ON THE FLECK AND WILLIS HOMOGENIZATION PROCEDURE IN STRAIN GRADIENT PLASTICITY

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Abstract. We revisit the homogenization process for a heterogeneous small strain gradient plasticity model considered in [5]. We derive a precise homogenized behavior, independently of any kind of periodicity assumption and demonstrate that it reduces to a model studied in [7] when periodicity is re-introduced.

Keywords:

Homogenization, gradient plasticity, H-convergence

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1. Introduction

Strain gradient plasticity models have been introduced in recent years to describe size effects in ductile metals which cannot be captured by standard plasticity theories (see [3, 4, 8, 9] and references therein). They are named after the inclusion in the model of the gradient of plastic strain, which is connected to the density of *geometrically necessary dislocations* present in the body (see [2]).

Let $\Omega \subseteq \mathbb{R}^N$ be the reference configuration of an elasto-plastic body undergoing infinitesimal displacements, and let us assume that the family of admissible configurations is given by the pairs (u,p), where $u : \Omega \to \mathbb{R}^N$ denotes the displacement of Ω, and $p : \Omega \to \mathrm{M}^N_D$ is the associated plastic strain. Note that p takes values in the space M^N_D of symmetric matrices with zero trace since it is usually assumed that plastic deformations take place without volume changes. The point of view of the *deformation theory* in plasticity is that, given external forces $f : \Omega \to \mathbb{R}^N$ and boundary displacements $\bar{u} : \partial\Omega \to \mathbb{R}^N$, the configuration (u,p) at equilibrium minimizes a total energy of the following type

$$(u,p) \mapsto \mathcal{E}(u,p) - \int_\Omega f \cdot u \, dx.$$

The term $\mathcal{E}(u,p)$ involves the gradient ∇p of the plastic strain. The precise form of $\mathcal{E}(u,p)$ is still suggested by phenomenological considerations, although in agreement with the general principles of thermodynamics (see [8, 9]). A prototype for $\mathcal{E}(u,p)$ is given by the expression

$$\mathcal{E}(u,p) := \frac{1}{2}\int_\Omega \mathbb{C}(x)(E(u)-p):(E(u)-p)\,dx + \int_\Omega V(\sqrt{|p|^2+\ell^2|\nabla p|^2})\,dx, \tag{1.1}$$

where $V : [0,+\infty[\to [0,+\infty[$ is a convex function. The first term is simply the elastic energy of Ω: $e := E(u) - p$ is the elastic strain of the configuration ($E(u) := \frac{1}{2}(\nabla u + \nabla u^t)$ is the symmetrized gradient of u) and $\mathbb{C}$ is the Hooke tensor. The second term is usually referred to as the *plastic potential* which is assumed to depend on the strain gradient plastic measure $\sqrt{|p|^2+\ell^2|\nabla p|^2}$. It involves the term ∇p via the material length-scale ℓ, which has the dimension of a length and which is assumed to have the order of magnitude of the distance at which interactions between dislocations take place.

In [5], N.A. Fleck and J.R. Willis studied the behavior of the deformation theory (1.1) when the elastic and plastic moduli highly oscillate in space and the response in the homogenization limit does not involve gradient terms. The deformation theory in the limit is supposed to involve an energy (independent of ∇p) of the form

$$\mathcal{E}^{hom}(u,p) = \int_\Omega F^{hom}(E(u)(x), p(x))\,dx. \tag{1.2}$$

Here the *effective energy density* $F^{hom}(e,q)$ is provided by minimizing the energy $\mathcal{E}(u,p)$ on a representative volume element, among displacement fields u satisfying the linear boundary condition $u = e \cdot x$ and plastic strains p with mean given by q. Fleck and Willis focused on the problem of finding suitable bounds for the effective energy (1.2). The case of a quadratic V, *i.e.*,

$$V(\sqrt{|p|^2 + \ell^2|\nabla p|^2}) := \frac{1}{2}b(x)[|p|^2 + \ell^2|\nabla p|^2],$$

is pivotal in the derivation of Hashin-Shtrikman type bounds and self-consistent estimates (see [5, Section 4]) which are used to infer estimates for $F^{hom}(e,p)$ in some nonlinear cases (V of power law type, see [5, Section 5]).

In [7], two of us, A. G. and A. M., provided among other things a mathematical framework for the derivation of (1.2) when V is quadratic and the elastic and plastic moduli oscillate *periodically*. We considered the energy defined on $(u,p) \in H^1(\Omega;\mathbb{R}^N) \times H^1(\Omega;\mathrm{M}^N_D)$ by

$$\mathcal{E}^\varepsilon(u,p) = \frac{1}{2}\int_\Omega \mathbb{C}\left(\frac{x}{\varepsilon}\right)(E(u)-p) : (E(u)-p)\,dx + \frac{1}{2}\int_\Omega b\left(\frac{x}{\varepsilon}\right)[|p|^2 + \varepsilon^2\ell^2|\nabla p|^2]\,dx, \tag{1.3}$$

where $\mathbb{C}$ and b are 1-periodic in each variable and satisfy suitable coercivity assumptions. Since the plastic and elastic moduli oscillate on a scale ε, the dissipative length-scale is accordingly given by $\varepsilon\ell$. The periodicity assumption was essential in order to use the method of *two-scale convergence* [13, 1], thanks to which we showed that the effective behaviour in the limit is given by an energy $\mathcal{E}^{hom} : H^1(\Omega;\mathbb{R}^N) \times L^2(\Omega;\mathrm{M}^N_D) \to \mathbb{R}$ of the form (1.2) whose energy density is provided, for every $e \in \mathrm{M}^N_{\mathrm{sym}}$ and $q \in \mathrm{M}^N_D$, by the formula

$$\begin{aligned} F^{hom}(e,q) := \min\Bigg\{ &\frac{1}{2}\int_Y \mathbb{C}(y)[e + E_y(W) - q] : [e + E_y(W) - q]\,dy \\ &+ \frac{1}{2}\int_Y b(y)[|Q|^2 + \ell^2|\nabla_y Q|^2]\,dy : (W,Q) \in H^1_{per}(Y;\mathbb{R}^N) \times H^1_{per}(Y;\mathrm{M}^N_D), \\ &\int_Y W(y)\,dy = 0,\ \int_Y Q(y)\,dy = q \Bigg\}. \end{aligned} \tag{1.4}$$

Here Y is the unit cell in $\mathbb{R}^N$, and the subscript *per* stands for "Y-periodic".

The aim of this paper is to deal with the homogenization of the deformation theory associated to (1.1) with V quadratic but without any periodicity assumption, as in the original problem of Fleck and Willis. This can be considered as the preliminary step if hoping to derive bounds for the effective energy. That, as was mentioned above, is the main concern in [5]. Moreover, as explained in Section 2, it can also be viewed as a first step in the study of the non-periodic homogenization of a quasi-static evolution under the effect of a vanishing strain gradient.

We first focus for simplicity on a "scalar" analogue of the problem. Specifically, we consider the asymptotic behavior of minimizers $(u^\varepsilon, \vartheta^\varepsilon) \in H^1_0(\Omega) \times H^1(\Omega)$ of the energy

$$\mathcal{F}^\varepsilon(u,\vartheta) := \frac{1}{2}\int_\Omega \left(A^\varepsilon(\nabla u - \vartheta\alpha^\varepsilon)\cdot(\nabla u - \vartheta\alpha^\varepsilon) + \varepsilon^2 B^\varepsilon\nabla\vartheta\cdot\nabla\vartheta + d^\varepsilon\vartheta^2\right)dx - \langle f, u\rangle,$$

where $A^\varepsilon, B^\varepsilon \in L^\infty(\Omega;\mathrm{M}^N_{\mathrm{sym}})$, $\alpha^\varepsilon \in L^\infty(\Omega;\mathbb{R}^N)$, $d^\varepsilon \in L^\infty(\Omega)$, $f \in H^{-1}(\Omega)$ and $\{\varepsilon\} := \{\varepsilon_j\}_j$ is an infinitesimal sequence.

The scalar valued function ϑ plays the role of the plastic strain, so that the introduction of the vector α^ε is needed to construct the "elastic" strain $\nabla u - \vartheta\alpha^\varepsilon$. The matrix A^ε is the analogue of the elastic moduli $\mathbb{C}$, while d^ε and B^ε play the role of the plastic moduli. Finally, $f \in H^{-1}(\Omega)$ is again connected to the external loads, and the boundary condition for the displacement u is taken homogeneous.

Rather than study the functionals $\mathcal{F}^\varepsilon$, we concentrate on the associated Euler-Lagrange equations satisfied by $(u^\varepsilon, \vartheta^\varepsilon)$

$$\begin{cases} -div[A^\varepsilon(\nabla u^\varepsilon - \vartheta^\varepsilon \alpha^\varepsilon)] = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega \\ -\varepsilon^2 div(B^\varepsilon \nabla \vartheta^\varepsilon) + d^\varepsilon \vartheta^\varepsilon - A^\varepsilon \alpha^\varepsilon \cdot (\nabla u^\varepsilon - \vartheta^\varepsilon \alpha^\varepsilon) = 0 & \text{in } \Omega \\ B^\varepsilon \nabla \vartheta^\varepsilon \cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where n denotes the exterior normal vector to Ω at a point of $\partial\Omega$. The third equation degenerates in $\nabla\vartheta^\varepsilon$ as $\varepsilon \to 0$.

We study equations (1.5) with the tool of H-convergence [12]. Under suitable coercivity and growth estimates for $A^\varepsilon, \alpha^\varepsilon, B^\varepsilon, d^\varepsilon$, we show that, up to the extraction of a subsequence of ε,

$$u^\varepsilon \rightharpoonup u \qquad \text{weakly in } H^1_0(\Omega) \qquad \text{and} \qquad \vartheta^\varepsilon \rightharpoonup \vartheta \qquad \text{weakly in } L^2(\Omega),$$

where (u, ϑ) satisfy

$$\begin{cases} -div[A^{hom}(\nabla u - \vartheta \alpha^{hom})] = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ d^{hom}\vartheta - A^{hom}\alpha^{hom} \cdot (\nabla u - \vartheta\alpha^{hom}) = 0 & \text{in } \Omega. \end{cases}$$

Here the *homogenized coefficients* $A^{hom}, \alpha^{hom}, d^{hom}$ depend only on $A^\varepsilon, \alpha^\varepsilon, B^\varepsilon, d^\varepsilon$ and not on f. From a variational point of view, the pair $(u, \vartheta) \in H^1_0(\Omega) \times L^2(\Omega)$ minimizes the functional

$$\mathcal{F}^{hom}(u, \vartheta) := \frac{1}{2}\int_\Omega \left(A^{hom}(\nabla u - \vartheta\alpha^{hom}) \cdot (\nabla u - \vartheta\alpha^{hom}) + d^{hom}\vartheta^2 \right) dx - \langle f, u \rangle.$$

The construction of the homogenized coefficients is carried out along the classical lines of H-convergence upon accounting for the degeneracy of the equation for ϑ^ε. This is performed following a technique similar to that used in [6], where a homogenization problem in elasticity with singular perturbations is studied.

We then revisit the actual setting of [5] and reformulate the result accordingly. Finally, we recover the results of [7] in the periodic case.

The structure of the paper is as follows. Section 2 is devoted to the description of the problem of the periodic homogenization of a quasi-static evolution with vanishing strain gradient effects as carried out in [7], and to its connection with the homogenization of the deformation theory (1.1). In Section 3 we formulate the precise assumptions on the coefficients $A^\varepsilon, \alpha^\varepsilon, B^\varepsilon, d^\varepsilon$ (see (3.2), (3.3), (3.4)) which are needed to perform the homogenization procedure for (1.5). These assumptions are tailored to the natural coercivity properties of the functional $\mathcal{F}^\varepsilon$, and, as a consequence, they are easily shown to be "stable" under homogenization. Section 4 is devoted to the construction of the homogenized coefficients A^{hom}, α^{hom} and d^{hom}(see Definition 4.4) and to the study of their main properties. This is done, following the approach of H-convergence, by constructing suitable auxiliary functions (see Proposition 4.1). In Section 5 we state and prove our main homogenization result (see Theorem 5.1). Finally, Section 6 revisits the actual setting of [5] and gives the desired homogenization result in that case (see Theorem 6.1). In the periodic case, the results in [7] are re-derived.

As far as notation is concerned, we denote by $\cdot$ the Euclidean scalar product, and by $|\cdot|$ the associated norm. M^N stands for the space of $N \times N$ matrices, while $\mathrm{M}^N_{\mathrm{sym}}$ denotes the subspace of symmetric matrices. $\mathcal{L}(\mathrm{M}^N_{\mathrm{sym}})$ denotes the space of linear maps from $\mathrm{M}^N_{\mathrm{sym}}$ into itself, and $\mathcal{L}_s(\mathrm{M}^N_{\mathrm{sym}})$ the subspace of all symmetric maps in $\mathcal{L}(\mathrm{M}^N_{\mathrm{sym}})$. For $A \in \mathrm{M}^N$, $|A|$ stands for its natural matrix norm, the Frobenius norm induced by the Euclidean structure of $\mathbb{R}^N$; the same notation also applies to elements of $\mathcal{L}(\mathrm{M}^N_{\mathrm{sym}})$. Moreover, if $A, B \in \mathrm{M}^N$, $A \geq B$ iff the matrix $A - B$ is non-negative, *i.e.,* $(A - B)\lambda \cdot \lambda \geq 0,\ \lambda \in \mathbb{R}^N$; the same notation also applies to elements of $\mathcal{L}(\mathrm{M}^N_{\mathrm{sym}})$. For $v, w \in \mathbb{R}^N$, $v \otimes w$ is the matrix whose components are given by $(v \otimes w)_{ij} = v_i w_j$. We denote by Id the identity matrix in $\mathrm{M}^N_{\mathrm{sym}}$ and by $\mathbb{I}d$ the identity element in $\mathcal{L}_s(\mathrm{M}^N_{\mathrm{sym}})$.

Also, if X, Y are two Banach spaces, we take as norm on $X \times Y$, $\|(f, g)\| := \sqrt{\|f\|^2_X + \|g\|^2_Y}$ and if $\mathcal{L}$ is a continuous bounded operator from X into Y, we denote its norm by $|\!|\!|\mathcal{L}|\!|\!|$. Finally, if X' is the topological dual of X, we denote by $\langle f, g\rangle$ the duality product between $f \in X'$ and $g \in X$.

2. Homogenization of a quasi-static evolution with vanishing strain gradient effects

In paper [7], two of us considered the problem of describing the effective behaviour under homogenization of quasi-static evolutions for a gradient plasticity model with vanishing gradient effects. The model is a variant of that of Gurtin and Anand [9]; it takes hardening into account. The link between oscillations of the plastic and elastic moduli and the gradient effects comes from the homogenization procedure of Fleck and Willis [5].

Employing the energetic approach to evolutions for rate independent systems (see [10]), the problem can be formalized in the following manner. A configuration of the elasto-plastic body Ω is given by a triplet (u, p, z) with

$$u \in H^1(\Omega;\mathbb{R}^N), \qquad p \in H^1(\Omega;\mathrm{M}^N_D), \qquad z \in L^2(\Omega),$$

where u denotes the displacement, p is the associated plastic strain, and z is a hardening internal variable.

The free energy of a configuration is given by

$$\mathcal{Q}(u,p,z) := \frac{1}{2}\int_\Omega \left[\mathbb{C}(x)(Eu(x)-p(x)) : (Eu(x)-p(x)) + z^2(x)\right] dx, \tag{2.1}$$

where the elasticity tensor $\mathbb{C} \in L^\infty(\Omega;\mathcal{L}_s(\mathrm{M}^N_{\mathrm{sym}}))$ is such that for a.e. $x \in \Omega$ and for every $M \in \mathrm{M}^N_{\mathrm{sym}}$

$$\alpha|M|^2 \le \mathbb{C}(x)M\cdot M \le \beta|M|^2 \tag{2.2}$$

with $\alpha, \beta > 0$.

The dissipation during an evolution $t \mapsto (u(t), p(t), z(t))$ defined on $[0, T]$ relative to a subinterval $[a, b]$ is given by

$$\mathcal{D}(p,z;a,b) := \sup\left\{\sum_{j=1}^k \mathcal{H}\left(p(t_j)-p(t_{j-1}), z(t_j)-z(t_{j-1})\right) \,:\, a = t_0 < \cdots < t_k = b\right\}. \tag{2.3}$$

Here the function $\mathcal{H}$ is related to the plastic stresses; indeed, the *higher order stresses* associated to $(p, \nabla p)$ (see [9] for their definition) belong to an admissible region which becomes larger and larger during the evolution, due to the hardening process. Hence, as is usual in plasticity, $\mathcal{H}$ is the convex conjugate of the support function of that region, that is

$$\mathcal{H}(p,z) := I_{\mathcal{C}}(p,z) + \int_\Omega b(x)z(x)\,dx, \tag{2.4}$$

where $I_{\mathcal{C}}$ denotes the indicator function of the cone

$$\mathcal{C} := \left\{(p,z) \in H^1(\Omega;\mathrm{M}^N_D)\times L^2(\Omega) : \sqrt{|p(x)|^2 + \ell^2|\nabla p(x)|^2} \le z(x) \text{ for a.e. } x \in \Omega\right\}. \tag{2.5}$$

Here the plastic modulus $b \in L^\infty(\Omega)$ is such that

$$\alpha \le b(x) \le \beta \qquad \text{for a.e. } x \text{ in } \Omega, \tag{2.6}$$

while $\ell > 0$ is a dissipative length scale. Note that for $\ell = 0$, the cone reduces to the usual cone of the Von Mises theory.

Assuming homogeneous boundary displacements on $\partial\Omega$, the family of admissible configurations of Ω is given by

$$\mathcal{A} := \{(u,p,z) \in H^1_0(\Omega;\mathbb{R}^N)\times H^1(\Omega;\mathrm{M}^N_D)\times L^2(\Omega) : (p,z) \in \mathcal{C}\}.$$

If external body forces acting on Ω are represented by the absolutely continuous function

$$f : [0,T] \to H^{-1}(\Omega;\mathbb{R}^N),$$

we define a *quasistatic evolution* as a map

$$\begin{array}{ccc}[0,T] & \to & H^1_0(\Omega;\mathbb{R}^N)\times H^1(\Omega;\mathrm{M}^N_D)\times L^2(\Omega)\\ \\ t & \mapsto & (u(t),p(t),z(t))\end{array}$$

satisfying the following conditions for every $t\in[0,T]$:

(a) *admissibility*: $(u(t),p(t),z(t))\in\mathcal{A}$;

(b) *global stability*: for every $(v,q,\xi)\in\mathcal{A}$

$$\mathcal{Q}(u(t),p(t),z(t))-\langle f(t),u(t)\rangle\le\mathcal{Q}(v,q,\xi)-\langle f(t),v\rangle+\mathcal{H}(q-p(t),\xi-z(t));$$

(c) *energy balance*: $t\mapsto(p(t),z(t))$ has bounded variation from $[0,T]$ to $H^1(\Omega;\mathrm{M}^N_D)\times L^2(\Omega)$ and

$$E(t)+\mathcal{D}(p,z;0,t)=E(0)-\int_0^t\langle\dot f(\tau),u(\tau)\rangle\,d\tau,$$

where

$$E(t):=\mathcal{Q}(u(t),p(t),z(t))-\langle f(t),u(t)\rangle.$$

Existence of quasi-static evolutions can be established employing the standard variational approach to rate independent evolutions formalized by Mielke and his co-authors [10].

Let $\varepsilon=\{\varepsilon_i\}$ be an infinitesimal sequence, $\mathbb{C}^\varepsilon\in L^\infty(\Omega;\mathcal{L}_s(\mathrm{M}^N_{\mathrm{sym}}))$ and $b^\varepsilon\in L^\infty(\Omega)$ satisfy (2.2) and (2.6), and let

$$t\mapsto(u^\varepsilon(t),p^\varepsilon(t),z^\varepsilon(t)),\qquad t\in[0,T]\tag{2.7}$$

be the quasi-static evolution for the previous model with elastic and plastic moduli given by $\mathbb{C}^\varepsilon$ and b^ε, and with dissipative length scale $\varepsilon\ell$. The homogenization problem with a vanishing strain gradient consists in describing what kind of equations are satisfied by the limit evolution (under weak convergence)

$$t\mapsto(u(t),p(t),z(t)),\qquad t\in[0,T],\tag{2.8}$$

where $p(t)$ is now simply an element of $L^2(\Omega;\mathrm{M}^N_D)$ since the control on its gradient degenerates as $\varepsilon\to0$.

In the footstep of [11] where the problem of the homogenization for a quasi-static evolution in standard plasticity with hardening is treated, it is shown in [7] that the study can be accomplished in the *periodic case*, that is for

$$\mathbb{C}^\varepsilon(x):=\mathbb{C}\left(\frac{x}{\varepsilon}\right)\qquad\text{and}\qquad b^\varepsilon(x):=b\left(\frac{x}{\varepsilon}\right),$$

where $\mathbb{C}\in L^\infty(\mathbb{R}^N;\mathcal{L}_s(\mathrm{M}^N_{\mathrm{sym}}))$ and $b\in L^\infty(\mathbb{R}^N)$ are 1-periodic in each variable and satisfy (2.2) and (2.6). The key mathematical tool is that of *two-scale convergence* [13, 1]. Recall that v_ε bounded in $L^2(\Omega)$ two-scale weakly converges to $V\in L^2(\Omega\times Y)$ as $\varepsilon\to0$ if

$$\lim_{\varepsilon\to0}\int_\Omega v_\varepsilon(x)\psi\left(x,\frac{x}{\varepsilon}\right)dx=\int_{\Omega\times Y}V(x,y)\psi(x,y)\,dxdy$$

for every smooth function $\psi(x,y)$ periodic in y, where Y denotes the unit cell in $\mathbb{R}^N$. The weak two-scale limit V thus depends on a further *microstructural* variable y which keeps track of the oscillations occurring along the sequence (v_ε). Notice that the mean of V with respect to $y\in Y$ provides the usual weak-L^2 limit of (v_ε).

It was shown in [7, Section 5] that, as $\varepsilon\to0$, the evolution (2.7) gives rise to an evolution

$$t\mapsto(u(t),U(t),P(t),Z(t)),\qquad t\in[0,T],\tag{2.9}$$

where

$$U(t)\in L^2(\Omega;H^1_{per,0}(Y;\mathbb{R}^N)),\qquad P(t)\in L^2(\Omega;H^1_{per}(Y;\mathrm{M}^N_D)),\qquad Z(t)\in L^2(\Omega\times Y),$$

the function $U(t)$ being the two-scale weak limit of $Eu^\varepsilon(t)$. Here the subscript *per* stands for "Y-periodic", while $per,0$ stands for "Y-periodic with zero mean". This evolution can be described in terms of admissibility, global stability and energy balance conditions involving suitable extensions to $\Omega\times Y$ of the free energy (2.1) and of the dissipation potential (2.4). In conclusion, the limit results in a quasi-static evolution for a model of strain gradient type with respect to the new microstructural variable $y \in Y$.

In view of the relation between two-scale weak convergence and weak-L^2 convergence, the connection between the two-scale evolution (2.9) and the homogenized (single scale) evolution (2.8) is given by

$$p(t,x) = \int_Y P(t,x,y)\,dy, \qquad z(t,x) = \int_Y Z(t,x,y)\,dy.$$

A description of $t \mapsto (u(t), p(t), z(t))$ in terms of a standard plasticity model seems hopeless. From a mathematical point of view, this is due to the fact that the two-scale version of the admissibility, global stability and energy balance conditions are nonlinear in y. We can thus conclude that the introduction of the microstructural variable y, *i.e.*, the use of two-scale convergence, provides information about the oscillations occurring during the homogenization procedure which in turn permits a description of the limit evolution in terms of a generalized plasticity model.

In the non-periodic case, the situation is completely different and the tool of two-scale convergence is no longer at our disposal. A convenient first simplifying step consists in passing from flow theory to *deformation theory*. This amounts to a characterization of the behavior of the body under the action of a time-monotone external load solely in terms of a single variational problem, that associated with the final value of the load. Deformation theory is generally assumed to provide useful information for deformation paths which are monotone in the components of the associated strains and stresses. Here, this should result in minimizing an energy of the Fleck and Willis type, *i.e.*,

$$(u,p) \mapsto \frac{1}{2}\int_\Omega \mathbb{C}^\varepsilon(E(u)-p) : (E(u)-p)\,dx + \int_\Omega V^\varepsilon\big(\sqrt{|p|^2 + \varepsilon^2\ell^2|\nabla p|^2}\big)\,dx - \langle f, u\rangle \tag{2.10}$$

for $(u,p) \in H^1_0(\Omega;\mathbb{R}^N) \times H^1(\Omega;\mathrm{M}^N_D)$. The precise form of V^ε is dictated, in the general case of nonlinear hardening, by the way in which the set of admissible generalized stresses evolves as a function of the plastic strain measure $\sqrt{|p|^2 + \varepsilon^2\ell^2|\nabla p|^2}$.

As mentioned in the introduction, the case of a quadratic V^ε is important for deriving bounds on the homogenized energy. The focus of the present paper is on the characterization as $\varepsilon \to 0$ of the minimizers of (2.10) when V^ε is quadratic using the tool of H-convergence. As a consequence, besides being a mathematical formalization of the Fleck and Willis homogenization procedure, our result can also be considered as a first step towards the study of the homogenization of a quasi-static evolution with vanishing strain gradients in the non-periodic case.

3. Setting of the problem

Let $\Omega \subseteq \mathbb{R}^N$ be an open and bounded set, $\varepsilon = (\varepsilon_j)$ an infinitesimal sequence and

$$A^\varepsilon, B^\varepsilon \in L^\infty(\Omega;\mathrm{M}^N_{\mathrm{sym}}), \qquad \alpha^\varepsilon \in L^\infty(\Omega;\mathbb{R}^N), \qquad d^\varepsilon \in L^\infty(\Omega).$$

In this paper we will concentrate on the study of the asymptotic behaviour as $\varepsilon \to 0$ of the minimizers $(u^\varepsilon, \vartheta^\varepsilon) \in H^1_0(\Omega) \times H^1(\Omega)$ of the energy $\mathcal{F}^\varepsilon : H^1_0(\Omega) \times H^1(\Omega) \to \mathbb{R}$

$$\mathcal{F}^\varepsilon(u,\vartheta) := \frac{1}{2}\int_\Omega \left(A^\varepsilon(\nabla u - \vartheta\alpha^\varepsilon)\cdot(\nabla u - \vartheta\alpha^\varepsilon) + \varepsilon^2 B^\varepsilon\nabla\vartheta\cdot\nabla\vartheta + d^\varepsilon\vartheta^2\right)\,dx - \langle f, u\rangle,$$

where $f \in H^{-1}(\Omega)$ and $\langle , \rangle$ denotes the duality pairing between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$. The associated Euler-Lagrange equations are given by

$$\begin{cases} -div[A^\varepsilon(\nabla u^\varepsilon - \vartheta^\varepsilon\alpha^\varepsilon)] = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega \\ -\varepsilon^2 div(B^\varepsilon\nabla\vartheta^\varepsilon) + d^\varepsilon\vartheta^\varepsilon - A^\varepsilon\alpha^\varepsilon\cdot(\nabla u^\varepsilon - \vartheta^\varepsilon\alpha^\varepsilon) = 0 & \text{in } \Omega \\ B^\varepsilon\nabla\vartheta^\varepsilon\cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where n denotes the outer normal to Ω at a point of $\partial\Omega$.

In order to perform the asymptotic analysis, we will assume throughout this section that, for all $\varepsilon > 0$ and a.e. in Ω, A^ε is invertible, $d^\varepsilon > 0$, and that, for some $0 < \beta \leq \gamma$, the following conditions hold for every $\lambda \in \mathbb{R}^N, \mu \in \mathbb{R}$:

$$\gamma|\lambda|^2 \geq B^\varepsilon \lambda \cdot \lambda \geq \beta|\lambda|^2, \tag{3.2}$$

$$A^\varepsilon(x)(\lambda - \mu\alpha^\varepsilon(x)) \cdot (\lambda - \mu\alpha^\varepsilon(x)) + d^\varepsilon(x)\mu^2 \geq \beta\{|\lambda|^2 + \mu^2\}, \tag{3.3}$$

and

$$(A^\varepsilon)^{-1}(x)\lambda \cdot \lambda + \frac{1}{d^\varepsilon(x)}(\alpha^\varepsilon(x) \cdot \lambda + \mu)^2 \geq \frac{1}{\gamma}\{|\lambda|^2 + \mu^2\}. \tag{3.4}$$

Remark 3.1. Notice that, in view of (3.3), (3.4),

$$\gamma Id \geq A^\varepsilon \geq \beta Id \qquad \text{and} \qquad \gamma \geq d^\varepsilon \geq \beta,$$

a.e. in Ω. Elementary algebraic manipulations– taking $\lambda = -A^\varepsilon\alpha^\varepsilon$, $\mu = d^\varepsilon + A^\varepsilon\alpha^\varepsilon \cdot \alpha^\varepsilon$ in (3.4) – would also establish that, a.e. in Ω,

$$\gamma \geq d^\varepsilon + A^\varepsilon\alpha^\varepsilon \cdot \alpha^\varepsilon \geq \beta \qquad \text{and} \qquad |A^\varepsilon\alpha^\varepsilon| \leq \gamma.$$ ¶

For future reference, we define the set

$$\mathcal{M}(\beta,\gamma;\Omega) := \{(A, \alpha, d) : \Omega \to \mathrm{M}^N_{\mathrm{sym}} \times \mathbb{R}^N \times \mathbb{R} \text{ measurable and such that } (3.3), (3.4) \text{ are satisfied a.e. in } \Omega\}. \tag{3.5}$$

In view of Remark 3.1, such a set is indeed a subset of $L^\infty(\Omega; \mathrm{M}^N_{\mathrm{sym}} \times \mathbb{R}^N \times \mathbb{R})$.

Remark 3.2. If $(A, \alpha, d) \in \mathcal{M}(\beta,\gamma;\Omega)$, then, minimizing (3.3) in μ at fixed λ immediately yields

$$A - \frac{A\alpha \otimes A\alpha}{d + A\alpha \cdot \alpha} \geq \beta Id.$$ ¶

The bounds (3.2)-(3.4) allow one to establish existence of solutions of (3.1), together with some uniform estimates.

Proposition 3.3. *For every $f \in H^{-1}(\Omega)$, equations (3.1) admit a unique solution $(u^\varepsilon, \vartheta^\varepsilon) \in H^1_0(\Omega) \times H^1(\Omega)$ and*

$$\sup_\varepsilon \left(\|\nabla u^\varepsilon\|_{L^2(\Omega;\mathbb{R}^N)} + \|\vartheta^\varepsilon\|_{L^2(\Omega)} + \|\varepsilon\nabla\vartheta^\varepsilon\|_{L^2(\Omega;\mathbb{R}^N)}\right) < +\infty. \tag{3.6}$$

Proof. Existence and uniqueness of the solution $(u^\varepsilon, \vartheta^\varepsilon)$ follow from an application of Lax-Milgram's lemma to the bilinear form on $H^1_0(\Omega) \times H^1(\Omega)$

$$\mathcal{B}^\varepsilon((u, \vartheta), (v, \eta)) := \int_\Omega A^\varepsilon(\nabla u - \vartheta\alpha) \cdot \nabla v + \varepsilon^2 B^\varepsilon \nabla\vartheta \cdot \nabla\eta + d^\varepsilon\vartheta\eta - A^\varepsilon\alpha^\varepsilon \cdot (\nabla u - \vartheta\alpha^\varepsilon)\eta \, dx.$$

That bilinear form is continuous in view of Remark 3.1. Coercivity of the associated quadratic form is is obtained upon appealing to (3.2) and (3.3),

$$\mathcal{B}^\varepsilon((u, \vartheta), (u, \vartheta)) \geq \beta \int_\Omega \left(|\nabla u|^2 + |\vartheta|^2 + \varepsilon^2|\nabla\vartheta|^2\right) dx, \tag{3.7}$$

so that coercivity follows by Poincaré's inequality.

Estimate (3.6) follows by the equality

$$\mathcal{B}^\varepsilon((u^\varepsilon, \vartheta^\varepsilon), (u^\varepsilon, \vartheta^\varepsilon)) = \langle f, u^\varepsilon \rangle$$

and taking (3.7) into account. □

4. Construction of the homogenized equation

The homogenized equations associated to (3.1) are constructed as is usual in the theory of H-convergence (see e.g. [12]) by means of auxiliary functions which enjoy "good" compactness properties. Equations (3.1) become degenerate in ϑ^ε as $\varepsilon \to 0$: as a consequence, the construction of the auxiliary functions requires some care. We will follow [6] where a similar problem has been treated in the context of singular perturbations in elasticity.

Proposition 4.1 (**Construction of the auxiliary functions**). *Under the assumptions* (3.2)-(3.4)*, there exists a subsequence of* $\{\varepsilon\}$ *(not relabeled) such that, for every* $\lambda \in \mathbb{R}^N$ *and* $\mu \in \mathbb{R}$*, there exist* $w^\varepsilon_{\lambda,\mu} \in H^1(\Omega)$ *and* $q^\varepsilon_{\lambda,\mu} \in H^1(\Omega)$ *with*

$$w^\varepsilon_{\lambda,\mu} \rightharpoonup \lambda \cdot x \qquad \text{weakly in } H^1(\Omega), \qquad q^\varepsilon_{\lambda,\mu} \rightharpoonup \mu \qquad \text{weakly in } L^2(\Omega), \tag{4.1}$$

$$-div[A^\varepsilon(\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon)] \qquad \text{compact in } H^{-1}(\Omega) \tag{4.2}$$

and

$$-\varepsilon^2 div(B^\varepsilon \nabla q^\varepsilon_{\lambda,\mu}) + d^\varepsilon q^\varepsilon_{\lambda,\mu} - A^\varepsilon\alpha^\varepsilon \cdot (\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon) \qquad \text{compact in } L^2(\Omega). \tag{4.3}$$

Moreover

$$\varepsilon \nabla q^\varepsilon_{\lambda,\mu} \qquad \text{is bounded in } L^2(\Omega;\mathbb{R}^N). \tag{4.4}$$

Proof. Let us fix $\Omega' \subseteq \mathbb{R}^N$ open, bounded and such that $\overline{\Omega} \subseteq \Omega'$. Let us extend $A^\varepsilon, \alpha^\varepsilon, d^\varepsilon, B^\varepsilon$ to Ω' by setting

$$A^\varepsilon := \beta Id, \quad \alpha^\varepsilon := 0, \quad d^\varepsilon := \beta, \quad B^\varepsilon := \beta Id \qquad \text{on } \Omega' \setminus \overline{\Omega}.$$

Notice that estimates (3.2)-(3.3) are still satisfied on Ω'.

We divide the proof into several steps.

Step 1. The map

$$\begin{array}{rcl} D^\varepsilon : [H^1_0(\Omega')]^2 & \longrightarrow & [H^{-1}(\Omega')]^2 \\ (u,\vartheta) & \mapsto & (-div[A^\varepsilon(\nabla u - \vartheta\alpha^\varepsilon)], -\varepsilon^2 div(B^\varepsilon\nabla\vartheta) + d^\varepsilon\vartheta - A^\varepsilon\alpha^\varepsilon \cdot (\nabla u - \vartheta\alpha^\varepsilon)) \end{array}$$

is an isomorphism between $[H^1_0(\Omega')]^2$ and $[H^{-1}(\Omega')]^2$, where, throughout the proof, $u \in H^1_0(\Omega')$ is equipped with the equivalent Dirichlet norm, *i.e.*, the quantity $\|\nabla u\|_{L^2(\Omega';\mathbb{R}^N)}$, thanks to Poincaré's inequality. Indeed, the quadratic form $\Phi^\varepsilon : [H^1_0(\Omega')]^2 \to \mathbb{R}$ given by

$$\Phi^\varepsilon(u,\vartheta) := \frac{1}{2}\int_{\Omega'} \left(A^\varepsilon(\nabla u - \vartheta\alpha^\varepsilon)\cdot(\nabla u - \vartheta\alpha^\varepsilon) + \varepsilon^2 B^\varepsilon\nabla\vartheta\cdot\nabla\vartheta + d^\varepsilon\vartheta^2\right) dx$$

satisfies

$$\Phi^\varepsilon(u,\vartheta) \geq \beta \int_{\Omega'} \left(|\nabla u|^2 + |\vartheta|^2 + |\varepsilon\nabla\vartheta|^2\right) dx. \tag{4.5}$$

Hence D^ε is an isomorphism as a consequence of the Lax-Milgram Lemma. Further, a straightforward computation shows that, for say $\varepsilon^2 < 1$,

$$\|D^\varepsilon\| \leq 5\gamma.$$

Also, given $f \in H^{-1}(\Omega')$ and $g \in L^2(\Omega')$ and setting $(u^\varepsilon, \vartheta^\varepsilon) := (D^\varepsilon)^{-1}(f,g)$,

$$\langle f, u^\varepsilon\rangle + \int_{\Omega'} g\vartheta^\varepsilon\, dx = \Phi^\varepsilon(u^\varepsilon,\vartheta^\varepsilon) \geq \beta\int_{\Omega'}\left(|\nabla u^\varepsilon|^2 + |\vartheta^\varepsilon|^2 + |\varepsilon\nabla\vartheta^\varepsilon|^2\right) dx,$$

so that

$$\int_{\Omega'}\left(|\nabla u^\varepsilon|^2 + |\vartheta^\varepsilon|^2 + |\varepsilon\nabla\vartheta^\varepsilon|^2\right) dx \leq \frac{1}{\beta^2}\left(\|f\|^2_{H^{-1}(\Omega')} + \|g\|^2_{L^2(\Omega')}\right). \tag{4.6}$$

Step 2. Let us consider the natural immersion $i : [H^1_0(\Omega')]^2 \to H^1_0(\Omega') \times L^2(\Omega')$, and

$$C^\varepsilon : H^{-1}(\Omega') \times L^2(\Omega') \to H^1_0(\Omega') \times L^2(\Omega')$$

given by

$$C^\varepsilon := i \circ (D^\varepsilon)^{-1} \circ i^*,$$

where i^* denotes the adjoint map of i.

Notice that, thanks to (4.6),

$$\sup_\varepsilon \|C^\varepsilon\| \leq \frac{1}{\beta}. \tag{4.7}$$

Step 3. Since $H^{-1}(\Omega') \times L^2(\Omega')$ is separable, and in view of (4.7), there exist a subsequence of ε (not relabeled) and a continuous linear operator

$$C : H^{-1}(\Omega') \times L^2(\Omega') \to H^1_0(\Omega') \times L^2(\Omega')$$

such that for every $(f, g) \in H^{-1}(\Omega') \times L^2(\Omega')$

$$C^\varepsilon(f, g) \rightharpoonup C(f, g) \qquad \text{weakly in } H^1_0(\Omega') \times L^2(\Omega').$$

Clearly

$$\|C\| \leq \frac{1}{\beta}.$$

Now, C is coercive. Indeed, let $(u^\varepsilon, \vartheta^\varepsilon) := C^\varepsilon(f, g)$. In view of Remark 3.1, for arbitrary $\varphi, \psi \in C^\infty_c(\Omega')$, we get, for some constant c depending only on γ,

$$\begin{aligned}
&\langle f, \varphi\rangle + \int_{\Omega'} g\psi \, dx \\
&\quad = \int_{\Omega'} A^\varepsilon(\nabla u^\varepsilon - \vartheta^\varepsilon \alpha^\varepsilon)\nabla\varphi \, dx + \int_{\Omega'} \left(-\varepsilon^2 div(B^\varepsilon \nabla\vartheta^\varepsilon) + d^\varepsilon\vartheta^\varepsilon - A^\varepsilon\alpha^\varepsilon \cdot (\nabla u^\varepsilon - \vartheta^\varepsilon\alpha^\varepsilon)\right)\psi \, dx \\
&\qquad \leq c\|(u^\varepsilon, \vartheta^\varepsilon)\|_{H^1_0(\Omega')\times L^2(\Omega')}\|(\varphi, \psi)\|_{H^1_0(\Omega')\times L^2(\Omega')} + \varepsilon^2\gamma\|\nabla\vartheta^\varepsilon\|_{H^1_0(\Omega')}\|\nabla\varphi\|_{H^1_0(\Omega')} \\
&\qquad\qquad \leq (c+\gamma)\left\{\|(u^\varepsilon, \vartheta^\varepsilon)\|_{H^1_0(\Omega')\times L^2(\Omega')} + \varepsilon^2\|\nabla\vartheta^\varepsilon\|_{H^1_0(\Omega')}\right\}\|(\varphi, \psi)\|_{H^1_0(\Omega')\times L^2(\Omega')},
\end{aligned}$$

so that, since from (4.6) $\varepsilon\|\nabla\vartheta^\varepsilon\|_{L^2(\Omega')}$ is bounded independently of ε, we obtain

$$\begin{aligned}
&\liminf_\varepsilon \|(u^\varepsilon, \vartheta^\varepsilon)\|_{H^1_0(\Omega')\times L^2(\Omega')} \\
&\qquad \geq \frac{1}{c+\gamma} \sup\left\{\frac{\langle f, \varphi\rangle + \int_{\Omega'} g\psi \, dx}{\|(\varphi, \psi)\|_{H^1_0(\Omega')\times L^2(\Omega')}} : (\varphi, \psi) \in [C^\infty_c(\Omega')]^2\right\} \\
&\qquad\qquad = \frac{1}{c+\gamma}\|(f, g)\|_{H^{-1}(\Omega')\times L^2(\Omega')}.
\end{aligned} \tag{4.8}$$

Then, from (4.5), we conclude, thanks to (4.8), that

$$\begin{aligned}
\langle (f, g), C(f, g)\rangle = \lim_\varepsilon \langle (f, g), C^\varepsilon(f, g)\rangle = \lim_\varepsilon \Phi^\varepsilon(u^\varepsilon, \vartheta^\varepsilon) \geq \beta \liminf_\varepsilon \|(u^\varepsilon, \vartheta^\varepsilon)\|^2_{H^1_0(\Omega')\times L^2(\Omega')} \\
\geq \frac{\beta}{(c+\gamma)^2}\|(f, g)\|^2_{H^{-1}(\Omega'))\times L^2(\Omega')},
\end{aligned}$$

hence the coercivity.

Step 4. Because the bounded linear operator C constructed in Step 3 is coercive, it admits an inverse C^{-1}. Consider $\varphi \in C^\infty_c(\Omega')$ such that $\varphi \equiv 1$ on Ω. We set for every $\lambda \in \mathbb{R}^N$ and $\mu \in \mathbb{R}$

$$(w^\varepsilon_{\lambda,\mu}, q^\varepsilon_{\lambda,\mu}) := C^\varepsilon C^{-1}(\varphi(x)\lambda \cdot x, \mu).$$

Then,

$$w^\varepsilon_{\lambda,\mu} \rightharpoonup \varphi(x)\lambda \cdot x \qquad \text{weakly in } H^1_0(\Omega'), \quad q^\varepsilon_{\lambda,\mu} \rightharpoonup \mu \qquad \text{weakly in } L^2(\Omega')$$

and

$$\big(-div[A^\varepsilon(\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon)], -\varepsilon^2 div(B^\varepsilon\nabla q^\varepsilon_{\lambda,\mu}) + d^\varepsilon q^\varepsilon_{\lambda,\mu} - A^\varepsilon\alpha^\varepsilon\cdot(\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon)\big) \\ = C^{-1}(\varphi(x)\lambda\cdot x, \mu) \qquad \text{in } H^{-1}(\Omega')\times L^2(\Omega').$$

Moreover, in view of (4.6), $\|\varepsilon\nabla\vartheta^\varepsilon_{\lambda,\mu}\|_{L^2(\Omega';\mathbb{R}^N)}$ is bounded.

Consider the restriction of $(w^\varepsilon_{\lambda,\mu}, q^\varepsilon_{\lambda,\mu})$ to Ω. Then (4.1), (4.2), (4.3) and (4.4) are satisfied. □

Let us still denote by ε the subsequence given by Proposition 4.1, and let $(w^\varepsilon_{\lambda,\mu}, q^\varepsilon_{\lambda,\mu})$ be the associated functions. The dependence of $(w^\varepsilon_{\lambda,\mu}, q^\varepsilon_{\lambda,\mu})$ upon (λ,μ) is clearly linear. It is therefore no restriction, up to the possible expense of extracting a further subsequence, to assume that

$$\begin{cases} (d^\varepsilon + A^\varepsilon\alpha^\varepsilon\cdot\alpha^\varepsilon)q^\varepsilon_{\lambda,\mu} \rightharpoonup d^0\mu + e^0\cdot\lambda & \text{weakly in } L^2(\Omega) \\ A^\varepsilon\alpha^\varepsilon\cdot\nabla w^\varepsilon_{\lambda,\mu} \rightharpoonup l^0\mu + f^0\cdot\lambda & \text{weakly in } L^2(\Omega) \\ q^\varepsilon_{\lambda,\mu}A^\varepsilon\alpha^\varepsilon \rightharpoonup \mu p^0 + H^0\lambda & \text{weakly in } L^2(\Omega;\mathbb{R}^N) \\ A^\varepsilon\nabla w^\varepsilon_{\lambda,\mu} \rightharpoonup \mu n^0 + A^0\lambda & \text{weakly in } L^2(\Omega;\mathbb{R}^N) \end{cases} \tag{4.9}$$

for suitable

$$d^0, l^0 \in L^2(\Omega), \qquad e^0, f^0, p^0, n^0 \in L^2(\Omega;\mathbb{R}^N), \qquad A^0, H^0 \in L^2(\Omega;\mathrm{M}^N). \tag{4.10}$$

Remark 4.2. In the following, we make repeated use of the following facts: if $v^\varepsilon \rightharpoonup v$ weakly in $H^1(\Omega)$ and $g^\varepsilon \rightharpoonup g$ weakly in $L^2(\Omega)$, then for every $\varphi \in C^\infty_c(\Omega)$

$$\lim_\varepsilon \int_\Omega \varphi A^\varepsilon(\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon)\cdot\nabla v^\varepsilon\,dx = \int_\Omega \varphi[(A^0 - H^0)\lambda + (n^0 - p^0)\mu]\cdot\nabla v\,dx \tag{4.11}$$

and

$$\lim_\varepsilon \int_\Omega \varphi[-\varepsilon^2 div(B^\varepsilon\nabla q^\varepsilon_{\lambda,\mu}) + d^\varepsilon q^\varepsilon_{\lambda,\mu} - A^\varepsilon\alpha^\varepsilon\cdot(\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon)]g^\varepsilon\,dx \\ = \int_\Omega \varphi[(e^0 - f^0)\cdot\lambda + (d^0 - l^0)\mu]g\,dx. \tag{4.12}$$

Equation (4.11) is a consequence of (4.2) and of the classical div-curl lemma (see [12]). Equation (4.12) is a consequence of (4.3) and of the fact that

$$-\varepsilon^2 div(B^\varepsilon\nabla q^\varepsilon_{\lambda,\mu}) + d^\varepsilon q^\varepsilon_{\lambda,\mu} - A^\varepsilon\alpha^\varepsilon\cdot(\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon) \to (e^0 - f^0)\cdot\lambda + (d^0 - l^0)\mu \qquad \text{strongly in } L^2(\Omega).$$

The term $-\varepsilon^2 div(B^\varepsilon\nabla q^\varepsilon_{\lambda,\mu})$ is essential for erasing possible oscillations, but it does not play any role in the identification of the limit since $\varepsilon\nabla q^\varepsilon_{\lambda,\mu}$ is bounded in $L^2(\Omega;\mathbb{R}^N)$ (see (4.4)). ¶

The following proposition contains some important properties of the functions defined in (4.10).

Proposition 4.3. *The following items hold true:*

(a) $n^0 - p^0 = e^0 - f^0$, *a.e. in* Ω*;*
(b) $A^0 - H^0 \in \mathrm{M}^N_{\mathrm{sym}}$ *(and is positive definite), a.e. in* Ω*;*
(c) *A.e. in* Ω*, and for any* $\lambda \in \mathbb{R}^N$ *and* $\mu \in \mathbb{R}$*,*

$$(A^0 - H^0)\lambda\cdot\lambda + 2(n^0 - p^0)\cdot\lambda\mu + (d^0 - l^0)\mu^2 \geq \beta\{|\lambda|^2 + \mu^2\}; \tag{4.13}$$

(d) *A.e. in* Ω *and for any* $\lambda \in \mathbb{R}^N$ *and* $\mu \in \mathbb{R}$*,*

$$(A^0 - H^0)\lambda\cdot\lambda + 2(n^0 - p^0)\cdot\lambda\mu + (d^0 - l^0)\mu^2 \geq \\ \frac{1}{\gamma}\left\{\big|(A^0 - H^0)\lambda + \mu(n^0 - p^0)\big|^2 + \big((d^0 - l^0)\mu + (n^0 - p^0)\cdot\lambda\big)^2\right\}. \tag{4.14}$$

Proof. We begin with item (a). Write, for every $\lambda \in \mathbb{R}^N$ and $\mu \in \mathbb{R}$,

$$\begin{aligned}A^\varepsilon(\nabla w^\varepsilon_{\lambda,0} - q^\varepsilon_{\lambda,0}\alpha^\varepsilon) \cdot \nabla w^\varepsilon_{0,\mu} - A^\varepsilon(\nabla w^\varepsilon_{0,\mu} - q^\varepsilon_{0,\mu}\alpha^\varepsilon) \cdot \nabla w^\varepsilon_{\lambda,0} \\ - (-\varepsilon^2 div(B^\varepsilon \nabla q^\varepsilon_{0,\mu}) + d^\varepsilon q^\varepsilon_{0,\mu} - A^\varepsilon\alpha^\varepsilon \cdot (\nabla w^\varepsilon_{0,\mu} - q^\varepsilon_{0,\mu}\alpha^\varepsilon))q^\varepsilon_{\lambda,0} \\ + (-\varepsilon^2 div(B^\varepsilon \nabla q^\varepsilon_{\lambda,0}) + d^\varepsilon q^\varepsilon_{\lambda,0} - A^\varepsilon\alpha^\varepsilon \cdot (\nabla w^\varepsilon_{\lambda,0} - q^\varepsilon_{\lambda,0}\alpha^\varepsilon))q^\varepsilon_{0,\mu} \\ = \varepsilon^2 div(B^\varepsilon \nabla q^\varepsilon_{0,\mu})q^\varepsilon_{\lambda,0} - \varepsilon^2 div(B^\varepsilon \nabla q^\varepsilon_{\lambda,0})q^\varepsilon_{0,\mu}.\end{aligned}$$

Multiplying by $\varphi \in C^\infty_c(\Omega)$ and integrating over Ω, we obtain that the right-hand side can be written as

$$-\varepsilon^2 \int_\Omega \left(B^\varepsilon \nabla q^\varepsilon_{0,\mu} \nabla(\varphi q^\varepsilon_{\lambda,0}) - B^\varepsilon \nabla q^\varepsilon_{\lambda,0} \nabla(\varphi q^\varepsilon_{0,\mu})\right) dx = -\varepsilon^2 \int_\Omega [q^\varepsilon_{\lambda,0} B^\varepsilon \nabla q^\varepsilon_{0,\mu} - q^\varepsilon_{0,\mu} B^\varepsilon \nabla q^\varepsilon_{\lambda,0}] \cdot \nabla\varphi \, dx.$$

Thanks to (4.4), the right-hand side vanishes as $\varepsilon \to 0$. Concerning the left-hand side, by Remark 4.2 we obtain

$$\int_\Omega \varphi[(-n^0 + p^0) \cdot \lambda\mu + (e^0 - f^0) \cdot \lambda\mu] \, dx = 0$$

from which, in view of the arbitrariness of φ, λ, μ we deduce that $n^0 - p^0 = e^0 - f^0$.

Let us now prove the first part of item (b). For every $\lambda, \lambda' \in \mathbb{R}^N$,

$$\begin{aligned}A^\varepsilon(\nabla w^\varepsilon_{\lambda,0} - q^\varepsilon_{\lambda,0}\alpha^\varepsilon) \cdot \nabla w^\varepsilon_{\lambda',0} - A^\varepsilon(\nabla w^\varepsilon_{\lambda',0} - q^\varepsilon_{\lambda',0}\alpha^\varepsilon) \cdot \nabla w^\varepsilon_{\lambda,0} \\ - (-\varepsilon^2 div(B^\varepsilon \nabla q^\varepsilon_{\lambda',0}) + d^\varepsilon q^\varepsilon_{\lambda',0} - A^\varepsilon\alpha^\varepsilon \cdot (\nabla w^\varepsilon_{\lambda',0} - q^\varepsilon_{\lambda',0}\alpha^\varepsilon))q^\varepsilon_{\lambda,0} \\ + (-\varepsilon^2 div(B^\varepsilon \nabla q^\varepsilon_{\lambda,0}) + d^\varepsilon q^\varepsilon_{\lambda,0} - A^\varepsilon\alpha^\varepsilon \cdot (\nabla w^\varepsilon_{\lambda,0} - q^\varepsilon_{\lambda,0}\alpha^\varepsilon))q^\varepsilon_{\lambda',0} \\ = \varepsilon^2 div(B^\varepsilon \nabla q^\varepsilon_{\lambda',0})q^\varepsilon_{\lambda,0} - \varepsilon^2 div(B^\varepsilon \nabla q^\varepsilon_{\lambda,0})q^\varepsilon_{\lambda',0}.\end{aligned}$$

Multiplying by $\varphi \in C^\infty_c(\Omega)$, integrating over Ω, and sending $\varepsilon \to 0$, we obtain, arguing as above,

$$\int_\Omega \varphi[(A^0 - H^0)\lambda \cdot \lambda' - (A^0 - H^0)\lambda' \cdot \lambda] \, dx = 0$$

from which we deduce that $A^0(x) - H^0(x)$ is symmetric for a.e. $x \in \Omega$.

We now come to item (c) and to the second part of item (b). For every $\varphi \in C^\infty_c(\Omega)$, $\varphi \geq 0$, we can write, in view of (3.2), (3.3),

$$\begin{aligned}\int_\Omega \varphi\{A^\varepsilon(\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon) \cdot (\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon) + \varepsilon^2 B^\varepsilon \nabla q^\varepsilon_{\lambda,\mu} \cdot \nabla q^\varepsilon_{\lambda,\mu} + d^\varepsilon (q^\varepsilon_{\lambda,\mu})^2\} \, dx \\ \geq \beta \int_\Omega \varphi \left\{|\nabla w^\varepsilon_{\lambda,\mu}|^2 + (q^\varepsilon_{\lambda,\mu})^2\right\} \, dx.\end{aligned}$$

In view of (4.4), the left hand-side also reads as

$$\int_\Omega \varphi\{A^\varepsilon(\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon) \cdot \nabla w^\varepsilon_{\lambda,\mu} + (-\varepsilon^2 div(B^\varepsilon \nabla q^\varepsilon_{\lambda,\mu}) + d^\varepsilon q^\varepsilon_{\lambda,\mu} - A^\varepsilon\alpha^\varepsilon \cdot (\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon))q^\varepsilon_{\lambda,\mu}\} dx + O(\varepsilon),$$

where $O(\varepsilon) \to 0$ as $\varepsilon \to 0$. By Remark 4.2, and taking into account that $n^0 - p^0 = e^0 - f^0$ we obtain that, with $\varepsilon \to 0$,

$$\int_\Omega \varphi\{(A^0 - H^0)\lambda \cdot \lambda + 2(n^0 - p^0) \cdot \lambda\mu + (d^0 - l^0)\mu^2\} \, dx \geq \beta\{|\lambda|^2 + \mu^2\} \int_\Omega \varphi \, dx$$

from which we infer that (4.13) holds true, a.e. in Ω.

We immediately deduce from that relation that $A^0 - H^0$ is positive definite, so that the second part of item (b) holds true.

Finally the proof of item (d) relies on (3.4). Indeed, test that inequality with $\lambda = A^\varepsilon(\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon)$, and $\mu = \left(d^\varepsilon q^\varepsilon_{\lambda,\mu} - A^\varepsilon\alpha^e \cdot (\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon)\right)$. For every $\varphi \in C^\infty_c(\Omega)$, $\varphi \geq 0$, we get, using

(3.2) once again,

$$\int_\Omega \varphi\{A^\varepsilon(\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon)\cdot(\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon) + \varepsilon^2 B^\varepsilon \nabla q^\varepsilon_{\lambda,\mu}\cdot\nabla q^\varepsilon_{\lambda,\mu} + d^\varepsilon(q^\varepsilon_{\lambda,\mu})^2\}\,dx$$
$$\geq \frac{1}{\gamma}\int_\Omega \varphi\left\{|A^\varepsilon(\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon)|^2 + \left((d^\varepsilon + A^\varepsilon\alpha^\varepsilon\cdot\alpha^\varepsilon)q^\varepsilon_{\lambda,\mu} - A^\varepsilon\alpha^e\cdot\nabla w^\varepsilon_{\lambda,\mu}\right)^2\right\}\,dx.$$

As before, we pass to the limit as $\varepsilon \to 0$. We obtain the result upon appealing to (4.9) and after localizing in x. □

We are now in a position to define the homogenized coefficients for problem (3.1).

Definition 4.4 (Homogenized coefficients). *Let $A^0, H^0, d^0, l^0, n^0, p^0$ be defined in* (4.9) *and* (4.10). *We set*

$$A^{hom} := A^0 - H^0, \qquad \alpha^{hom} = -[A^{hom}]^{-1}(n^0 - p^0), \qquad d^{hom} := d^0 - l^0 - A^{hom}\alpha^{hom}\cdot\alpha^{hom}.$$

Note that the definition makes sense in view of item (b) in Proposition 4.3. We also have the following

Proposition 4.5. *$(A^{hom}, \alpha^{hom}, d^{hom}) \in \mathcal{M}(\beta,\gamma;\Omega)$ defined in* (3.5).

Proof. In view of (4.13), we immediately obtain that, a.e. in Ω, and for any $\lambda \in \mathbb{R}^N$ and $\mu \in \mathbb{R}$

$$A^{hom}(\lambda - \mu\alpha^{hom})\cdot(\lambda - \mu\alpha^{hom}) + d^{hom}\mu^2 \geq \beta\{|\lambda|^2 + \mu^2\}. \tag{4.15}$$

We now rewrite (4.14) similarly. We obtain, a.e. in Ω, and for any $\lambda \in \mathbb{R}^N$ and $\mu \in \mathbb{R}$

$$A^{hom}(\lambda - \mu\alpha^{hom})\cdot(\lambda - \mu\alpha^{hom}) + d^{hom}\mu^2 \geq \frac{1}{\gamma}\{|A^{hom}(\lambda - \mu\alpha^{hom})|^2 + \left(d^{hom}\mu - A^{hom}\alpha^{hom}\cdot(\lambda - \mu\alpha^{hom})\right)^2\}. \tag{4.16}$$

For any $(\bar\lambda, \bar\mu) \in \mathbb{R}^N \times \mathbb{R}$ and a.e. $x \in \Omega$, we solve

$$\begin{cases} A^{hom}(\lambda - \mu\alpha^{hom}) = \bar\lambda \\ d^{hom}\mu - A^{hom}\alpha^{hom}\cdot(\lambda - \mu\alpha^{hom}) = \bar\mu, \end{cases}$$

getting the possibly x-dependent solution

$$\lambda = (A^{hom})^{-1}\bar\lambda + \frac{\alpha^{hom}}{d^{hom}}(\bar\mu + \alpha^{hom}\cdot\bar\lambda) \qquad \text{and} \qquad \mu = \frac{1}{d^{hom}}(\bar\mu + \alpha^{hom}\cdot\bar\lambda).$$

Replacing in (4.16), we finally get, a.e. in Ω and for any $\bar\lambda \in \mathbb{R}^N$ and $\bar\mu \in \mathbb{R}$

$$(A^{hom})^{-1}\bar\lambda\cdot\bar\lambda + \frac{1}{d^{hom}}(\bar\mu + \alpha^{hom}\cdot\bar\lambda)^2 \geq \frac{1}{\gamma}\{|\bar\lambda|^2 + \bar\mu^2\}. \tag{4.17}$$

In view of (4.15), (4.17), the result follows. □

5. The homogenization result

The following theorem is the main result of the paper.

Theorem 5.1. *Assume* (3.2) *and that $(A^\varepsilon, \alpha^\varepsilon, d^\varepsilon) \in \mathcal{M}(\beta,\gamma;\Omega)$ defined in* (3.5). *Let $\{\varepsilon\}$ denote the subsequence given by Proposition 4.1, and let $(A^{hom}, \alpha^{hom}, d^{hom}) \in \mathcal{M}(\beta,\gamma;\Omega)$ be the associated homogenized coefficients given in Definition 4.4 (and Proposition 4.5).*

For every $f \in H^{-1}(\Omega)$, let $(u^\varepsilon, \vartheta^\varepsilon) \in H^1_0(\Omega) \times H^1(\Omega)$ be the solution of

$$\begin{cases} -div[A^\varepsilon(\nabla u^\varepsilon - \vartheta^\varepsilon\alpha^\varepsilon)] = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega \\ -\varepsilon^2 div(B^\varepsilon\nabla\vartheta^\varepsilon) + d^\varepsilon\vartheta^\varepsilon - A^\varepsilon\alpha^\varepsilon\cdot(\nabla u^\varepsilon - \vartheta^\varepsilon\alpha^\varepsilon) = 0 & \text{in } \Omega \\ B^\varepsilon\nabla\vartheta^\varepsilon\cdot n = 0 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

(see Proposition 3.3). Then, as $\varepsilon \to 0$,

$$u^\varepsilon \rightharpoonup u \qquad \text{weakly in } H^1_0(\Omega) \qquad \text{and} \qquad \vartheta^\varepsilon \rightharpoonup \vartheta \qquad \text{weakly in } L^2(\Omega),$$

where $(u,\vartheta) \in H^1_0(\Omega) \times L^2(\Omega)$ is such that

$$\begin{cases} -div[A^{hom}(\nabla u - \vartheta \alpha^{hom})] = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ d^{hom}\vartheta \; - \; A^{hom}\alpha^{hom} \; \cdot \; (\nabla u - \vartheta\alpha^{hom}) = 0 & \text{in } \Omega. \end{cases} \tag{5.2}$$

In particular, $u \in H^1_0(\Omega)$ solves

$$\begin{cases} -div\left[\left(A^{hom} - \dfrac{A^{hom}\alpha^{hom} \otimes A^{hom}\alpha^{hom}}{d^{hom} + A^{hom}\alpha^{hom}\cdot\alpha^{hom}}\right)\nabla u\right] = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

Proof. Let us fix a subsequence of ε (not relabeled) such that

$$u^\varepsilon \rightharpoonup u \qquad \text{weakly in } H^1_0(\Omega) \qquad \text{and} \qquad \vartheta^\varepsilon \rightharpoonup \vartheta \qquad \text{weakly in } L^2(\Omega),$$

for some $(u,\vartheta) \in H^1_0(\Omega) \times L^2(\Omega)$. This is possible in view of estimate (3.6) and Poincaré's inequality. For every $\lambda \in \mathbb{R}^N$ and $\mu \in \mathbb{R}$ we consider the auxiliary functions $(w^\varepsilon_{\lambda,\mu}, q^\varepsilon_{\lambda,\mu})$ given by Proposition 4.1.

For every $\varphi \in C^\infty_c(\Omega)$,

$$\begin{aligned} 0 &= \int_\Omega \varphi\left(-\varepsilon^2 div(B^\varepsilon\nabla\vartheta^\varepsilon) + d^\varepsilon\vartheta^\varepsilon - A^\varepsilon\alpha^\varepsilon\cdot(\nabla u^\varepsilon - \vartheta^\varepsilon\alpha^\varepsilon)\right) q^\varepsilon_{\lambda,\mu}\,dx \\ &= \varepsilon^2\int_\Omega q^\varepsilon_{\lambda,\mu} B^\varepsilon\nabla\vartheta^\varepsilon\cdot\nabla\varphi\,dx - \varepsilon^2\int_\Omega \vartheta^\varepsilon B^\varepsilon\nabla\varphi\cdot\nabla q^\varepsilon_{\lambda,\mu}\,dx \\ &\quad + \int_\Omega \varphi\left(-\varepsilon^2 div(B^\varepsilon\nabla q^\varepsilon_{\lambda,\mu}) + d^\varepsilon q^\varepsilon_{\lambda,\mu} - A^\varepsilon\alpha^\varepsilon\cdot(\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon)\right)\vartheta^\varepsilon\,dx \\ &\quad - \int_\Omega \varphi A^\varepsilon(\nabla u^\varepsilon - \vartheta^\varepsilon\alpha^\varepsilon)\cdot\nabla w^\varepsilon_{\lambda,\mu}\,dx + \int_\Omega \varphi A^\varepsilon(\nabla w^\varepsilon_{\lambda,\mu} - q^\varepsilon_{\lambda,\mu}\alpha^\varepsilon)\cdot\nabla u^\varepsilon\,dx \end{aligned} \tag{5.4}$$

where the first equality holds in view of the third equation in (3.1). The first two terms on the right-hand side of the second equality vanish as $\varepsilon \to 0$ since, thanks to (3.2), (3.6), (4.1) and (4.4),

$$\sup_\varepsilon\left(\|B^\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^N_{\rm sym})} + \|\vartheta^\varepsilon\|_{L^2(\Omega)} + \|\varepsilon\nabla\vartheta^\varepsilon\|_{L^2(\Omega;\mathbb{R}^N)} + \|q^\varepsilon_{\lambda,\mu}\|_{L^2(\Omega)} + \|\varepsilon\nabla q^\varepsilon_{\lambda,\mu}\|_{L^2(\Omega;\mathbb{R}^N)}\right) < \infty.$$

Setting

$$\sigma^\varepsilon := A^\varepsilon(\nabla u^\varepsilon - \vartheta^\varepsilon\alpha^\varepsilon),$$

we can assume that there exists $\sigma \in L^2(\Omega;\mathbb{R}^N)$ such that up to a subsequence

$$\sigma^\varepsilon \rightharpoonup \sigma \qquad \text{weakly in } L^2(\Omega;\mathbb{R}^N).$$

Clearly, in view of the first equation in (5.1),

$$-div\,\sigma = f \qquad \text{in } \Omega. \tag{5.5}$$

Thanks to Remark 4.2, we can pass to the limit in the right hand-side of the second equality in (5.4) obtaining, since $e^0 - f^0 = n^0 - p^0$ in view of Proposition 4.3, and φ is arbitrary,

$$\begin{cases} \sigma = (A^0 - H^0)\nabla u + (n^0 - p^0)\vartheta \\ \vartheta = -\dfrac{n^0 - p^0}{d^0 - l^0}\cdot\nabla u, \end{cases}$$

a.e. in Ω. In particular,

$$\sigma = \left(A^0 - H^0 - \frac{(n^0 - p^0)\otimes(n^0 - p^0)}{d^0 - l^0}\right)\nabla u.$$

In view of the very definition of $A^{hom}, \alpha^{hom}, d^{hom}$, and of equation (5.5), we get that (u,ϑ) is a solution of (5.2). In particular u is a solution of the elliptic problem (5.3). Since, according to Remark 3.2, the matrix in (5.3) is coercive, u, hence ϑ, are uniquely determined. As a consequence, there is no need to pass to further subsequences in the previous argument and the proof is concluded. $\square$

Remark 5.2. The tool of H-convergence can be employed also in the case of a *non* symmetric matrix A^ε. Namely, one can study the asymptotic behaviour of solutions of the more general problem

$$\begin{cases} -div[A^\varepsilon(\nabla u^\varepsilon - \vartheta^\varepsilon\alpha^\varepsilon)] = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega \\ -\varepsilon^2 div(B^\varepsilon\nabla\vartheta^\varepsilon) + d^\varepsilon\vartheta^\varepsilon - (A^\varepsilon)^T\alpha^\varepsilon\cdot(\nabla u^\varepsilon - \vartheta^\varepsilon\alpha^\varepsilon) = 0 & \text{in } \Omega \\ B^\varepsilon\nabla\vartheta^\varepsilon\cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.6}$$

where the matrix A^ε is not required to be symmetric. This can be done at the expense of considering additional auxiliary functions $(\tilde{w}_{\lambda,\mu}, \tilde{q}_{\lambda,\mu})$ associated to the transposed system and computing the relative homogenized coefficients, as in Section 4. Note that problem (5.6) reduces to (5.1) in case A^ε symmetric. ¶

6. The Fleck & Willis setting

As alluded to in the introduction, in the presence of body loads $f \in H^{-1}(\Omega;\mathbb{R}^N)$ and in the absence of surface loads or boundary displacements, the strain gradient plasticity model considered in [5] amounts to considering the minimum configurations $(u^\varepsilon, p^\varepsilon)$ in $H^1_0(\Omega;\mathbb{R}^N)\times H^1(\Omega;\mathrm{M}^N_D)$ for an energy of the form

$$(u,p)\mapsto \mathcal{E}^\varepsilon(u,p) - \int_\Omega f\cdot u\,dx.$$

The internal energy $\mathcal{E}^\varepsilon(u,p)$ in the quadratic case is given by the expression

$$\mathcal{E}^\varepsilon(u,p) := \frac{1}{2}\int_\Omega \mathbb{C}^\varepsilon(x)(E(u)-p):(E(u)-p)\,dx + \frac{1}{2}\int_\Omega b^\varepsilon(x)[|p|^2 + \varepsilon^2\ell^2|\nabla p|^2]\,dx. \tag{6.1}$$

The associated Euler-Lagrange equations are

$$\begin{cases} -div[\mathbb{C}^\varepsilon(E(u^\varepsilon)-p^\varepsilon)] = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ -\varepsilon^2\ell^2 div(b^\varepsilon\nabla p^\varepsilon) + b^\varepsilon p^\varepsilon - \mathbb{C}^\varepsilon(E(u^\varepsilon)-p^\varepsilon) = 0 & \text{in } \Omega \\ \dfrac{\partial p^\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.2}$$

where n denotes the exterior normal vector to Ω at a point of $\partial\Omega$.

The natural assumptions in strain gradient plasticity are as follows: $\mathbb{C}^\varepsilon$ and b^ε are measurable, $\mathbb{C}^\varepsilon \in \mathcal{L}_s(\mathrm{M}^N_{\mathrm{sym}})$ a.e. in Ω, and, for some $0<\beta<\gamma$,

$$\beta \le b^\varepsilon \le \gamma \qquad \text{and} \qquad \beta\mathbb{I}d \le \mathbb{C}^\varepsilon \le \gamma\mathbb{I}d, \tag{6.3}$$

$\mathbb{I}d$ denoting the identity in $\mathcal{L}_s(\mathrm{M}^N_{\mathrm{sym}})$. Under assumption (6.3), the existence of $\hat\gamma > \hat\beta > 0$ such that, for a.e. $x\in\Omega$ and for every $(e,p)\in \mathrm{M}^N_{\mathrm{sym}}\times\mathrm{M}^N_D$,

$$\begin{cases} \mathbb{C}^\varepsilon(x)(e-p)\cdot(e-p) & + & b^\varepsilon(x)|p|^2 & \ge & \hat\beta(|e|^2+|p|^2) \\ (\mathbb{C}^\varepsilon)^{-1}(x)e\cdot e & + & \dfrac{1}{b^\varepsilon(x)}|e+p|^2 & \ge & \dfrac{1}{\hat\gamma}\{|e|^2+|p|^2\} \end{cases}$$

is immediate. Then, the proof of Theorem 5.1 can be reproduced word for word in this new setting at the expense of the use of Korn's inequality (which, in $H^1_0(\Omega;\mathbb{R}^N)$, does not require any smoothness assumptions on the domain Ω).

Upon defining, for every $e \in \mathrm{M}^N_{\mathrm{sym}}$ and $q \in \mathrm{M}^N_D$, $W^\varepsilon_{e,q} \in H^1(\Omega;\mathbb{R}^N)$ and $Q^\varepsilon_{e,q} \in H^1(\Omega;\mathrm{M}^N_D)$ such that

$$(6.4)\qquad \begin{cases} W^\varepsilon_{e,q} \rightharpoonup ex \quad \text{weakly in } H^1(\Omega;\mathbb{R}^N), \quad Q^\varepsilon_{e,q} \rightharpoonup q \quad \text{weakly in } L^2(\Omega;\mathrm{M}^N_D) \\ -div[\mathbb{C}^\varepsilon(E(W^\varepsilon_{e,q}) - Q^\varepsilon_{e,q})] \quad \text{compact in } H^{-1}(\Omega;\mathbb{R}^N) \\ -\varepsilon^2\ell^2 div(b^\varepsilon\nabla Q^\varepsilon_{e,q}) + b^\varepsilon Q^\varepsilon_{e,q} - \mathbb{C}^\varepsilon(E(W^\varepsilon_{e,q}) - Q^\varepsilon_{e,q}) \quad \text{compact in } L^2(\Omega;\mathrm{M}^N_D) \\ \varepsilon\|\nabla Q^\varepsilon_{e,q}\|_{L^2(\Omega;(\mathrm{M}^N_D)^N)} \text{ is bounded,} \end{cases}$$

we set, up to the possible expense of extracting a further subsequence,

$$(6.5)\qquad \begin{cases} (b^\varepsilon Id + \mathbb{C}^\varepsilon)Q^\varepsilon_{e,q} \rightharpoonup \mathbb{D}^0 q + \mathbb{E}^0 e & \text{weakly in } L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}) \\ \mathbb{C}^\varepsilon Q^\varepsilon_{e,q} \rightharpoonup \mathbb{P}^0 q + \mathbb{H}^0 e & \text{weakly in } L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}) \\ \mathbb{C}^\varepsilon E(W^\varepsilon_{e,q}) \rightharpoonup \mathbb{N}^0 q + \mathbb{A}^0 e & \text{weakly in } L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}) \end{cases}$$

for suitable

$$\mathbb{D}^0, \mathbb{E}^0, \mathbb{P}^0, \mathbb{N}^0, \mathbb{A}^0, \mathbb{H}^0 \in L^2(\Omega;\mathcal{L}(\mathrm{M}^N_{\mathrm{sym}})).$$

It is then easily checked that the analogue of Proposition 4.3 holds true. In particular,

$$(6.6)\qquad \begin{cases} \mathbb{N}^0 - \mathbb{P}^0 = (\mathbb{E}^0 - \mathbb{A}^0)^T \in L^\infty(\Omega;\mathcal{L}(\mathrm{M}^N_{\mathrm{sym}})) \\ \mathbb{A}^0 - \mathbb{H}^0 \in L^\infty(\Omega;\mathcal{L}_s(\mathrm{M}^N_{\mathrm{sym}})) \text{ and } \mathbb{A}^0 - \mathbb{H}^0 \geq \hat\beta \mathbb{I}d \\ \mathbb{D}^0 - \mathbb{N}^0 \in L^\infty(\Omega;\mathcal{L}_s(\mathrm{M}^N_{\mathrm{sym}})) \text{ and } \mathbb{D}^0 - \mathbb{N}^0 \geq \hat\beta \mathbb{I}d. \end{cases}$$

In view of (6.6), the following definitions are meaningful:

$$(6.7)\quad \mathbb{C}^{hom} := \mathbb{A}^0 - \mathbb{H}^0,\ \mathbb{A}^{hom} := -(\mathbb{C}^{hom})^{-1}(\mathbb{N}^0 - \mathbb{P}^0),\ \mathbb{D}^{hom} := \mathbb{D}^0 - \mathbb{N}^0 - (\mathbb{A}^{hom})^T\mathbb{C}^{hom}\mathbb{A}^{hom},$$

and

$$\mathbb{C}^{hom} - \mathbb{C}^{hom}\mathbb{A}^{hom}\left(\mathbb{D}^{hom} + (\mathbb{A}^{hom})^T\mathbb{C}^{hom}\mathbb{A}^{hom}\right)^{-1}(\mathbb{A}^{hom})^T\mathbb{C}^{hom} \geq \hat\beta\mathbb{I}d.$$

We finally get the following result:

Theorem 6.1. *Assuming* (6.3), *there exists a subsequence of* $\{\varepsilon\}$ *(not relabeled) such that, as* $\varepsilon \to 0$, $(u^\varepsilon, p^\varepsilon)$ *solutions of* (6.2) *satisfy*

$$u^\varepsilon \rightharpoonup u \quad \textit{weakly in } H^1_0(\Omega;\mathbb{R}^N) \quad \textit{and} \quad p^\varepsilon \rightharpoonup p \quad \textit{weakly in } L^2(\Omega;\mathrm{M}^N_D),$$

where $(u,p) \in H^1_0(\Omega;\mathbb{R}^N) \times L^2(\Omega;\mathrm{M}^N_D)$ *is such that*

$$(6.8)\qquad \begin{cases} -div[\mathbb{C}^{hom}(E(u) - \mathbb{A}^{hom}p)] = f & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega \\ \mathbb{D}^{hom}p - (\mathbb{A}^{hom})^T\mathbb{C}^{hom}(E(u) - \mathbb{A}^{hom}p) = 0 & \textit{in } \Omega. \end{cases}$$

In particular, $u \in H^1_0(\Omega;\mathbb{R}^N)$ *solves*

$$\begin{cases} -div\left[\left(\mathbb{C}^{hom} - (\mathbb{C}^{hom}\mathbb{A}^{hom}\left(\mathbb{D}^{hom} + (\mathbb{A}^{hom})^T\mathbb{C}^{hom}\mathbb{A}^{hom}\right)^{-1}(\mathbb{A}^{hom})^T\mathbb{C}^{hom}\right)E(u)\right] = f & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega. \end{cases}$$

Remark 6.2. Note that the solution (u,p) of (6.8) can be equivalently seen as the minimizer on $H^1_0(\Omega;\mathbb{R}^N) \times L^2(\Omega;\mathrm{M}^N_D)$ of

$$\mathcal{F}^{hom}(\bar u, \bar p) := \frac{1}{2}\int_\Omega \left(\mathbb{C}^{hom}(E(\bar u) - \mathbb{A}^{hom}\bar p)\cdot(E(\bar u) - \mathbb{A}^{hom}\bar p) + \mathbb{D}^{hom}\bar p\cdot\bar p\right)\,dx - \int_\Omega f\cdot\bar u\,dx.$$

¶

Remark 6.3. In the particular case where $\mathbb{C}^\varepsilon = \hat{\mathbb{C}}$, *i.e.*, when the elastic moduli do not oscillate, we deduce easily from (6.5) and (6.6) that

$$\mathbb{A}^0 = \mathbb{P}^0 = \hat{\mathbb{C}}, \qquad \mathbb{E}^0 = \mathbb{N}^0 = \mathbb{H}^0 = 0,$$

and that for every $q \in \mathrm{M}^N_D$

$$\mathbb{D}^0 q = \hat{\mathbb{C}} q + \lim_\varepsilon b^\varepsilon Q^\varepsilon_{0,q}.$$

We then obtain

$$\mathbb{C}^{hom} = \hat{\mathbb{C}}, \qquad \mathbb{A}^{hom} = Id,$$

and for every $q \in \mathrm{M}^N_D$

$$\mathbb{D}^{hom} q = \lim_\varepsilon b^\varepsilon Q^\varepsilon_{0,q}.$$

We conclude that

$$\mathcal{F}^{hom}(\bar{u}, \bar{p}) := \frac{1}{2} \int_\Omega \left(\hat{\mathbb{C}}(E(\bar{u}) - \bar{p}) \cdot (E(\bar{u}) - \bar{p}) + \mathbb{D}^{hom} \bar{p} \cdot \bar{p} \right) \, dx - \int_\Omega f \cdot \bar{u} \, dx.$$

The elastic part of the functional remains thus unchanged in the homogenization limit. Because the tensor $\mathbb{D}^{hom}$ occurring in the plastic part is constructed with the help of b^ε and $Q^\varepsilon_{0,q}$, it will depend on the plastic and elastic moduli of Ω, as well as on the characteristic length scale ℓ. ¶

In the periodic case, which is the focus of [7], the functions b^ε and $\mathbb{C}^\varepsilon$ are given as $b(\frac{x}{\varepsilon})$, resp. $\mathbb{C}(\frac{x}{\varepsilon})$, where $y \mapsto (b(y), \mathbb{C}(y))$ is an e.g. bounded measurable function on $Y := (0,1)^N$, extended by periodicity to all of $\mathbb{R}^N$. In our setting, it is sufficient to identify the functions $W^\varepsilon_{e,q}, Q^\varepsilon_{e,q}$. A straightforward computation would establish that, denoting by $W_{e,q}(y) \in H^1_{per,0}(Y; \mathbb{R}^N), Q_{e,q}(y) \in H^1_{per}(Y; \mathrm{M}^N_D)$ with $\int_Y Q_{e,q} \, dy = q$ the minimum of (1.4), then a good candidate for (6.4) is

$$W^\varepsilon_{e,q}(x) := ex + \varepsilon W_{e,q}\left(\frac{x}{\varepsilon}\right) \qquad \text{and} \qquad Q^\varepsilon_{e,q}(x) := Q_{e,q}\left(\frac{x}{\varepsilon}\right).$$

Thanks to Riemann-Lebesgue's lemma, the energy $F^{hom}(e,q)$ defined in (1.4) is the weak L^1-limit of

$$\frac{1}{2}\left(\mathbb{C}\left(\frac{x}{\varepsilon}\right)(E(W^\varepsilon_{e,q}) - Q^\varepsilon_{e,q}) \cdot (E(W^\varepsilon_{e,q}) - Q^\varepsilon_{e,q}) + b\left(\frac{x}{\varepsilon}\right)\left(|Q^\varepsilon_{e,q}|^2 + \varepsilon^2 \ell^2 |\nabla Q^\varepsilon_{e,q}|^2\right)\right).$$

But that weak limit is in turn that of

$$\begin{aligned} \frac{1}{2}\Big(\mathbb{C}\left(\frac{x}{\varepsilon}\right)&(E(W^\varepsilon_{e,q}) - Q^\varepsilon_{e,q}) \cdot E(W^\varepsilon_{e,q}) \\ &+ \left[b\left(\frac{x}{\varepsilon}\right) Q^\varepsilon_{e,q} - \mathbb{C}\left(\frac{x}{\varepsilon}\right)(E(W^\varepsilon_{e,q}) - Q^\varepsilon_{e,q}) - \varepsilon^2 \ell^2 div\left(b^\varepsilon\left(\frac{x}{\varepsilon}\right)\nabla Q^\varepsilon_{e,q}\right)\right] Q^\varepsilon_{e,q}\Big), \end{aligned} \tag{6.9}$$

as immediately seen upon testing both limits against a $\mathbb{C}^\infty_c(\Omega)$-function and appealing to the fourth relation in (6.4).

In view of (6.5), (6.7), the weak limit of (6.9) is found to be

$$\frac{1}{2}\left(\mathbb{C}^{hom}(e - \mathbb{A}^{hom} q) \cdot (e - \mathbb{A}^{hom} q) + \mathbb{D}^{hom} q \cdot q\right),$$

so that $\mathcal{F}^{hom}$ in Remark 6.2 equivalently reads as

$$\mathcal{F}^{hom}(\bar{u}, \bar{p}) := \frac{1}{2} \int_\Omega F^{hom}\left(E(\bar{u}), \bar{p}\right) \, dx - \int_\Omega f \cdot \bar{u} \, dx,$$

which is precisely the result obtained in [7].

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