheta}{\rho} \boldsymbol{e}_\rho \otimes \boldsymbol{e}_\rho \, .$$

In particular, $T \in L^1_{loc}(H; \mathrm{Sym}_1)$.

It is worth noting that T admits a divergence-free extension to all of $\mathbb{R}^2$ by setting

$$T(x, y) = -\frac{2f}{\pi} \frac{|x|}{(x^2 + y^2)^2} \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}$$

or

$$T(\rho, \vartheta) = -\frac{2f}{\pi} \frac{|\cos \vartheta|}{\rho} \boldsymbol{e}_\rho \otimes \boldsymbol{e}_\rho \, .$$

Then again $T \in L^1_{loc}(\mathbb{R}^2; \mathrm{Sym}_1)$ and one can prove that indeed $\operatorname{div} T = \mathbf{0}$ on $\mathbb{R}^2$ in the sense of distributions.

We will see in a moment that, despite the integrability of T and the regularity of its divergence, the normal trace of T on the boundary of a rectangle passing through the origin is a singular measure (in the present case, a Dirac delta).

5.1. Singularity on a side

Let now $\Omega =]0, b[\times]c, d[$ with $c < 0$ and $d > 0$. We want to prove that T satisfies *i–iii* of Theorem 4.2 on Ω. In proving this, it may help to consider that T admits a potential $\boldsymbol{g}$ (since $\operatorname{div} T = \mathbf{0}$ in the sense of distributions), indeed

$$T = \begin{bmatrix} \dfrac{\partial g_1}{\partial y} & -\dfrac{\partial g_1}{\partial x} \\[2mm] \dfrac{\partial g_2}{\partial y} & -\dfrac{\partial g_2}{\partial x} \end{bmatrix}$$

where

$$\boldsymbol{g} = \frac{f}{\pi} \left[\left(-\arctan \frac{y}{|x|} - \frac{|x| y}{x^2 + y^2} \right) \boldsymbol{e}_1 + \frac{|x| x}{x^2 + y^2} \boldsymbol{e}_2 \right] \quad \in L^\infty(\mathbb{R}^2; \mathbb{R}^2) \, .$$

Being T a C^∞-tensor field outside the origin, it is clear that the only side which can be tricky is the segment $\{0\} \times]c, d[$. Moreover, since T is integrable, the Disintegration Theorem is simply Fubini's Theorem and $\lambda_1 = \lambda_2 = \mathscr{L}^1$.

i. One has

$$\begin{aligned}\int_c^d |T(x,y)\boldsymbol{e}_1|\,dy &= \int_c^d \left|\frac{\partial g_1}{\partial y}\boldsymbol{e}_1 + \frac{\partial g_2}{\partial y}\boldsymbol{e}_2\right| dy \\ &\leqslant -\int_c^d \frac{\partial g_1}{\partial y} + \int_c^0 \frac{\partial g_2}{\partial y}dy - \int_0^d \frac{\partial g_2}{\partial y}dy \\ &\leqslant -\Big(g_1(x,d) - g_1(x,c)\Big) + \Big(g_2(x,0) - g_2(x,c)\Big) - \Big(g_2(x,d) - g_2(x,0)\Big)\end{aligned}$$

which is uniformly bounded in x, whence

$$\limsup_{r\to 0^+} \frac{1}{r}\int_0^r \int_c^d |T(x,y)\boldsymbol{e}_1|\,dy\,dx < +\infty\,.$$

ii. The disintegration is simply Fubini's Theorem, hence $\lambda_1^{(s)} = 0$.

iii. For every $\boldsymbol{v} \in C([c,d];\mathbb{R}^2)$, we have to check that $x = 0$ is a right Lebesgue point for the function

$$x \mapsto \int_c^d \boldsymbol{v}(y)\cdot T(x,y)\boldsymbol{e}_1\,dy\,.$$

Indeed, as noted in Remark 4.2, it is sufficient to consider $\boldsymbol{v} \in C_0^\infty\big(\mathbb{R};\mathbb{R}^2\big)$. Let us perform an integration by parts:

$$\begin{aligned}\int_c^d \boldsymbol{v}(y)\cdot T(x,y)\boldsymbol{e}_1\,dy &= \int_c^d \left(\frac{\partial g_1}{\partial y}v_1 + \frac{\partial g_2}{\partial y}v_2\right) dy \\ &= g_1(x,d)v_1(d) - g_1(x,c)v_1(c) + g_2(x,d)v_2(d) - g_2(x,c)v_2(c) \\ &\quad - \int_c^d (g_1 v_1' + g_2 v_2')\,dy\,.\end{aligned}$$

Since

$$\lim_{x\to 0^+} \boldsymbol{g}(x,c) = \frac{f}{2}\boldsymbol{e}_1\,, \qquad \lim_{x\to 0^+} \boldsymbol{g}(x,d) = -\frac{f}{2}\boldsymbol{e}_1\,,$$

by the Dominated Convergence Theorem one has for the last integral

$$\begin{aligned}\lim_{x\to 0^+} \int_c^d (g_1 v_1' + g_2 v_2')\,dy &= \lim_{x\to 0^+}\int_c^0 g_1 v_1'\,dy + \lim_{x\to 0^+}\int_0^d g_1 v_1'\,dy \\ &= \frac{f}{2}\int_c^0 v_1'\,dy - \frac{f}{2}\int_0^d v_1'\,dy = \frac{f}{2}\Big(2v_1(0) - v_1(c) - v_1(d)\Big)\,.\end{aligned}$$

Therefore

$$\lim_{x\to 0^+} \int_c^d \boldsymbol{v}(y)\cdot T(x,y)\boldsymbol{e}_1\,dy = -\frac{f}{2}\Big(v_1(d)+v_1(c)\Big)$$
$$-\frac{f}{2}\Big(2v_1(0)-v_1(c)-v_1(d)\Big) = -fv_1(0)$$

and 0 is a right Lebesgue point with Lebesgue value $\varphi_{\boldsymbol{v}} = -fv_1(0)$.

Finally, from the previous computation one finds that $\boldsymbol{\nu}_a = f\boldsymbol{e}_1\delta_0$, where the right-hand side denotes the Dirac measure on $[c,d]$ concentrated in 0.

5.2. Singularity at a vertex

Consider now a rectangle of the form $\Omega =]0,b[\times]0,d[$. With the same computations of the previous subsection, one can prove that also in this case T satisfies *i*–*iii* of Theorem 4.2 on Ω. Let us pay more attention to *iii*, which gives us the value of the normal trace. There are now two tricky sides, indeed $\{0\}\times]0,d[$ and $]0,b[\times\{0\}$. For instance, let us study the first. As above, we compute for $\boldsymbol{v}\in C_0^\infty(\mathbb{R};\mathbb{R}^2)$ the limit

$$\lim_{x\to 0^+} \int_0^d \boldsymbol{v}(y)\cdot T(x,y)\boldsymbol{e}_1\,dy\,,$$

taking into account that

$$\lim_{x\to 0^+} \boldsymbol{g}(x,0) = \frac{f}{\pi}\boldsymbol{e}_2\,.$$

This time we find $\nu_a = \left(\frac{f}{2}\boldsymbol{e}_1 + \frac{f}{\pi}\boldsymbol{e}_2\right)\delta_0$. Again, the measure concentrates on the vertex $(0,0)$, but now it has also a non trivial component along $\boldsymbol{e}_2$.

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