TWO-SCALE HOMOGENIZATION FOR A MODEL IN STRAIN GRADIENT PLASTICITY

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Abstract. Using the tool of two-scale convergence, we provide a rigorous mathematical setting for the homogenization result obtained by Fleck and Willis (*J. Mech. Phys. Solids*, 2004) concerning the effective plastic behaviour of a strain gradient composite material. Moreover, moving from deformation theory to flow theory, we prove a convergence result for the homogenization of quasistatic evolutions in the presence of isotropic linear hardening.

Keywords: strain gradient plasticity, periodic homogenization, two-scale convergence, quasistatic evolutions.

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1. Introduction

Strain gradient plasticity models have been deeply studied in recent years in order to understand size effects taking place in ductile metals (see [4, 5, 9, 10] and references therein). The gradient of the plastic strain is connected with the density of *geometrically necessary dislocations* inside the body (see [2]) and its inclusion in the model aims at capturing their interactions. However, the way in which such a term affects the equations of the model is still suggested by phenomenological considerations, although in agreement with the general principles of thermodynamics (see [9, 10]). Strain gradient terms may play both a *dissipative* and an *energetic* role; if the configuration of an elastoplastic body $\Omega \subseteq \mathbb{R}^N$ subject to small displacements $u : \Omega \to \mathbb{R}^N$ entails a plastic strain $p : \Omega \to \mathrm{M}^N_D$ (here M^N_D stands for the space of symmetric deviatoric matrices, see Section 2), an overall plastic strain measure usually employed to compute the dissipation during an evolution is given by the quantity

$$\sqrt{|\dot{p}|^2 + \ell^2 |\nabla \dot{p}|^2}.$$

Here ℓ is a *dissipative* length-scale, which has the dimension of a length and the order of magnitude of the distance at which interactions between dislocations take place. In particular, for polycrystals, ℓ is comparable with the size of the grains of the material.

In 2004 Fleck and Willis [6] studied the behaviour of composite materials with highly oscillating elastic and plastic moduli, whose response in the homogenization limit does not involve gradient terms. More precisely, they considered a strain gradient deformation theory whose associated energy is given by

$$\mathcal{E}(u,p) = \frac{1}{2}\int_\Omega \mathbb{C}(x)(Eu-p):(Eu-p)\,dx + \int_\Omega b(x)[|p|^2 + \ell^2|\nabla p|^2]\,dx, \tag{1.1}$$

where the elastic tensor $\mathbb{C}$ and the yielding function b highly oscillate in space, and Eu denotes the symmetrized gradient of u. Since the interactions modeled by strain gradients tend to vanish in the homogenization limit, Fleck and Willis focused on the problem of finding suitable bounds for the effective energy (independent of ∇p)

$$\mathcal{E}^{\mathit{eff}}(u,p) = \int_\Omega F^{\mathit{eff}}(Eu(x),p(x))\,dx \tag{1.2}$$

governing the behaviour of the homogenized body. Here the *effective energy density* $F^{\mathit{eff}}(\bar{A},\bar{p})$ is provided by minimizing the energy (1.1) on a representative volume element, among displacement fields u satisfying the linear boundary condition $u = \bar{A}\cdot x$ and plastic strains p with mean given by $\bar{p}$. In the particular case when $\mathbb{C}$ is constant and only the yielding function b oscillates, the effective energy density becomes

$$F^{\mathit{eff}}(\bar{A},\bar{p}) = \frac{1}{2}\mathbb{C}(\bar{A}-\bar{p}):(\bar{A}-\bar{p}) + V^{\mathit{eff}}(\bar{p}), \tag{1.3}$$

where the elastic part is clearly identified, but the effective plastic potential V^{eff} depends also on the elastic properties of Ω. Indeed, its expression involves an operator Γ (introduced by Willis in [22]) associated to an elasticity problem which depends on $\mathbb{C}$ (see Theorem 4.7).

The first aim of our paper is to provide a rigorous mathematical framework in order to establish (1.2) in the case when $\mathbb{C}$ and b oscillate in a periodic way. We consider energies of the form

$$\mathcal{E}_\varepsilon(u,p) = \frac{1}{2}\int_\Omega \mathbb{C}\left(\frac{x}{\varepsilon}\right)(Eu-p):(Eu-p)\,dx + \int_\Omega b\left(\frac{x}{\varepsilon}\right)[|p|^2 + \varepsilon^2\ell^2|\nabla p|^2]\,dx, \tag{1.4}$$

defined on $H^1(\Omega;\mathbb{R}^N)\times H^1(\Omega;\mathrm{M}^N_D)$, where $\mathbb{C}$ and b are periodic and satisfy suitable coercivity assumptions. The plastic and elastic moduli oscillate on a scale ε; accordingly, the dissipative length-scale is given by $\varepsilon\ell$, with $\ell > 0$.

We study the asymptotic behaviour of $\mathcal{E}_\varepsilon$ as $\varepsilon \to 0$ in the framework of *two-scale convergence*. This remarkable notion (see Section 3 for the precise definition and the main properties) has been introduced by Nguetseng [16] and Allaire [1] in order to study periodic homogenization in linearized elasticity. However, for the purposes of our paper we employ a formulation of two-scale convergence due to Mielke and Timofte [15], which is based on suitable *unfolding* and *folding* operators. Considering for $\varepsilon > 0$ the decomposition

$$\mathbb{R}^N = \varepsilon\mathbb{Z}^N + \varepsilon Y, \qquad Y := \left[-\frac{1}{2},\frac{1}{2}\right[^N,$$

for a family of functions $v_\varepsilon \in L^p(\Omega)$ whose supports are uniformly compactly contained in Ω (for the general case we refer to Section 3), the unfolding $\mathcal{T}_\varepsilon(v_\varepsilon) \in L^p(\Omega\times Y)$ turns out to be defined as

$$\mathcal{T}_\varepsilon(v_\varepsilon)(x,y) = v_\varepsilon(\mathcal{N}_\varepsilon(x) + \varepsilon y), \tag{1.5}$$

$\mathcal{N}_\varepsilon(x)$ being the center of the cell of the grid containing x, and the *two-scale weak limit* of v_ε is given by $V \in L^p(\Omega\times Y)$ such that

$$\mathcal{T}_\varepsilon(v_\varepsilon) \rightharpoonup V \qquad \text{weakly in } L^p(\Omega\times Y).$$

Noticeably, a *microstructural* variable y appears in order to keep track of the oscillations of the functions of the family. In Section 4 we will prove that the asymptotic behaviour of (1.4) along a family $(u_\varepsilon, p_\varepsilon)_{\varepsilon>0}$ can be inferred from the two-scale energy

$$\mathcal{E}(u,U,P) = \frac{1}{2}\int_{\Omega\times Y} \mathbb{C}(y)(Eu + E_yU - P) : (Eu + E_yU - P)\,dxdy + \int_{\Omega\times Y} b(y)[|P|^2 + \ell^2|\nabla_y P|^2]\,dxdy, \tag{1.6}$$

where $U \in L^2(\Omega; H^1_{per,0}(Y;\mathbb{R}^N))$, periodic and with null average in y, is connected with the two-scale weak limit of Eu_ε (see Proposition 3.3), while $P \in L^2(\Omega; H^1_{per}(Y;\mathrm{M}^N_D))$, periodic in y, is associated with the two-scale weak limit of p_ε and $\varepsilon\nabla p_\varepsilon$ (see Theorem 3.5). By employing (1.6), we will show (Theorem 4.3 and Theorem 4.5) that the configurations $(u_\varepsilon, p_\varepsilon)$ which minimize (under suitable boundary conditions for the displacement)

$$(u,p) \mapsto \mathcal{E}_\varepsilon(u,p) - \int_\Omega f\cdot u\,dx,$$

where $f \in L^2(\Omega;\mathbb{R}^N)$ stands for the density per unit volume of external body forces, converge as $\varepsilon \to 0$ in the weak topology of $H^1(\Omega;\mathbb{R}^N) \times L^2(\Omega;\mathrm{M}^N_D)$ to the minimizer of

$$(u,p) \mapsto \mathcal{E}^{\mathit{eff}}(u,p) - \int_\Omega f\cdot u\,dx,$$

where $\mathcal{E}^{\mathit{eff}}$ is of the form (1.2). Moreover, concerning the effective energy density we obtain the formula

$$F^{\mathit{eff}}(\bar A,\bar p) := \min\Big\{\frac{1}{2}\int_Y \mathbb{C}(y)[\bar A + E_yU - P] : [\bar A + E_yU - P]\,dy + \int_Y b(y)[|P|^2 + \ell^2|\nabla_y P|^2]\,dy : (U,P) \in H^1_{per,0}(Y;\mathbb{R}^N)\times H^1_{per}(Y;\mathrm{M}^N_D),\ \int_Y P(y)\,dy = \bar p\Big\} \tag{1.7}$$

in which the representative volume element is precisely the unit cell Y. In the case when the elasticity tensor is constant and oscillations do occur only in the yielding function, we obtain a characterization of Willis' operator Γ in terms of the function U appearing in (1.6) (see Definition 4.6 and Theorem 4.7).

The key tool in investigating the asymptotic behaviour as $\varepsilon \to 0$ of the energies (1.4) is Theorem 3.5, where an asymptotic and approximation result in a two-scale sense concerning functions v_ε bounded in $H^1(\Omega)$ with $\varepsilon\nabla v_\varepsilon$ bounded in $H^1(\Omega;\mathbb{R}^N)$ is given.

In Section 5 we move from deformation theory to flow theory, considering quasistatic evolutions for the model associated to (1.1) in the presence of isotropic linear hardening. Setting again the problem in the case of periodic oscillations for the elastic and plastic moduli, we study the asymptotic behaviour of quasistatic evolutions with vanishing strain gradient effects, employing the energetic approach to evolutions for rate independent systems introduced by Mielke and his school (see [13] and references therein). In this framework, the analysis of the deformation theory can be considered as a preliminary step for the study of the corresponding flow theory. We show in Theorem 5.8 that the homogenization of quasistatic evolutions can be understood moving to a two-scale setting and considering a suitable notion of quasistatic evolution within this context (see Definition 5.4): even if strain gradient effects tend to vanish, the model turns out to be of strain gradient type with respect to the microstructural variable y. The passage to a single scale setting seems to lead to an evolution which cannot be described in terms of standard plasticity models associated to the effective energy (1.2) (see Remark 5.9).

The paper is organized as follows. In Section 2 we state the notation employed throughout the paper, while in Section 3 we recall the definition and the basic properties of two-scale convergence which will be essential in Section 4 when dealing with the two-scale approach to the homogenization procedure of Fleck and Willis. Finally, Section 5 is devoted to the homogenization of strain gradient quasistatic evolutions with isotropic linear hardening.

2. Notation and preliminaries

In this section we introduce the notation and recall some basic definitions concerning the functional spaces employed in the rest of the paper. In the following, $B_r(x)$ will denote the open ball of center $x \in \mathbb{R}^N$ and radius $r > 0$. If $E \subset \mathbb{R}^N$, we will denote its volume by $|E|$, and 1_E will stand for its characteristic function, i.e., $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \notin E$.

Matrices. On the space of $N \times N$ matrices $A = (a_{ij})$ with $a_{ij} \in \mathbb{R}$ we will consider the scalar product

$$A : B := \sum_{i,j} a_{ij} b_{ij}.$$

The associated norm of A is denoted by $|A|$.

We will denote by $\mathrm{M}^N_{\mathrm{sym}}$ the subspace of symmetric matrices, and by M^N_D the subspace of $\mathrm{M}^N_{\mathrm{sym}}$ of deviatoric matrices A, that is such that $\mathrm{tr}A := \sum_i a_{ii} = 0$.

The symmetrized gradient of a $\mathbb{R}^N$-valued function $u(x)$ is defined as

$$Eu := \frac{\nabla u + \nabla u^T}{2},$$

where $(\nabla u)_{ij} = \frac{\partial u_i}{\partial x_j}$ is the gradient of u and ∇u^T denotes its transpose.

The gradient of a matrix-valued function $A(x) = (a_{ij}(x))$ is defined as the third-order tensor

$$(\nabla A)_{ijk} := \frac{\partial a_{ij}}{\partial x_k}.$$

We will consider on the space of third order tensors $\mathbb{A} = (a_{ijk})$ the norm

$$|\mathbb{A}| := \sqrt{\sum_{i,j,k} a^2_{ijk}}.$$

We say that $\mathbb{A} = (a_{ijk})$ is *symmetric-deviatoric in its first two subscripts* if

$$a_{ijk} = a_{jik} \qquad \text{and} \qquad \sum_p a_{ppk} = 0,$$

and we write $\mathbb{A} \in \mathbb{M}^N_D$.

Functional spaces. Throughout the paper, given $E \subseteq \mathbb{R}^N$ measurable and X a finite dimensional normed space, $L^p(E; X)$ with $p \in [1; +\infty[$ will stand for the space of p-summable functions with values in X. $L^\infty(E; X)$ will denote the space of essentially bounded maps from E to X, and $\|\cdot\|_\infty$ will be the associated sup-norm. Given $A \subseteq \mathbb{R}^N$ open, $W^{1,p}(A; X)$ will denote the usual Sobolev space of functions in $L^p(A; X)$ whose distributional derivatives are p-summable. For $p = 2$ we write $H^1(A; X)$ in place of $W^{1,2}(A; X)$. If $X = \mathbb{R}$, as usual we will write $L^p(E)$ and $W^{1,p}(A)$.

Let us set

$$Y := \left[-\frac{1}{2}, \frac{1}{2}\right[^N. \tag{2.1}$$

We will refer to Y as the *unit cell*, and write $W^{1,p}(Y; X)$ in place of $W^{1,p}(\mathrm{int}(Y); X)$. Moreover we set

$$W^{1,p}_{per}(Y; X) := \{u \in W^{1,p}(Y; X) : u \text{ admits a } Y\text{-periodic extension to } \mathbb{R}^N\},$$

$$H^1_{per}(Y; X) := W^{1,2}_{per}(Y; X),$$

and

$$W^{1,p}_{per,0}(Y; X) := \left\{u \in W^{1,p}_{per}(Y; X) : \int_Y u\, dy = 0\right\},$$

$$H^1_{per,0}(Y; X) := W^{1,2}_{per,0}(Y; X).$$

Notice that $u \in W^{1,p}_{per}(Y;X)$ can be characterized in terms of traces on the faces of Y. For $i = 1,\dots,N$ set

$$\partial_i^{\pm} Y := \left\{ y \in \overline{Y} : y_i = \pm\frac{1}{2} \right\}, \tag{2.2}$$

and let $\gamma_i^{\pm}$ denote the trace operator from $W^{1,p}(Y;X)$ to $L^p(\partial_i^{\pm}Y;X)$. The spaces $L^p(\partial_i^{\pm}Y;X)$ can be identified naturally with $L^p(]-1/2,1/2[^{N-1};X)$: it turns out easily that $u \in W^{1,p}_{per}(Y;X)$ if and only if $u \in W^{1,p}(Y;X)$ and $\gamma_i^+(u) = \gamma_i^-(u)$ for every $i = 1,\dots,N$.

3. Two-scale convergence

Introduced in the seminal papers of Nguetseng [16] and Allaire [1] about twenty years ago, two-scale convergence is nowadays a pretty well-known notion. Dealing with periodic functions with a precise scale parameter, it revealed as a powerful tool in performing periodic homogenization.

In this section we recall some basic facts concerning two-scale convergence, and we prove an approximation result (Theorem 3.5) which will be essential for our analysis of the homogenization procedure in strain gradient plasticity proposed by Fleck and Willis [6]. Since we need to manage problems which involve also the boundary of a body, we use a refined definition of two-scale convergence introduced by Mielke and Timofte in [15], where the reader can find a detailed study of the subject.

Performing for $\varepsilon > 0$ and $x \in \mathbb{R}^N$ the unique decomposition

$$x = \mathcal{N}_\varepsilon(x) + \varepsilon\mathcal{R}_\varepsilon(x)\,, \qquad \frac{1}{\varepsilon}\mathcal{N}_\varepsilon(x) \in \mathbb{Z}^N\,, \quad \mathcal{R}_\varepsilon(x) \in Y\,,$$

where Y is the unit cell defined in (2.1), let $\mathcal{D}_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N \times Y$ and $\mathcal{S}_\varepsilon : \mathbb{R}^N \times Y \to \mathbb{R}^N$ be defined as

$$\mathcal{D}_\varepsilon(x) := (\mathcal{N}_\varepsilon(x), \mathcal{R}_\varepsilon(x)) \quad \text{and} \quad \mathcal{S}_\varepsilon(x,y) := \mathcal{N}_\varepsilon(x) + \varepsilon y\,.$$

Let Ω be a bounded measurable subset of $\mathbb{R}^N$ with $|\partial\Omega| = 0$, and let $p \in [1,+\infty[$. Let us denote by $v1_\Omega$ the extension of $v \in L^p(\Omega;X)$ to all $\mathbb{R}^N$ with value 0 outside Ω. The *unfolding operator* $\mathcal{T}_\varepsilon : L^p(\Omega;X) \to L^p(\mathbb{R}^N \times Y;X)$ is the isometry defined as

$$\mathcal{T}_\varepsilon(v) := (v1_\Omega) \circ \mathcal{S}_\varepsilon\,. \tag{3.1}$$

For any $U \in L^p(\mathbb{R}^N \times Y;X)$ let

$$\mathcal{P}_\varepsilon(U)(x,y) := \frac{1}{\varepsilon^N}\int_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} U(\xi,y)\,d\xi$$

be the projection of U on the space of piecewise constant functions with respect to $x \in \mathbb{R}^N$. Then the *folding operator* $\mathcal{F}_\varepsilon : L^p(\mathbb{R}^N \times Y;X) \to L^p(\Omega;X)$ is given by

$$\mathcal{F}_\varepsilon(U) := (\mathcal{P}_\varepsilon(1_{[\Omega\times Y]_\varepsilon}U) \circ \mathcal{D}_\varepsilon)\Big|_\Omega\,, \tag{3.2}$$

where $[\Omega \times Y]_\varepsilon := \{(x,y) \in \mathbb{R}^N \times Y : \mathcal{S}_\varepsilon(x,y) \in \Omega\}$.

Definition 3.1 (Two-scale convergence). *Let X be a finite-dimensional normed space, $(u_\varepsilon)_{\varepsilon>0}$ a family in $L^p(\Omega;X)$, and $U \in L^p(\mathbb{R}^N \times Y;X)$ with $U = 0$ a.e. outside $\Omega \times Y$.*

We say that u_ε converges two-scale weakly *to U for $\varepsilon \to 0$, and write*

$$u_\varepsilon \overset{w\text{-}2}{\rightharpoonup} U \qquad \text{in } L^p(\Omega \times Y;X)$$

if $\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup U$ weakly in $L^p(\mathbb{R}^N \times Y;X)$.

We say that u_ε converges two-scale strongly *to U for $\varepsilon \to 0$, and write*

$$u_\varepsilon \overset{s\text{-}2}{\to} U \qquad \text{in } L^p(\Omega \times Y;X)$$

if $\mathcal{T}_\varepsilon u_\varepsilon \to U$ strongly in $L^p(\mathbb{R}^N \times Y;X)$.

Hence, within the approach of Mielke and Timofte, weak and strong two-scale convergence reduce to the classical weak and strong convergence in the double-variable space $L^p(\mathbb{R}^N \times Y)$. We will write $U \in L^p(\Omega \times Y)$ referring to the function U of the previous definition.

The following properties are quite easy to prove (see [15, Propositions 2.4-2.6]).

Proposition 3.2 (**Basic properties of two-scale convergence).** *Let X be a finite-dimensional normed space, $(u_\varepsilon)_{\varepsilon>0}$ a family in $L^p(\Omega;X)$, and $U\in L^p(\mathbb{R}^N\times Y;X)$ with $U=0$ a.e. outside $\Omega\times Y$ and $p\in]1,+\infty[$.*

(1) *If $(u_\varepsilon)_{\varepsilon>0}$ is two-scale weakly convergent in $L^p(\Omega\times Y;X)$ as $\varepsilon\to 0$, then it is bounded in $L^p(\Omega;X)$ as $\varepsilon\to 0$.*
(2) *If $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^p(\Omega;X)$, then $(u_\varepsilon)_{\varepsilon>0}$ is two-scale weakly convergent in $L^p(\Omega\times Y;X)$ along a suitable sequence $\varepsilon_n\to 0$.*
(3) *$u_\varepsilon \overset{s\text{-}2}{\to} U$ two-scale strongly in $L^p(\Omega\times Y;X)$ as $\varepsilon\to 0$ if and only if $u_\varepsilon \overset{w\text{-}2}{\rightharpoonup} U$ two-scale weakly in $L^p(\Omega\times Y;X)$ and $\|u_\varepsilon\|_{L^p(\Omega;X)}\to\|U\|_{L^p(\mathbb{R}^N\times Y;X)}$ as $\varepsilon\to 0$.*
(4) *$\mathcal{F}_\varepsilon(U)\overset{s\text{-}2}{\to} U$ two-scale strongly in $L^p(\Omega\times Y;X)$ as $\varepsilon\to 0$.*
(5) *If $u_\varepsilon \overset{w\text{-}2}{\rightharpoonup} U$ two-scale weakly in $L^p(\Omega\times Y;X)$ as $\varepsilon\to 0$, then*

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\psi\left(x,\frac{x}{\varepsilon}\right)dx=\int_{\Omega\times Y}U(x,y)\psi(x,y)\,dxdy \tag{3.3}$$

for every $\psi\in L^{p'}(\Omega;C^0_{per}(\overline{Y};X))$ ($p':=p/p-1$). Moreover, if $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^p(\Omega;X)$ as $\varepsilon\to 0$, then the two facts are equivalent.
(6) *If $u_\varepsilon \overset{w\text{-}2}{\rightharpoonup} U$ two-scale weakly in $L^p(\Omega\times Y)$ and $v_\varepsilon \overset{s\text{-}2}{\to} V$ two-scale strongly in $L^{p'}(\Omega\times Y)$,*

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon v_\varepsilon\,dx=\int_{\Omega\times Y}UV\,dxdy.$$

(7) *If $u_\varepsilon \overset{s\text{-}2}{\to} U$ two-scale strongly in $L^p(\Omega\times Y)$, and if $m_\varepsilon\in L^\infty(\Omega)$ is such that $\mathcal{T}_\varepsilon(m_\varepsilon)\to M$ a.e. in $\mathbb{R}^N\times Y$, then*

$$m_\varepsilon u_\varepsilon \overset{s\text{-}2}{\to} MU \qquad \textit{two-scale strongly in } L^p(\Omega\times Y).$$

Notice that taking ψ independent of y in (3.3), it follows that

$$u_\varepsilon(x)\rightharpoonup u(x):=\int_Y U(x,y)\,dy \qquad \text{weakly in } L^p(\Omega), \tag{3.4}$$

i.e., the average with respect to y of the two-scale weak limit U yields the usual weak limit of u_ε in $L^p(\Omega)$.

Let us now recall the main results on the two-scale convergence of derivatives of a Sobolev function. For $u\in L^p(\Omega)$, we denote with the same symbol u the function in $L^p(\Omega\times Y)$ such that $(x,y)\mapsto u(x)$. Then the following proposition holds (see [15, Proposition 2.9]).

Proposition 3.3 (**Two-scale convergence of gradients).** *Let $(u_\varepsilon)_{\varepsilon>0}$ be a family in $W^{1,p}(\Omega)$, $p\in]1,+\infty[$, such that $u_\varepsilon\rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ for $\varepsilon\to 0$. Then*

$$u_\varepsilon \overset{s\text{-}2}{\to} u \qquad \textit{two-scale strongly in } L^p(\Omega\times Y)$$

and there exists $U\in L^p(\Omega;W^{1,p}_{per,0}(Y))$ such that along a suitable sequence $\varepsilon_n\to 0$

$$\nabla u_{\varepsilon_n} \overset{w\text{-}2}{\rightharpoonup} \nabla u+\nabla_y U \quad \textit{two-scale weakly in } L^p(\Omega\times Y;\mathbb{R}^N).$$

Conversely, for every $u\in W^{1,p}(\Omega)$ and $U\in L^p(\Omega;W^{1,p}_{per,0}(Y;\mathbb{R}^N))$, there exists a family $(u_\varepsilon)_{\varepsilon>0}$ in $W^{1,p}(\Omega)$ such that for $\varepsilon\to 0$

$$u_\varepsilon\rightharpoonup u \quad \textit{weakly in } W^{1,p}(\Omega) \tag{3.5}$$

and

$$\nabla u_\varepsilon \overset{s\text{-}2}{\to} \nabla u+\nabla_y U \quad \textit{two-scale strongly in } L^p(\Omega\times Y;\mathbb{R}^N). \tag{3.6}$$

The previous result clearly holds also for the case of sequences, and can be extended to the case of functions taking values in a finite dimensional normed space.

Remark 3.4. The previous proposition entails the following variant which takes into account boundary conditions. Let Ω have a Lipschitz boundary. For every $u \in W^{1,p}(\Omega)$ and $U \in L^p(\Omega; W^{1,p}_{per,0}(Y))$, there exists a family $(u_\varepsilon)_{\varepsilon>0}$ in $W^{1,p}(\Omega)$ with $u_\varepsilon = u$ on $\partial\Omega$ (in the sense of traces) for every $\varepsilon > 0$ and such that (3.5) and (3.6) hold for $\varepsilon \to 0$. Indeed, if $(\tilde{u}_\varepsilon)_{\varepsilon>0}$ is the family given by Proposition 3.3, it is sufficient to consider

$$u_\varepsilon := \varphi_\varepsilon \tilde{u}_\varepsilon + (1-\varphi_\varepsilon)u,$$

where $\varphi_\varepsilon \in C^\infty_c(\Omega)$, $0 \le \varphi_\varepsilon \le 1$, is such that for $\varepsilon \to 0$

$$\varphi_\varepsilon \nearrow 1 \qquad \text{pointwise in } \Omega,$$

and (recall that by compact embedding $\tilde{u}_\varepsilon \to u$ strongly in $L^p(\Omega)$)

$$\|\nabla\varphi_\varepsilon\|_\infty \|\tilde{u}_\varepsilon - u\|_{L^p(\Omega)} \to 0.$$

In view of the analysis of the homogenization theory of Fleck and Willis, the following result is essential.

Theorem 3.5. *Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded with $|\partial\Omega| = 0$, and let $p \in [1, +\infty[$. The following facts hold.*

(a) *If $p > 1$ and $(u_\varepsilon)_{\varepsilon>0}$ is such that $u_\varepsilon \in W^{1,p}(\Omega)$ with*

$$\|u_\varepsilon\|_{L^p(\Omega)} + \varepsilon\|\nabla u_\varepsilon\|_{L^p(\Omega;\mathbb{R}^N)} \le C$$

for some $C > 0$, then there exist $\varepsilon_n \to 0$ and $U \in L^p(\Omega; W^{1,p}_{per}(Y))$ such that

$$u_{\varepsilon_n} \overset{w\text{-}2}{\rightharpoonup} U \qquad \text{two-scale weakly in } L^p(\Omega \times Y)$$

$$\varepsilon_n \nabla u_{\varepsilon_n} \overset{w\text{-}2}{\rightharpoonup} \nabla_y U \qquad \text{two-scale weakly in } L^p(\Omega \times Y; \mathbb{R}^N)\,.$$

(b) *For every $U \in L^p(\Omega; W^{1,p}_{per}(Y))$ there exists a family $(u_\varepsilon)_{\varepsilon>0}$ in $W^{1,p}(\Omega)$ such that for $\varepsilon \to 0$*

$$u_\varepsilon \overset{s\text{-}2}{\to} U \qquad \text{two-scale strongly in } L^p(\Omega \times Y)$$

$$\varepsilon \nabla u_\varepsilon \overset{s\text{-}2}{\to} \nabla_y U \qquad \text{two-scale strongly in } L^p(\Omega \times Y; \mathbb{R}^N).$$

Proof. Concerning point (a), since a bound on the norms of u_ε and $\varepsilon\nabla u_\varepsilon$ is available, the conclusion can be inferred from [1, Proposition 1.14] taking into account Proposition 3.2. For the reader's convenience, we give a direct proof.

Let us fix $\varepsilon_n \to 0$. By compactness, there exists $\varepsilon_n \to 0$ such that

$$u_{\varepsilon_n} \overset{w\text{-}2}{\rightharpoonup} U \qquad \text{two-scale weakly in } L^p(\Omega \times Y)$$

$$\varepsilon_n \nabla u_{\varepsilon_n} \overset{w\text{-}2}{\rightharpoonup} V \qquad \text{two-scale weakly in } L^p(\Omega \times Y; \mathbb{R}^N)$$

for some $U \in L^p(\Omega \times Y)$ and $V \in L^p(\Omega \times Y; \mathbb{R}^N)$. We claim that for a.e. $x \in \Omega$

$$V(x,\cdot) = \nabla_y U(x,\cdot) \qquad \text{in } Y \tag{3.7}$$

and

$$U(x,\cdot) \in W^{1,p}_{per}(Y)\,. \tag{3.8}$$

Indeed taking into account (3.3) we have that for every $\psi \in L^{p'}(\Omega; C^0_{per}(\overline{Y}))$,

$$\lim_{n\to\infty} \int_\Omega u_{\varepsilon_n}(x)\psi\left(x, \frac{x}{\varepsilon_n}\right) dx = \int_{\Omega\times Y} U(x,y)\psi(x,y)\,dxdy\,, \tag{3.9}$$

and

$$\lim_{n\to\infty} \int_\Omega \varepsilon_n \nabla u_{\varepsilon_n}(x)\psi\left(x, \frac{x}{\varepsilon_n}\right) dx = \int_{\Omega\times Y} V(x,y)\psi(x,y)\,dxdy\,. \tag{3.10}$$

Let us consider a test ψ of the form

$$\psi(x,y) := g(x)f(y)$$

with $g \in C^1_c(\Omega)$ and $f \in C^1_{per}(\overline{Y})$. Since for $i = 1, \dots, N$

$$\int_\Omega \varepsilon_n \partial_i u_{\varepsilon_n}(x) g(x) f\left(\frac{x}{\varepsilon_n}\right) dx = -\int_\Omega \varepsilon_n u_{\varepsilon_n}(x) \left[\partial_i g(x) f\left(\frac{x}{\varepsilon_n}\right) + \frac{1}{\varepsilon_n} g(x) \partial_i f\left(\frac{x}{\varepsilon_n}\right)\right] dx$$
$$= -\varepsilon_n \int_\Omega u_{\varepsilon_n}(x) \partial_i g(x) f\left(\frac{x}{\varepsilon_n}\right) dx - \int_\Omega u_{\varepsilon_n}(x) g(x) \partial_i f\left(\frac{x}{\varepsilon_n}\right) dx\,,$$

taking the limit as $n \to \infty$, in view of (3.9) and (3.10) we deduce that

$$\int_{\Omega \times Y} V_i(x,y) g(x) f(y)\, dxdy = -\int_{\Omega \times Y} U(x,y) g(x) \partial_i f(y)\, dxdy\,.$$

Letting g and f vary in countable dense subsets of $L^{p'}(\Omega)$ and $W^{1,p'}_{per}(Y)$ respectively, we deduce that for a.e. $x \in \Omega$

$$\int_Y V_i(x,y) f(y)\, dy = -\int_Y U(x,y) \partial_i f(y)\, dy \qquad \text{for every } f \in W^{1,p'}_{per}(Y), \quad i = 1, \dots, N, \tag{3.11}$$

so that claim (3.7) follows.

Choosing f independent of the coordinate y_i for $i = 1, \dots, N$ in (3.11) entails that for a.e. $x \in \Omega$

$$\int_Y \partial_i U(x,y) f(y)\, dy = 0,$$

so that in view of the integration by parts and the arbitrariness of f we deduce

$$\gamma_i^+(U(x,\cdot)) = \gamma_i^-(U(x,\cdot)),$$

where $\gamma_i^\pm$ denote the trace on the faces $y_i = \pm 1/2$. We conclude that claim (3.8) holds, so that the proof of point (a) is concluded.

Let us come to point (b). By means of a diagonal argument, it suffices to consider U belonging to the dense subset given by $C^1_c(\Omega; C^1_{per}(\bar{Y}))$. The result follows by setting

$$u_\varepsilon(x) := U\left(x, \frac{x}{\varepsilon}\right) \in C^1_c(\Omega).$$

Indeed, since

$$\mathcal{T}_\varepsilon(u_\varepsilon)(x,y) = U(\mathcal{N}_\varepsilon(x), y),$$

denoting by L the Lipschitz constant of U we have

$$|\mathcal{T}_\varepsilon(u_\varepsilon)(x,y) - U(x,y)| \le L|x - \mathcal{N}_\varepsilon(x)|$$

so that

$$\lim_{\varepsilon \to 0} \|\mathcal{T}_\varepsilon(u_\varepsilon) - U\|_{L^p(\mathbb{R}^N \times Y)} = 0.$$

We deduce for $\varepsilon \to 0$

$$u_\varepsilon \overset{s\text{-}2}{\to} U \qquad \text{two-scale strongly in } L^p(\Omega \times Y).$$

By the same arguments, since

$$\nabla u_\varepsilon(x) = \nabla_x U\left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon} \nabla_y U\left(x, \frac{x}{\varepsilon}\right),$$

we infer that for $\varepsilon \to 0$

$$\varepsilon \nabla u_\varepsilon \overset{s\text{-}2}{\to} \nabla_y U \qquad \text{two-scale strongly in } L^p(\Omega \times Y; \mathbb{R}^N),$$

so that the proof is concluded. □

Remark 3.6 (The vector valued case). The proof of Theorem 3.5 can be adapted to the case where U takes values in a finite dimensional normed space, since it is sufficient to work component by component. For our applications to strain gradient plasticity, we will consider functions taking values in the space M^N_D of symmetric deviatoric matrices defined in Section 2.

Remark 3.7 (The approximation result under an admissibility constraint). As a consequence of Theorem 3.5 (extended to the vectorial setting according to the previous remark) we get the following approximation result which will be used when dealing with the homogenization of quasistatic evolutions in Section 5.

Let M^N_D and $\mathbb{M}^N_D$ denote the set deviatoric matrices defined in Section 2, and let $\ell > 0$. If $P \in L^2(\Omega; H^1_{per}(Y; \mathrm{M}^N_D))$ and $Z \in L^2(\Omega \times Y)$ satisfy

$$\sqrt{|P|^2 + \ell^2 |\nabla_y P|^2} \leq Z \qquad \text{a.e. in } \Omega \times Y, \tag{3.12}$$

for every $\varepsilon > 0$ we can find $p_\varepsilon \in H^1(\Omega; \mathrm{M}^N_D)$ and $z_\varepsilon \in L^2(\Omega)$ such that

$$\sqrt{|p_\varepsilon|^2 + \varepsilon^2 \ell^2 |\nabla p_\varepsilon|^2} \leq z_\varepsilon \qquad \text{a.e. in } \Omega,$$

and as $\varepsilon \to 0$

$$p_\varepsilon \overset{s\text{-}2}{\to} P \qquad \text{two-scale strongly in } L^2(\Omega \times Y; \mathrm{M}^N_D) \tag{3.13}$$

$$\varepsilon \nabla p_\varepsilon \overset{s\text{-}2}{\to} \nabla_y P \qquad \text{two-scale strongly in } L^2(\Omega \times Y; \mathbb{M}^N_D) \tag{3.14}$$

$$z_\varepsilon \overset{s\text{-}2}{\to} Z \qquad \text{two-scale strongly in } L^2(\Omega \times Y).$$

Indeed, let $(p_\varepsilon)_{\varepsilon>0}$ be a family in $H^1(\Omega)$ satisfying (3.13) and (3.14) accoring to Theorem 3.5. Notice that thanks to Jensen's inequality, (3.12) entails

$$\sqrt{|\mathcal{F}_\varepsilon(P)|^2 + \ell^2 |\mathcal{F}_\varepsilon(\nabla_y P)|^2} \leq \mathcal{F}_\varepsilon(Z) \qquad \text{a.e. in } \Omega.$$

We deduce that

$$\begin{aligned} \sqrt{|p_\varepsilon|^2 + \varepsilon^2 \ell^2 |\nabla p_\varepsilon|^2} &\leq \sqrt{|\mathcal{F}_\varepsilon(P)|^2 + \ell^2 |\mathcal{F}_\varepsilon(\nabla_y P)|^2} + |p_\varepsilon - \mathcal{F}_\varepsilon(P)| + \ell |\varepsilon \nabla p_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y P)| \\ &\leq \mathcal{F}_\varepsilon(Z) + |p_\varepsilon - \mathcal{F}_\varepsilon(P)| + \ell |\varepsilon \nabla p_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y P)|. \end{aligned}$$

Since by Proposition 3.2

$$\begin{aligned} \mathcal{F}_\varepsilon(P) &\overset{s\text{-}2}{\to} P && \text{two-scale strongly in } L^2(\Omega \times Y; \mathrm{M}^N_D) \\ \mathcal{F}_\varepsilon(\nabla_y P) &\overset{s\text{-}2}{\to} \nabla_y P && \text{two-scale strongly in } L^2(\Omega \times Y; \mathbb{M}^N_D) \\ \mathcal{F}_\varepsilon(z_\varepsilon) &\overset{s\text{-}2}{\to} Z && \text{two-scale strongly in } L^2(\Omega \times Y), \end{aligned}$$

the result follows by choosing

$$z_\varepsilon := \mathcal{F}_\varepsilon(Z) + |p_\varepsilon - \mathcal{F}_\varepsilon(P)| + \ell |\varepsilon \nabla p_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y P)|.$$

4. A two-scale framework for the homogenization result of Fleck and Willis

The aim of this section is to show that variational arguments based on two-scale convergence provide, in a periodic setting, a rigorous mathematical framework for the homogenization result of Fleck and Willis [6] in strain gradient plasticity.

4.1. The homogenization result of Fleck and Willis.

Let us briefly describe the result of Fleck and Willis. Let $\Omega \subset \mathbb{R}^N$ be the reference configuration of an elastoplastic body subject to infinitesimal displacements. The configuration of Ω is given by a pair (u, p) where $u : \Omega \to \mathbb{R}^N$ stands for the displacement and $p : \Omega \to \mathrm{M}^N_D$ denotes the plastic strain taking values in the space of symmetric deviatoric matrices M^N_D. The elastic strain of Ω associated to the configuration (u, p) is then given by

$$e = Eu - p,$$

where Eu denotes the symmetrized gradient of u.

The elastic properties of Ω are encoded in the elasticity tensor $\mathbb{C} : \Omega \to \mathrm{Lin}(\mathrm{M}^N_{\mathrm{sym}}; \mathrm{M}^N_{\mathrm{sym}})$ which is assumed to satisfy the coercivity condition

$$\alpha |M|^2 \leq \mathbb{C}(x) M : M \leq \beta |M|^2 \tag{4.1}$$

for a.e. $x \in \Omega$ and for every $M \in \mathrm{M}^N_{\mathrm{sym}}$, where $0 < \alpha < \beta < +\infty$.

The plastic behaviour of Ω is determined by a yielding function $b : \Omega \to [0, +\infty[$ such that for a.e. $x \in \mathbb{R}^N$

$$b(x) > c > 0. \tag{4.2}$$

The strain gradient deformation theory considered by Fleck and Willis [6] amounts in the minimization of the following energy

$$\mathcal{E}(u,p) := \frac{1}{2}\int_\Omega \mathbb{C}(x)(Eu-p) : (Eu-p)\,dx + \int_\Omega b(x)[|p|^2 + \ell^2|\nabla p|^2]\,dx \tag{4.3}$$

under suitable boundary conditions and external loads. Here $\ell > 0$ denotes a *dissipative* length scale which depends on the material under consideration.

Fleck and Willis consider the case of a composite material Ω whose homogenized response under external loads and prescribed boundary conditions can be described without employing gradients of the plastic strain p. The homogenized deformation theory involves the *effective energy*

$$\mathcal{E}^{\mathit{eff}}(u,p) = \int_\Omega F^{\mathit{eff}}(Eu(x), p(x))\,dx, \tag{4.4}$$

where the *effective potential* $F^{\mathit{eff}}(\bar{A}, \bar{p})$ is given by minimizing the energy (4.3) on a representative volume element, among displacement fields u satisfying the linear boundary condition $u = \bar{A} \cdot x$ and plastic strains p whose mean is given precisely by $\bar{p}$.

In the following subsection, we provide a two-scale approach to the homogenization procedure in a periodic setting which justifies in a rigorous mathematical way the effective energy $\mathcal{E}^{\mathit{eff}}$ (see Theorem 4.3) and provides a cell problem for the energy density F^{eff} (see Theorem 4.5).

4.2. **Two-scale analysis for the homogenization result of Fleck and Willis.** Let the reference configuration $\Omega \subseteq \mathbb{R}^N$ be open, bounded and with Lipschitz boundary. In particular $|\partial\Omega| = 0$.

Let us consider the periodic setting in which the elasticity tensor and the plastic yielding function are provided by

$$\mathbb{C} \in L^\infty(\mathbb{R}^N; \mathrm{Lin}(\mathrm{M}^N_{\mathrm{sym}}; \mathrm{M}^N_{\mathrm{sym}})) \qquad \text{and} \qquad b \in L^\infty(\mathbb{R}^N)$$

such that for every $i = 1, \ldots, N$ and for a.e. $x \in \mathbb{R}^N$

$$\mathbb{C}(x + e_i) = \mathbb{C}(x) \qquad \text{and} \qquad b(x + e_i) = b(x),$$

where $\{e_i : i = 1, \ldots, N\}$ denotes the canonical basis of $\mathbb{R}^N$. We assume that the coercivity conditions (4.1) and (4.2) hold.

The form of the energy involved in (4.3) suggests the following functional framework for a configuration of Ω:

$$u \in H^1(\Omega; \mathbb{R}^N) \qquad \text{and} \qquad p \in H^1(\Omega; \mathrm{M}^N_D).$$

The homogenization procedure involves the study of the asymptotic behaviour as $\varepsilon \to 0$ of an energy of the type

$$(u,p) \mapsto \frac{1}{2}\int_\Omega \mathbb{C}\left(\frac{x}{\varepsilon}\right)(Eu-p) : (Eu-p)\,dx + \int_\Omega b\left(\frac{x}{\varepsilon}\right)[|p|^2 + \varepsilon^2\ell^2|\nabla p|^2]\,dx.$$

Here the elasticity tensor and the yielding function oscillate periodically on a scale ε. Accordingly, the dissipative length scale is given by $\varepsilon\ell$ with $\ell > 0$, so that the strain gradient effects tend to vanish in the limit.

Let us consider the functional

$$\mathcal{E}_\varepsilon : H^1(\Omega; \mathbb{R}^N) \times L^2(\Omega; \mathrm{M}^N_D) \to [0, +\infty]$$

defined as

$$\mathcal{E}_\varepsilon(u,p) := \frac{1}{2}\int_\Omega \mathbb{C}\left(\frac{x}{\varepsilon}\right)(Eu-p) : (Eu-p)\,dx + \int_\Omega b\left(\frac{x}{\varepsilon}\right)[|p|^2 + \varepsilon^2\ell^2|\nabla p|^2]\,dx$$

if $p \in H^1(\Omega; \mathrm{M}^N_D)$, and $\mathcal{E}_\varepsilon(u,p) = +\infty$ if $p \in L^2(\Omega; \mathrm{M}^N_D) \setminus H^1(\Omega; \mathrm{M}^N_D)$.

In view of the coercivity assumptions on $\mathbb{C}$ and b, the inequality

$$\mathcal{E}_\varepsilon(u_\varepsilon, p_\varepsilon) \le C$$

together with boundary conditions for u_ε entails naturally a bound for u_ε in $H^1(\Omega;\mathbb{R}^N)$ and for p_ε in $L^2(\Omega;\mathbb{M}^N_D)$. As a consequence, from a mathematical point of view, the problem of the computation of the effective energy (4.4) can be rephrased as the problem of studying the asymptotic behaviour as $\varepsilon \to 0$ with respect to the weak topology of $H^1(\Omega;\mathbb{R}^N) \times L^2(\Omega;\mathbb{M}^N_D)$ of the minimizers of $\mathcal{E}_\varepsilon$, subject to suitable external body forces and boundary conditions.

This goal will be accomplished in Theorem 4.3 by means of a preliminary analysis involving two-scale convergence arguments.

Let us consider the functional

$$\mathcal{E} : H^1(\Omega;\mathbb{R}^N) \times L^2(\Omega; H^1_{per,0}(Y;\mathbb{R}^N)) \times L^2(\Omega; H^1_{per}(Y;\mathbb{M}^N_D)) \to [0,+\infty[$$

given by

$$\mathcal{E}(u,U,P) := \frac{1}{2}\int_{\Omega\times Y} \mathbb{C}(y)(Eu + E_yU - P) : (Eu + E_yU - P)\,dxdy \\ + \int_{\Omega\times Y} b(y)[|P|^2 + \ell^2|\nabla_y P|^2]\,dxdy,$$

where E_y stands for the symmetrized gradient with respect to y.

The following result provides an asymptotic link between $\mathcal{E}_\varepsilon$ and $\mathcal{E}$.

Proposition 4.1. *The following facts hold.*

(a) *Lower estimate: if $\varepsilon_n \to 0$ and $(u_{\varepsilon_n}, p_{\varepsilon_n})_{n\in\mathbb{N}}$ is a sequence in $H^1(\Omega;\mathbb{R}^N) \times H^1(\Omega;\mathbb{M}^N_D)$ such that*

$$\begin{aligned} u_{\varepsilon_n} &\rightharpoonup u && \text{weakly in } H^1(\Omega;\mathbb{R}^N) \\ Eu_{\varepsilon_n} &\overset{w\text{-}2}{\rightharpoonup} Eu + E_yU && \text{two-scale weakly in } L^2(\Omega\times Y;\mathbb{M}^N_{\rm sym}) \\ p_{\varepsilon_n} &\overset{w\text{-}2}{\rightharpoonup} P && \text{two-scale weakly in } L^2(\Omega\times Y;\mathbb{M}^N_D) \\ \varepsilon_n\nabla p_{\varepsilon_n} &\overset{w\text{-}2}{\rightharpoonup} \nabla_y P && \text{two-scale weakly in } L^2(\Omega\times Y;\mathbb{M}^N_D), \end{aligned}$$

then

$$\mathcal{E}(u,U,P) \le \liminf_{n\to\infty} \mathcal{E}_{\varepsilon_n}(u_{\varepsilon_n}, p_{\varepsilon_n}).$$

(b) *Recovering family: for every*

$$(u,U,P) \in H^1(\Omega;\mathbb{R}^N) \times L^2(\Omega; H^1_{per,0}(Y;\mathbb{R}^N)) \times L^2(\Omega; H^1_{per}(Y;\mathbb{M}^N_D))$$

there exists a family $(u_\varepsilon, p_\varepsilon)_{\varepsilon>0}$ in $H^1(\Omega;\mathbb{R}^N) \times H^1(\Omega;\mathbb{M}^N_D)$ such that $u_\varepsilon = u$ on $\partial\Omega$ and for $\varepsilon \to 0$

$$\begin{aligned} u_\varepsilon &\rightharpoonup u && \text{weakly in } H^1(\Omega;\mathbb{R}^N) \\ Eu_\varepsilon &\overset{s\text{-}2}{\to} Eu + E_yU && \text{two-scale strongly in } L^2(\Omega\times Y;\mathbb{M}^N_{\rm sym}) \\ p_\varepsilon &\overset{s\text{-}2}{\to} P && \text{two-scale strongly in } L^2(\Omega\times Y;\mathbb{M}^N_D) \\ \varepsilon\nabla p_\varepsilon &\overset{s\text{-}2}{\to} \nabla_y P && \text{two-scale strongly in } L^2(\Omega\times Y;\mathbb{M}^N_D), \end{aligned}$$

so that in particular

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon, p_\varepsilon) = \mathcal{E}(u,U,P).$$

Proof. Point (a) follows immediately observing that

$$\mathcal{E}_{\varepsilon_n}(u_{\varepsilon_n}, p_{\varepsilon_n}) = \frac{1}{2}\int_{\mathbb{R}^N\times Y} \mathbb{C}(y)(\mathcal{T}_{\varepsilon_n}(Eu_{\varepsilon_n}) - \mathcal{T}_{\varepsilon_n}(p_{\varepsilon_n})) : (\mathcal{T}_{\varepsilon_n}(Eu_{\varepsilon_n}) - \mathcal{T}_{\varepsilon_n}(p_{\varepsilon_n}))\,dxdy \\ + \int_{\mathbb{R}^N\times Y} b(y)[|\mathcal{T}_{\varepsilon_n}(p_{\varepsilon_n})|^2 + \ell^2|\mathcal{T}_{\varepsilon_n}(\varepsilon_n\nabla p_{\varepsilon_n})|^2]\,dxdy, \tag{4.5}$$

where $\mathcal{T}_{\varepsilon_n}$ is the unfolding operator (3.1), and applying the usual lower semicontinuity for quadratic functionals under weak convergence in $L^2(\mathbb{R}^N \times Y)$.

Concerning point (b), by Theorem 3.5 there exists $p_\varepsilon \in H^1(\Omega; \mathrm{M}^N_D)$ such that for $\varepsilon \to 0$

$$p_\varepsilon \overset{s\text{-}2}{\to} P \qquad \text{two-scale strongly in } L^2(\Omega \times Y; \mathrm{M}^N_D)$$

and

$$\varepsilon \nabla p_\varepsilon \overset{s\text{-}2}{\to} \nabla_y P \qquad \text{two-scale strongly in } L^2(\Omega \times Y; \mathbb{M}^N_D).$$

Moreover, by Proposition 3.3 and Remark 3.4, there exists $u_\varepsilon \in H^1(\Omega; \mathbb{R}^N)$ with $u_\varepsilon = u$ on $\partial\Omega$ for every $\varepsilon > 0$ and such that for $\varepsilon \to 0$

$$u_\varepsilon \rightharpoonup u \qquad \text{weakly in } H^1(\Omega; \mathbb{R}^N)$$

and

$$Eu_\varepsilon \overset{s\text{-}2}{\to} Eu + E_y U \qquad \text{two-scale strongly in } L^2(\Omega \times Y; \mathrm{M}^N_{\mathrm{sym}}).$$

The convergence of the energies follows from the representation formula (4.5). $\square$

In order to move toward a single scale setting, let us introduce the functional

$$\mathcal{E}^{\mathit{eff}} : H^1(\Omega; \mathbb{R}^N) \times L^2(\Omega; \mathrm{M}^N_D) \to [0, +\infty[$$

defined by

$$\mathcal{E}^{\mathit{eff}}(u,p) := \min_{(U,P)} \Big\{ \mathcal{E}(u,U,P) : (U,P) \in L^2(\Omega; H^1_{per,0}(Y; \mathbb{R}^N)) \times L^2(\Omega; H^1_{per}(Y; \mathrm{M}^N_D)), \\ \int_Y P(x,y)\, dy = p(x) \text{ for a.e. } x \in \Omega \Big\}. \tag{4.6}$$

Notice that U is left free in the minimization, while $P(x,y)$ satisfies a constraint on the mean with respect to the *microstructural* variable y.

The minimum in the previous formula is indeed attained as is shown in the following lemma.

Lemma 4.2. *Let $(u,p) \in H^1(\Omega; \mathbb{R}^N) \times L^2(\Omega; \mathrm{M}^N_D)$. Then there exists a unique pair*

$$(U,P) \in L^2(\Omega; H^1_{per,0}(Y; \mathbb{R}^N)) \times L^2(\Omega; H^1_{per}(Y; \mathrm{M}^N_D))$$

with $\int_Y P(x,y)\, dy = p(x)$ for a.e. $x \in \Omega$ such that $\mathcal{E}^{\mathit{eff}}(u,p) = \mathcal{E}(u,U,P)$.

Proof. Let (U_n, P_n) be a minimizing sequence for problem (4.6) relative to (u,p). By comparison with the admissible pair given by $(0,p)$, we immediately get that for n large

$$\frac{1}{2} \int_{\Omega \times Y} \mathbb{C}(y)(Eu + E_y U_n - P_n) : (Eu + E_y U_n - P_n)\, dxdy + \int_{\Omega \times Y} b(y)[|P_n|^2 + \ell^2 |\nabla_y P_n|^2]\, dxdy \\ \le \mathcal{E}(u,0,p) + 1.$$

In view of the coercivity assumptions on $\mathbb{C}$ and b, up to a subsequence we have that

$$P_n \rightharpoonup P \qquad \text{weakly in } L^2(\Omega; H^1_{per}(Y; \mathrm{M}^N_D)) \tag{4.7}$$

and in view of Korn's inequality for periodic functions with zero mean

$$U_n \rightharpoonup U \qquad \text{weakly in } L^2(\Omega; H^1_{per,0}(Y; \mathbb{R}^N)),$$

for some $P \in L^2(\Omega; H^1_{per}(Y; \mathrm{M}^N_D))$ and $U \in L^2(\Omega; H^1_{per,0}(Y; \mathbb{R}^N))$. In particular we deduce by lower semicontinuity

$$\mathcal{E}(u,U,P) \le \liminf_{n\to\infty} \mathcal{E}(u,U_n,P_n).$$

Notice that $P(x,y)$ satisfies the constraint concerning the mean with respect to y. Indeed, by (4.7) and since P_n satisfies the constraint, for every $\varphi \in L^2(\Omega)$ we have

$$\int_{\Omega \times Y} P(x,y)\varphi(x)\, dxdy = \lim_{n\to\infty} \int_{\Omega \times Y} P_n(x,y)\varphi(x)\, dxdy = \int_\Omega p(x)\varphi(x)\, dx,$$

hence

$$\int_Y P(x,y)\, dy = p(x) \qquad \text{for a.e. } x \in \Omega.$$

The pair (U, P) is thus admissible for (u, p) in (4.6) and

$$\mathcal{E}^{\mathit{eff}}(u,p) \le \mathcal{E}(u,U,P) \le \liminf_{n\to\infty} \mathcal{E}(v,U_n,P_n) = \mathcal{E}^{\mathit{eff}}(u,p).$$

Then (U, P) is a solution of the minimization problem. Its uniqueness follows by the strict convexity of the functional together with Korn's inequality. □

The following theorem shows that $\mathcal{E}^{\mathit{eff}}$ is indeed the effective energy we need to understand the asymptotic behaviour of Ω when the strain gradient effects vanish. Let us assume that the boundary displacement is given by the trace on $\partial\Omega$ of a given Sobolev function $\bar{u} \in H^1(\Omega;\mathbb{R}^N)$. Moreover, let us consider body forces acting on Ω whose density per unit volume is given by a function $f \in L^2(\Omega;\mathbb{R}^N)$.

Theorem 4.3 (The homogenization result of Fleck and Willis). *For every $\varepsilon > 0$ let $(u_\varepsilon, p_\varepsilon)$ be the minimizer of*

$$(u,p) \mapsto \mathcal{E}_\varepsilon(u,p) - \int_\Omega f\cdot u\,dx$$

on $H^1(\Omega;\mathbb{R}^N)\times H^1(\Omega;\mathrm{M}^N_D)$ with $u=\bar{u}$ on $\partial\Omega$. Then for $\varepsilon\to 0$

$$u_\varepsilon \rightharpoonup u_0 \qquad \textit{weakly in } H^1(\Omega;\mathbb{R}^N)$$

and

$$p_\varepsilon \rightharpoonup p_0 \qquad \textit{weakly in } L^2(\Omega;\mathrm{M}^N_D),$$

where (u_0,p_0) is the unique minimizer of

$$(u,p) \mapsto \mathcal{E}^{\mathit{eff}}(u,p) - \int_\Omega f\cdot u\,dx \tag{4.8}$$

on $H^1(\Omega;\mathbb{R}^N)\times L^2(\Omega;\mathrm{M}^N_D)$ with $u=\bar{u}$ on $\partial\Omega$. Moreover

$$\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon,p_\varepsilon) = \mathcal{E}^{\mathit{eff}}(u_0,p_0).$$

Proof. By comparison with the admissible configuration $(\bar{u}, 0)$ and in view of the coercivity assumptions on $\mathbb{C}$ and b we get

$$\|Eu_\varepsilon - p_\varepsilon\|^2_{L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})} + \|p_\varepsilon\|^2_{L^2(\Omega;\mathrm{M}^N_D)} + \|\varepsilon\nabla p_\varepsilon\|^2_{L^2(\Omega;\mathbb{M}^N_D)} \le \tilde{C}\left(1+\int_\Omega |f||u_\varepsilon|\,dx\right),$$

where $\tilde{C} > 0$. In view of Korn's inequality we easily obtain

$$\|u_\varepsilon\|^2_{H^1(\Omega;\mathrm{M}^N_{\mathrm{sym}})} + \|p_\varepsilon\|^2_{L^2(\Omega;\mathrm{M}^N_D)} + \|\varepsilon\nabla p_\varepsilon\|^2_{L^2(\Omega;\mathbb{M}^N_D)} \le C$$

with $C > 0$. We deduce that there exists $\varepsilon_n \to 0$ such that

$$u_{\varepsilon_n} \rightharpoonup u_0 \qquad \text{weakly in } H^1(\Omega;\mathbb{R}^N)$$

and thanks to Proposition 3.3

$$Eu_{\varepsilon_n} \overset{w\text{-}2}{\rightharpoonup} Eu_0 + E_yU_0 \qquad \text{two-scale weakly in } L^2(\Omega\times Y;\mathrm{M}^N_{\mathrm{sym}})$$

for some $(u_0,U_0)\in H^1(\Omega;\mathbb{R}^N)\times L^2(\Omega;H^1_{per,0}(Y;\mathbb{R}^N))$. Moreover, in view of Theorem 3.5, we infer that there exists $P_0\in L^2(\Omega;H^1_{per}(Y;\mathrm{M}^N_D))$ such that up to a subsequence

$$p_{\varepsilon_n} \overset{w\text{-}2}{\rightharpoonup} P_0 \qquad \text{two-scale weakly in } L^2(\Omega\times Y;\mathrm{M}^N_D)$$

and

$$\varepsilon_n\nabla p_{\varepsilon_n} \overset{w\text{-}2}{\rightharpoonup} \nabla_y P_0 \qquad \text{two-scale weakly in } L^2(\Omega\times Y;\mathbb{M}^N_D).$$

The configuration (u_0,U_0,P_0) is a minimizer of the functional

$$(u,U,P) \mapsto \mathcal{E}(u,U,P) - \int_\Omega f\cdot u\,dx \tag{4.9}$$

where u satisfies $u = \bar{u}$ on $\partial\Omega$. Indeed for every admissible (v, V, Q), by Proposition 4.1 there exists a family $(v_\varepsilon, q_\varepsilon)_{\varepsilon>0}$ in $H^1(\Omega;\mathbb{R}^N) \times H^1(\Omega;\mathrm{M}^N_D)$ with $v_\varepsilon = \bar{u}$ on $\partial\Omega$ such that for $\varepsilon \to 0$

$$\begin{aligned} v_\varepsilon &\rightharpoonup v && \text{weakly in } H^1(\Omega;\mathbb{R}^N) \\ Ev_\varepsilon &\overset{s\text{-}2}{\to} Ev + E_yV && \text{two-scale strongly in } L^2(\Omega\times Y;\mathrm{M}^N_{\mathrm{sym}}) \\ q_\varepsilon &\overset{s\text{-}2}{\to} Q && \text{two-scale strongly in } L^2(\Omega\times Y;\mathrm{M}^N_D) \\ \varepsilon\nabla q_\varepsilon &\overset{s\text{-}2}{\to} \nabla_y Q && \text{two-scale strongly in } L^2(\Omega\times Y;\mathbb{M}^N_D), \end{aligned}$$

and

$$\mathcal{E}_\varepsilon(v_\varepsilon, q_\varepsilon) \to \mathcal{E}(v, V, Q).$$

Since by minimality of $(u_{\varepsilon_n}, p_{\varepsilon_n})$

$$\mathcal{E}_{\varepsilon_n}(u_{\varepsilon_n}, p_{\varepsilon_n}) - \int_\Omega f\cdot u_{\varepsilon_n}\,dx \le \mathcal{E}_{\varepsilon_n}(v_{\varepsilon_n}, q_{\varepsilon_n}) - \int_\Omega f\cdot v_{\varepsilon_n}\,dx,$$

passing to the limit we obtain in view of point (a) of Proposition 4.1

$$\begin{aligned} \mathcal{E}(u_0, U_0, P_0) - \int_\Omega f\cdot u_0\,dx &\le \liminf_{n\to\infty}\left(\mathcal{E}_{\varepsilon_n}(u_{\varepsilon_n}, p_{\varepsilon_n}) - \int_\Omega f\cdot u_{\varepsilon_n}\,dx\right) \\ &\le \limsup_{n\to\infty}\left(\mathcal{E}_{\varepsilon_n}(u_{\varepsilon_n}, p_{\varepsilon_n}) - \int_\Omega f\cdot u_{\varepsilon_n}\,dx\right) \\ &\le \limsup_{n\to\infty}\left(\mathcal{E}_{\varepsilon_n}(v_{\varepsilon_n}, q_{\varepsilon_n}) - \int_\Omega f\cdot v_{\varepsilon_n}\,dx\right) = \mathcal{E}(v, V, Q) - \int_\Omega f\cdot v\,dx. \end{aligned} \tag{4.10}$$

We infer that (u_0, U_0, P_0) is a minimizer of (4.9). Since the minimizer is unique by strict convexity, we conclude that for $\varepsilon \to 0$

$$\begin{aligned} u_\varepsilon &\rightharpoonup u_0 && \text{weakly in } H^1(\Omega;\mathbb{R}^N) \\ Eu_\varepsilon &\overset{w\text{-}2}{\rightharpoonup} Eu_0 + E_yU_0 && \text{two-scale weakly in } L^2(\Omega\times Y;\mathrm{M}^N_{\mathrm{sym}}) \\ p_\varepsilon &\overset{w\text{-}2}{\rightharpoonup} P_0 && \text{two-scale weakly in } L^2(\Omega\times Y;\mathrm{M}^N_D) \\ \varepsilon\nabla p_\varepsilon &\overset{w\text{-}2}{\rightharpoonup} \nabla_y P_0 && \text{two-scale weakly in } L^2(\Omega\times Y;\mathbb{M}^N_D), \end{aligned} \tag{4.11}$$

and thanks to (4.10) with the choice $(v, V, Q) = (u_0, U_0, P_0)$

$$\mathcal{E}_\varepsilon(u_\varepsilon, p_\varepsilon) \to \mathcal{E}(u_0, U_0, P_0). \tag{4.12}$$

Let us set for almost every $x \in \Omega$

$$p_0(x) := \int_Y P_0(x, y)\,dy.$$

Clearly we have $p_0 \in L^2(\Omega;\mathrm{M}^N_D)$. By (3.4) we get

$$p_\varepsilon \rightharpoonup p_0 \qquad \text{weakly in } L^2(\Omega;\mathrm{M}^N_D). \tag{4.13}$$

The result follows provided that we show that the pair (u_0, p_0) is the unique minimizer of (4.8) under the boundary condition $u = \bar{u}$ on $\partial\Omega$ with

$$\mathcal{E}^{\mathit{eff}}(u_0, p_0) = \mathcal{E}(u_0, U_0, P_0). \tag{4.14}$$

For every $(u, p) \in H^1(\Omega;\mathbb{R}^N) \times L^2(\Omega;\mathrm{M}^N_D)$ with $u = \bar{u}$ on $\partial\Omega$, letting (U, P) be the associated pair according to Lemma 4.2, we have

$$\begin{aligned} \mathcal{E}^{\mathit{eff}}(u_0, p_0) - \int_\Omega f\cdot u_0\,dx \le \mathcal{E}(u_0, U_0, P_0) - \int_\Omega f\cdot u_0\,dx &\le \mathcal{E}(u, U, P) - \int_\Omega f\cdot u\,dx \\ &= \mathcal{E}^{\mathit{eff}}(u, p) - \int_\Omega f\cdot u\,dx, \end{aligned}$$

so that the minimality of (u_0, p_0) follows. The uniqueness holds in view of the strict convexity of $\mathcal{E}$. Indeed, if $(\tilde{u}, \tilde{p})$ were another minimizer, and $(\tilde{U}, \tilde{P})$ the associated pair according to Lemma

4.2, in view of the preceding inequalities we would get that $(\tilde{u}, \tilde{U}, \tilde{P})$ is a minimizer of (4.9). Since $\mathcal{E}$ is strictly convex, we would infer that $(\tilde{u}, \tilde{U}, \tilde{P}) = (u_0, U_0, P_0)$ so that in particular $\tilde{u} = u_0$ and $\tilde{p} = p_0$. This entails also that (U_0, P_0) is the pair associated to (u_0, p_0) according to Lemma 4.2, so that (4.14) holds, and the proof is concluded. □

Remark 4.4. The previous theorem suggests that $\mathcal{E}_\varepsilon$ Γ-converges in the sense of De Giorgi to $\mathcal{E}^{\mathit{eff}}$ as $\varepsilon \to 0$. This is indeed the case provided that we consider the weak topology and we restrict the functionals to the pairs (u, p) such that $u = \bar{u}$ on $\partial\Omega$. In this way, the convergence of the minimizers $(u_\varepsilon, p_\varepsilon)$ along a suitable sequence $\varepsilon_n \to 0$ to a minimizer of the effective energy turns out to be a standard result of Γ-convergence. The two-scale analysis enables us to deduce that the the limit energy has a unique minimizer, so that the convergence holds indeed along the entire family.

In the rest of the section we concentrate on the representation formula (4.4) for $\mathcal{E}^{\mathit{eff}}$.

Theorem 4.5 (Representation formula for the effective energy). *For every $(u, p) \in H^1(\Omega; \mathbb{R}^N) \times L^2(\Omega; \mathrm{M}^N_D)$ we have*

$$\mathcal{E}^{\mathit{eff}}(u, p) = \int_\Omega F^{\mathit{eff}}(Eu(x), p(x))\, dx, \tag{4.15}$$

where for $(\bar{A}, \bar{p}) \in \mathrm{M}^N_{\mathrm{sym}} \times \mathrm{M}^N_D$

$$\begin{aligned} F^{\mathit{eff}}(\bar{A}, \bar{p}) := \min \Bigg\{ &\frac{1}{2} \int_Y \mathbb{C}(y)[\bar{A} + E_y U - P] : [\bar{A} + E_y U - P]\, dy \\ &+ \int_Y b(y)[|P|^2 + \ell^2 |\nabla_y P|^2]\, dy : (U, P) \in H^1_{per,0}(Y; \mathbb{R}^N) \times H^1_{per}(Y; \mathrm{M}^N_D), \int_Y P(y)\, dy = \bar{p} \Bigg\}. \end{aligned} \tag{4.16}$$

Proof. Firstly let us prove that the minimum problem in (4.16) admits indeed a unique solution. This follows by the direct method of the Calculus of Variations. If $(U_n, P_n)_{n \in \mathbb{N}}$ is a minimizing sequence for the problem, by comparison with the admissible pair $(0, \bar{p})$, taking into account the coercivity assumptions on $\mathbb{C}$ and b, one deduces easily that for n large

$$\|E_y U_n\|^2_{L^2(Y; \mathrm{M}^N_{\mathrm{sym}})} + \|P_n\|^2_{H^1(Y; \mathrm{M}^N_D)} \le C(|\bar{A}|^2 + |\bar{p}|^2),$$

where $C > 0$ is a suitable constant. By Korn's inequality we get that U_n is bounded in $H^1(Y; \mathbb{R}^N)$. Up to a subsequence we have that

$$U_n \rightharpoonup U \qquad \text{weakly in } H^1_{per,0}(Y; \mathbb{R}^N)$$

and

$$P_n \rightharpoonup P \qquad \text{weakly in } H^1_{per}(Y; \mathrm{M}^N_D).$$

Moreover, since $P_n \to P$ strongly in $L^2(Y; \mathrm{M}^N_D)$ we get that

$$\int_Y P(y)\, dy = \lim_{n \to \infty} \int_Y P_n(y)\, dy = \bar{p}.$$

We conclude that the pair (U, P) is admissible. By lower semicontinuity, we infer that (U, P) is a minimizer for the functional in (4.16) and satisfies

$$\|E_y U\|^2_{L^2(Y; \mathrm{M}^N_{\mathrm{sym}})} + \|P\|^2_{H^1(Y; \mathrm{M}^N_D)} \le C(|\bar{A}|^2 + |\bar{p}|^2). \tag{4.17}$$

The uniqueness is ensured by strict convexity.

Let us come to the representation formula (4.15). By Lemma 4.2, for every $(u, p) \in H^1(\Omega; \mathbb{R}^N) \times L^2(\Omega; \mathrm{M}^N_D)$ we have

$$\mathcal{E}^{\mathit{eff}}(u, p) = \mathcal{E}(u, U, P)$$

for some $U \in L^2(\Omega; H^1_{per,0}(Y; \mathbb{R}^N))$ and $P \in L^2(\Omega; H^1_{per}(Y; \mathrm{M}^N_D))$ such that

$$p(x) = \int_Y P(x, y)\, dy \qquad \text{for a.e. } x \in \Omega.$$

Notice that for a.e. $x \in \Omega$ the pair $(U(x,\cdot), P(x,\cdot))$ is admissible for the computation of $F^{\mathit{eff}}(Eu(x), p(x))$. By the very definition of F^{eff} we deduce that

$$\begin{aligned}(4.18)\quad & \mathcal{E}^{\mathit{eff}}(u,p) = \mathcal{E}(u,U,P) \\ &= \int_\Omega \left[\frac{1}{2}\int_Y \mathbb{C}(y)[Eu(x)+E_yU-P] : [Eu(x)+E_yU-P]\,dy + \int_Y b(y)[|P|^2+\ell^2|\nabla_y P|^2]\,dy\right] dx \\ &\geq \int_\Omega F^{\mathit{eff}}(Eu(x),p(x))\,dx.\end{aligned}$$

On the other hand, for a.e. $x \in \Omega$ let

$$(U_x, P_x)$$

be the unique solution of problem (4.16) defining $F^{\mathit{eff}}(Eu(x), p(x))$. By (4.17) we deduce that

$$\|E_yU_x\|^2_{L^2(Y;\mathrm{M}^N_{\mathrm{sym}})} + \|P_x\|^2_{H^1(Y;\mathrm{M}^N_D)} \leq C(|Eu(x)|^2 + |p(x)|^2),$$

where C does not depend on x. We infer that, the measurability with respect to x coming from the uniqueness of the minimizer,

$$U(x,y) := U_x(y) \in L^2(\Omega; H^1_{per,0}(Y;\mathbb{R}^N))$$

and

$$P(x,y) := P_x(y) \in L^2(\Omega; H^1_{per}(Y;\mathrm{M}^N_D)).$$

It follows that

$$\int_\Omega F^{\mathit{eff}}(Eu(x),p(x))\,dx = \mathcal{E}(u,U,P),$$

and by the very definition of $\mathcal{E}^{\mathit{eff}}$

$$\int_\Omega F^{\mathit{eff}}(Eu(x),p(x))\,dx = \mathcal{E}(u,U,P) \geq \mathcal{E}^{\mathit{eff}}(u,p). \tag{4.19}$$

In view of (4.18) and (4.19), the representation formula (4.15) follows. □

Let us now investigate the representation formula (4.16) in the particular case when the elastic moduli do not oscillate, i.e.,

$$\mathbb{C} \text{ is constant.}$$

The only term responsible for the homogenization is thus the yielding function b. As shown in [6], the representation formula for F^{eff} involves an operator Γ introduced by Willis in [22] which in our context can be characterized as follows.

Definition 4.6. *For every $q \in L^2(Y;\mathrm{M}^N_{\mathrm{sym}})$ let V be the unique minimizer of*

$$V \mapsto \int_Y \mathbb{C}E_yV : [E_yV - 2q]\,dy$$

on $H^1_{per,0}(Y;\mathbb{R}^N)$. We set

$$\Gamma(\mathbb{C}q) := E_yV$$

so that Γ is a well defined operator from $L^2(Y;\mathrm{M}^N_{\mathrm{sym}})$ into itself.

The effective energy density assumes the following form.

Theorem 4.7. *If $\mathbb{C}$ is constant, then for every $(\bar{A},\bar{p}) \in \mathrm{M}^N_{\mathrm{sym}} \times \mathrm{M}^N_D$ we have*

$$F^{\mathit{eff}}(\bar{A},\bar{p}) = \frac{1}{2}\mathbb{C}(\bar{A}-\bar{p}) : (\bar{A}-\bar{p}) + V^{\mathit{eff}}(\bar{p})$$

where

$$\begin{aligned}V^{\mathit{eff}}(\bar{p}) := \min\Bigg\{&\frac{1}{2}\int_Y [\mathbb{C}(P(y)-\bar{p}) : (P(y)-\bar{p}) - P(y) : \mathbb{C}\Gamma(\mathbb{C}P)(y)]\,dy \\ &+ \int_Y b(y)[|P|^2+\ell^2|\nabla P|^2\,dy : P \in H^1_{per}(Y;\mathrm{M}^N_D), \int_Y P(y)\,dy = \bar{p}\Bigg\},\end{aligned}$$

and the operator Γ is given in Definition 4.6.

Proof. If (U,P) is admissible for the computation of $F^{\mathit{eff}}(\bar{A},\bar{p})$ according to Theorem 4.5, since $\mathbb{C}$ is constant, the mean of P on Y is $\bar{p}$, and in view of an integration by parts we have

$$
\begin{aligned}
&\frac{1}{2}\int_Y \mathbb{C}[\bar{A}+E_yU-P]:[\bar{A}+E_yU-P]\,dy\\
&\qquad=\frac{1}{2}\int_Y \mathbb{C}[\bar{A}-\bar{p}+E_yU-(P-\bar{p})]:[\bar{A}-\bar{p}+E_yU-(P-\bar{p})]\,dy\\
&\qquad=\frac{1}{2}\mathbb{C}(\bar{A}-\bar{p}):(\bar{A}-\bar{p})+\int_Y \mathbb{C}(\bar{A}-\bar{p}):E_yU\,dy+\frac{1}{2}\int_Y \mathbb{C}E_yU:E_yU\,dy\\
&\qquad\quad-\int_Y \mathbb{C}E_yU:(P-\bar{p})\,dy+\frac{1}{2}\int_Y \mathbb{C}(P-\bar{p}):(P-\bar{p})\,dy\\
&\qquad=\frac{1}{2}\mathbb{C}(\bar{A}-\bar{p}):(\bar{A}-\bar{p})+\frac{1}{2}\int_Y \mathbb{C}(P-\bar{p}):(P-\bar{p})\,dy\\
&\qquad\quad+\frac{1}{2}\int_Y \mathbb{C}E_yU:[E_yU-2P]\,dy.
\end{aligned}
$$

Taking into account the representation formula (4.16) for F^{eff} we deduce

$$
\begin{aligned}
F^{\mathit{eff}}(\bar{A},\bar{p})=&\frac{1}{2}\mathbb{C}(\bar{A}-\bar{p}):(\bar{A}-\bar{p})\\
&+\min_{(U,P)}\Big[\frac{1}{2}\int_Y \mathbb{C}(P-\bar{p}):(P-\bar{p})\,dy+\frac{1}{2}\int_Y \mathbb{C}E_yU:[E_yU-2P]\,dy\\
&\qquad+\int_Y b(y)[|P|^2+\ell^2|\nabla P|^2]\,dy\Big].
\end{aligned}
$$

We take the minimum on U with P fixed: since U appears only in the second term which attains the minimum for V such that $E_yV=\Gamma(\mathbb{C}P)$, with associated value

$$-\frac{1}{2}\int_Y P:\mathbb{C}\Gamma(\mathbb{C}P)\,dy,$$

the representation formula follows. □

Remark 4.8. When $\mathbb{C}$ is constant the effective energy assumes the form

$$\mathcal{E}^{\mathit{eff}}(u,p)=\frac{1}{2}\int_\Omega \mathbb{C}(Eu(x)-p(x)):(Eu(x)-p(x))\,dx+\int_\Omega V^{\mathit{eff}}(p(x))\,dx,$$

so that it is the sum of an elastic energy and a plastic potential.

The formula suggests that the plastic potential carries an information about the dissipation involved in the plastic process. Notice that V^{eff} does not only depend on the yielding function b, but also on the elasticity tensor $\mathbb{C}$, even if this one is assumed to be constant. This implies that some qualitative properties of the plastic potential, such as growth behaviour at infinity for example, can be different in the homogenized limit.

This fact shows that some problems can occur when dealing with the homogenization of quasistatic evolutions taking the point of view of the energetic approach to rate-independent processes developed by Mielke and his school [13]. Indeed the approach is based on the analysis of deformation-theory type problems where the plastic potential has a linear growth: since the linear behaviour can be lost in the homogenized limit, the effective plastic potential cannot be interpreted as a dissipation.

We finally note that the interplay between elastic and plastic parts in the definition of V^{eff} is due to the compatibility condition between elastic and plastic strains, whose sum must be the symmetrized gradient of a displacement (in our treatment such a condition is automatically satisfied since we write $Eu-p$ for the elastic strain of u). Such a condition entails that a decoupling of the problem in elastic and plastic parts cannot be carried out.

5. Two-scale homogenization of a strain gradient flow theory with isotropic linear hardening

In this section we study the homogenization of a quasistatic evolution for the strain gradient plasticity model studied in Section 4. We consider an evolution with isotropic linear hardening, so that displacements and plastic strains can be described within the mathematical framework of Sobolev spaces introduced before: without hardening, strain localizations may take place, and plastic strains should be described within the theory of functions of bounded variation (see [8]). The model corresponds to a particular case of the one proposed by Gurtin and Anand [10], since we consider only dissipation effects associated to the gradient of the plastic strain. We employ the energetic formulation of rate independent processes due to Mielke and his school (see [13] and references therein).

5.1. **Energetic formulation of a quasistatic evolution.** Let the reference configuration of the elastoplastic body be given by $\Omega \subseteq \mathbb{R}^N$ bounded open set with Lipschitz boundary. Let $\partial_D \Omega$ be a measurable subset of $\partial\Omega$ with positive surface measure.

A configuration of Ω is given by a triple (u,p,z) with

$$u \in H^1(\Omega;\mathbb{R}^N), \qquad p \in H^1(\Omega;\mathrm{M}^N_D), \qquad z \in L^2(\Omega),$$

where u denotes the displacement, p is the associated plastic strain, and z is a hardening internal variable. Here M^N_D denotes the space of symmetric deviatoric matrices (see Section 2).

Within the small displacements and small strains approximation, let us consider the free energy

$$\mathcal{Q}(u,p,z) := \frac{1}{2}\int_\Omega \left[\mathbb{C}(x)(Eu(x)-p(x)) : (Eu(x)-p(x)) + z^2(x)\right] dx,$$

where $\mathbb{C} \in L^\infty(\Omega; \mathrm{Lin}(\mathrm{M}^N_{\mathrm{sym}};\mathrm{M}^N_{\mathrm{sym}}))$ denotes an elasticity tensor satisfying the coercivity assumption (4.1).

During the evolution, the *higher order stresses* associated to $(p,\nabla p)$ (see [10] for their definition) belong to an admissible region $\mathcal{S}_{yield}$ which becomes larger and larger thanks to the hardening process. We keep track of this fact by considering the convex conjugate of the support function of $\mathcal{S}_{yield}$ which is given by

$$\mathcal{H}(p,z) := I_{\mathcal{C}}(p,z) + \int_\Omega b(x)z(x)\,dx,$$

with $I_{\mathcal{C}}$ denoting the indicator function of the cone

$$\mathcal{C} := \left\{(p,z) \in H^1(\Omega;\mathrm{M}^N_D) \times L^2(\Omega) : \sqrt{|p(x)|^2 + \ell^2|\nabla p(x)|^2} \le z(x) \text{ for a.e. } x \in \Omega\right\},$$

where $\ell > 0$ is a dissipative length scale, and the yielding function $b \in L^\infty(\Omega)$ satisfies the coercivity assumption (4.2).

The dissipation during an evolution $t \mapsto (u(t),p(t),z(t))$ defined on $[0,T]$ relative to a subinterval $[a,b]$ is given in terms of $\mathcal{H}$ by

$$\mathcal{D}(p,z;a,b) := \sup\left\{\sum_{j=1}^k \mathcal{H}\left(p(t_j)-p(t_{j-1}), z(t_j)-z(t_{j-1})\right) \,:\, a=t_0<\cdots<t_k=b\right\}. \tag{5.1}$$

An admissible boundary displacement is given by the trace on $\partial_D\Omega$ of a function $\psi \in H^1(\Omega;\mathbb{R}^N)$. Let the family of admissible configurations of Ω relative to the boundary displacement ψ be given by

$$\mathcal{A}(\psi) := \{(u,p,z) \in H^1(\Omega;\mathbb{R}^N) \times H^1(\Omega;\mathrm{M}^N_D) \times L^2(\Omega) : u=\psi \text{ on } \partial_D\Omega,\ (p,z)\in\mathcal{C}\},$$

where the equality on $\partial_D\Omega$ is intended in the sense of traces.

Let us assume that the prescribed boundary displacements on $\partial_D\Omega$ are given by the absolutely continuous function

$$\psi : [0,T] \to H^1(\Omega;\mathbb{R}^N), \tag{5.2}$$

while body and traction forces acting on Ω are given by the absolutely continuous function

$$l : [0,T] \to (H^1(\Omega;\mathbb{R}^N))^*. \tag{5.3}$$

We will denote by $\dot{l}$ the derivative with respect to t which exists almost everywhere on $[0,T]$.

The energetic formulation of a quasistatic evolution for our model of strain gradient plasticity with isotropic linear hardening is the following.

Definition 5.1 (Energetic formulation of a quasistatic evolution). *Let $t \mapsto \psi(t)$ and $t \mapsto l(t)$ be assigned boundary displacements and external loads according to* (5.2) *and* (5.3) *respectively. A map*

$$\begin{array}{rcl} [0,T] & \to & H^1(\Omega;\mathbb{R}^N) \times H^1(\Omega;\mathrm{M}^N_D) \times L^2(\Omega) \\ t & \mapsto & (u(t),p(t),z(t)) \end{array}$$

is a quasistatic evolution if the following conditions hold for every $t \in [0,T]$.

(a) *Admissibility: $(u(t),p(t),z(t)) \in \mathcal{A}(\psi(t))$.*

(b) *Global stability: for every $(v,q,\xi) \in \mathcal{A}(\psi(t))$*

$$\mathcal{Q}(u(t),p(t),z(t)) - \langle l(t),u(t)\rangle \le \mathcal{Q}(v,q,\xi) - \langle l(t),v\rangle + \mathcal{H}(q-p(t),\xi-z(t)). \tag{5.4}$$

(c) *Energy balance: the function $t \mapsto (p(t),z(t))$ has bounded variation from $[0,T]$ to $H^1(\Omega;\mathrm{M}^N_D) \times L^2(\Omega)$ and*

$$E(t) + \mathcal{D}(p,z;0,t) = E(0) - \int_0^t \langle \dot{l}(\tau),u(\tau)\rangle\, d\tau,$$

where

$$E(t) := \mathcal{Q}(u(t),p(t),z(t)) - \langle l(t),u(t)\rangle,$$

and $\mathcal{D}(p,z;0,t)$ is defined in (5.1).

By general results concerning quasistatic evolutions (see [13] or [14]), the following result holds.

Theorem 5.2. *Let $(u_0,p_0,z_0) \in \mathcal{A}(\psi(0))$ satisfy the global stability condition* (5.4)*. Then there exists a unique quasistatic evolution $t \mapsto (u(t),p(t),z(t))$ such that $(u(0),p(0),z(0)) = (u_0,p_0,z_0)$. Moreover the maps $t \mapsto u(t)$, $t \mapsto p(t)$ and $t \mapsto z(t)$ are absolutely continuous from $[0,T]$ to $H^1(\Omega;\mathbb{R}^N)$, $H^1(\Omega;\mathrm{M}^N_D)$ and $L^2(\Omega)$ respectively.*

Remark 5.3 (Connection with the flow rule formulation). The energetic formulation of the evolution is equivalent to the ordinary one involving balance equations for the stresses and the flow rule for the plastic strains. Concerning this issue, the reader is referred to [3] for the case of ordinary plasticity and to [8] for the model of Gurtin and Anand.

Let us briefly summarize the results concerning our framework (for technical details we refer to the above mentioned papers). If the external loads are given by

$$\langle l(t),u\rangle = \int_\Omega f(t)\cdot u\, dx + \int_{\partial_N \Omega} g(t)\cdot u\, dS(x)$$

for suitable body forces $f(t)$ on Ω and traction forces $g(t)$ on $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$, the Cauchy stress tensor

$$\sigma(t) := \mathbb{C}(Eu(t) - p(t))$$

turns out to satisfy for every $t \in [0,T]$ the standard balance equation

$$\begin{cases} -\operatorname{div}\sigma(t) = f(t) & \text{in } \Omega \\ \sigma(t)\cdot n = g(t) & \text{on } \partial_N\Omega. \end{cases}$$

The higher order stresses $(T_p(t),\mathbb{K}_p(t)) \in L^2(\Omega;\mathrm{M}^N_D) \times L^2(\Omega;\mathbb{M}^N_D)$ associated to $(p(t),\nabla p(t))$ satisfy for every $t \in [0,T]$

$$(T_p(t,x),\mathbb{K}_p(t,x)) \in \mathcal{S}_{yield}(t,x) \qquad \text{for a.e. } x \in \Omega,$$

where the admissible region $\mathcal{S}_{yield}(t,x)$ is given by

$$\mathcal{S}_{yield}(t,x) := \left\{ (A,\mathbb{B}) \in \mathrm{M}^N_D \times \mathbb{M}^N_D : \sqrt{|A|^2 + \frac{1}{\ell^2}|\mathbb{B}|^2} \leq S_Y(t,x) \right\}$$

with $S_Y(t,x) := b(x) + z(t,x)$. Moreover they are related to the Cauchy stress tensor by means of the balance equation

$$\begin{cases} T_p(t) = \sigma_D(t) + \operatorname{div} \mathbb{K}_p(t) & \text{in } \Omega \\ \mathbb{K}_p(t) \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\sigma_D(t)$ denotes the deviatoric part of $\sigma(t)$. Finally, for a.e. $t \in [0,T]$ and for a.e. $x \in \Omega$ the following flow rule holds: if

$$\sqrt{|T_p(t,x)|^2 + \frac{1}{\ell^2}|\mathbb{K}_p(t,x)|^2} < S_Y(t,x),$$

then $(\dot p(t,x), \nabla \dot p(t,x)) = (0,0)$, while if

$$\sqrt{|T_p(t,x)|^2 + \frac{1}{\ell^2}|\mathbb{K}_p(t,x)|^2} = S_Y(t,x),$$

then

$$\begin{cases} \dot p(t,x) = \lambda(t,x) \dfrac{T_p(t,x)}{\sqrt{|T_p(t,x)|^2 + \frac{1}{\ell^2}|\mathbb{K}_p(t,x)|^2}} \\[2ex] \nabla \dot p(t,x) = \lambda(t,x) \dfrac{\ell^{-2}\mathbb{K}_p(t,x)}{\sqrt{|T_p(t,x)|^2 + \frac{1}{\ell^2}|\mathbb{K}_p(t,x)|^2}} \\[2ex] \dot z(t,x) = \lambda(t,x) \end{cases}$$

with $\lambda(t,x) \geq 0$.

Notice that for $\ell = 0$, the terms involving ∇p disappear, and the theory formally reduces to the usual von Mises plasticity theory: indeed we have $\mathbb{K}_p(t,x) = 0$ and $\sigma_D(t,x) = T_p(t,x)$, with

$$|\sigma_D(t,x)| \leq S_Y(t,x).$$

Moreover, plasticity develops if $\sigma_D(t,x)$ reaches the yield surface, that is if $|\sigma_D(t,x)| = S_Y(t,x)$, and in such a case

$$\begin{cases} \dot p(t,x) = \lambda(t,x) \frac{\sigma_D(t,x)}{|\sigma_D(t,x)|} \\ \dot z(t,x) = \lambda(t,x) \end{cases}$$

with $\lambda(t,x) \geq 0$.

5.2. **Homogenization of a quasistatic evolution.** In this subsection we study the asymptotic behavior of a quasistatic evolution of our strain gradient plasticity model with isotropic linear hardening, in which the elastic and plastic moduli highly oscillate in a periodic way.

Let us assume that the elasticity tensor and the plastic yielding function are provided by

$$x \mapsto \mathbb{C}\left(\frac{x}{\varepsilon}\right), \qquad x \mapsto b\left(\frac{x}{\varepsilon}\right), \tag{5.5}$$

where $\varepsilon > 0$ and

$$\mathbb{C} \in L^\infty(\mathbb{R}^N; \mathrm{Lin}(\mathrm{M}^N_{\mathrm{sym}}; \mathrm{M}^N_{\mathrm{sym}})) \qquad \text{and} \qquad b \in L^\infty(\mathbb{R}^N)$$

are such that for every $i = 1, \dots, N$ and for a.e. $x \in \mathbb{R}^N$

$$\mathbb{C}(x + e_i) = \mathbb{C}(x) \qquad \text{and} \qquad b(x + e_i) = b(x).$$

Here $\{e_i : i = 1, \dots, N\}$ denotes the canonical basis of $\mathbb{R}^N$. We assume that the coercivity conditions (4.1) and (4.2) hold almost everywhere on $\mathbb{R}^N$.

We are interested in the asymptotic behaviour of quasistatic evolutions with the choice (5.5), and, as we did in Section 4, with the dissipative length scale of the form $\varepsilon\ell$ with $\ell > 0$. Again, it is convenient to move to a two-scale setting.

A configuration of $\Omega \times Y$, where Y is the unit cell (2.1), is given by

$$(u, U, P, Z) \in H^1(\Omega;\mathbb{R}^N) \times L^2(\Omega; H^1_{per,0}(Y;\mathbb{R}^N)) \times L^2(\Omega; H^1_{per}(Y;\mathbb{R}^N)) \times L^2(\Omega \times Y).$$

The associated free energy becomes

$$\tilde{\mathcal{Q}}(u, U, P, Z) = \frac{1}{2}\int_{\Omega\times Y} \left[\mathbb{C}(y)(Eu + E_yU - P) : (Eu + E_yU - P) + Z^2(x, y)\right] dxdy,$$

while the dissipation functional assumes the form

$$\tilde{\mathcal{H}}(P, Z) := I_{\tilde{\mathcal{C}}}(P, Z) + \int_{\Omega\times Y} b(y)Z\, dxdy$$

with

$$\tilde{\mathcal{C}} := \Big\{(P, Z) \in L^2(\Omega; H^1_{per}(Y; \mathrm{M}^N_D)) \times L^2(\Omega \times Y) : \\ \sqrt{|P(x, y)|^2 + \ell^2|\nabla_y P(x, y)|^2} \le Z(x, y) \text{ for a.e. } (x, y) \in \Omega \times Y\Big\}.$$

Let $\tilde{\mathcal{D}}$ be the dissipation associated with $\tilde{\mathcal{H}}$ following the procedure defined in (5.1).

The family of admissible configurations relative to the boundary displacement ψ is given by

$$\tilde{\mathcal{A}}(\psi) := \Big\{(u, U, P, Z) \in H^1(\Omega;\mathbb{R}^N)\times L^2(\Omega; H^1_{per,0}(Y;\mathbb{R}^N))\times L^2(\Omega; H^1_{per}(Y;\mathbb{R}^N))\times L^2(\Omega\times Y) : \\ u = \psi \text{ on } \partial_D\Omega \text{ and } (P, Z) \in \tilde{\mathcal{C}}\Big\}.$$

Definition 5.4 (Two-scale quasistatic evolution). *Let $t \mapsto \psi(t)$ and $t \mapsto l(t)$ be assigned boundary displacements and external loads according to* (5.2) *and* (5.3) *respectively. A map*

$$\begin{array}{ccc} [0, T] & \to & H^1(\Omega;\mathbb{R}^N) \times L^2(\Omega; H^1_{per,0}(Y;\mathbb{R}^N)) \times L^2(\Omega; H^1_{per}(Y; \mathrm{M}^N_D)) \times L^2(\Omega \times Y) \\ t & \mapsto & (u(t), U(t), P(t), Z(t)) \end{array}$$

is a quasistatic evolution if the following conditions hold for every $t \in [0, T]$.

(a) *Admissibility: $(u(t), U(t), P(t), Z(t)) \in \tilde{\mathcal{A}}(\psi(t))$.*
(b) *Global stability: for every $(v, V, Q, \Xi) \in \tilde{\mathcal{A}}(\psi(t))$*

$$\tilde{\mathcal{Q}}(u(t), U(t), P(t), Z(t)) - \langle l(t), u(t)\rangle \le \tilde{\mathcal{Q}}(v, V, Q, \Xi) - \langle l(t), v\rangle + \tilde{\mathcal{H}}(Q - P(t), \Xi - Z(t)). \tag{5.6}$$

(c) *Energy balance: the function $t \mapsto (P(t), Z(t))$ has bounded variation from $[0, T]$ to $L^2(\Omega; H^1_{per}(Y; \mathrm{M}^N_D)) \times L^2(\Omega \times Y)$ and*

$$\tilde{E}(t) + \tilde{\mathcal{D}}(P, Z; 0, t) = \tilde{E}(0) - \int_0^t \langle \dot{l}(\tau), u(\tau)\rangle\, d\tau,$$

where

$$\tilde{E}(t) := \tilde{\mathcal{Q}}(u(t), U(t), P(t), Z(t)) - \langle l(t), u(t)\rangle.$$

The following existence result holds (see [13] or [14]).

Theorem 5.5. *Let $(u_0, U_0, P_0, Z_0) \in \tilde{\mathcal{A}}(\psi(0))$ satisfy the global stability condition* (5.6). *Then there exists a unique quasistatic evolution $t \mapsto (u(t), U(t), P(t), Z(t))$ such that*

$$(u(0), U(0), P(0), Z(0)) = (u_0, U_0, P_0, Z_0).$$

Moreover the maps $t \mapsto u(t)$, $t \mapsto U(t)$, $t \mapsto P(t)$ and $t \mapsto Z(t)$ are absolutely continuous from $[0, T]$ to $H^1(\Omega;\mathbb{R}^N)$, $L^2(\Omega; H^1_{per,0}(Y; \mathrm{M}^N_D))$, $L^2(\Omega; H^1_{per}(Y; \mathrm{M}^N_D))$ and $L^2(\Omega \times Y)$ respectively.

Coming back to our model with the choices (5.5) for the elasticity tensor and the yielding function, with dissipative length scale $\varepsilon\ell$, let us denote by $\mathcal{Q}_\varepsilon$, $\mathcal{H}_\varepsilon$, $\mathcal{D}_\varepsilon$, E_ε the associated free energy, dissipation functionals and total energy respectively. Moreover $\mathcal{C}_\varepsilon$ will denote the cone associated to $\mathcal{H}_\varepsilon$, and $\mathcal{A}_\varepsilon(\psi)$ the family of admissible configurations relative to ψ.

For every $\varepsilon > 0$ let
$$(u^0_\varepsilon, p^0_\varepsilon, z^0_\varepsilon) \in \mathcal{A}_\varepsilon(\psi(0))$$
be globally stable initial configurations such that
$$\begin{cases} u^0_\varepsilon \rightharpoonup u_0 & \text{weakly in } H^1(\Omega;\mathbb{R}^N) \\ Eu^0_\varepsilon \overset{w\text{-}2}{\rightharpoonup} Eu_0 + E_y U_0 & \text{two-scale weakly in } L^2(\Omega\times Y;\mathrm{M}^N_{\mathrm{sym}}) \\ p^0_\varepsilon \overset{w\text{-}2}{\rightharpoonup} P_0 & \text{two-scale weakly in } L^2(\Omega\times Y;\mathrm{M}^N_D) \\ \varepsilon\nabla p^0_\varepsilon \overset{w\text{-}2}{\rightharpoonup} \nabla_y P_0 & \text{two-scale weakly in } L^2(\Omega\times Y;\mathbb{M}^N_D) \\ z^\varepsilon_0 \overset{w\text{-}2}{\rightharpoonup} Z_0 & \text{two-scale weakly in } L^2(\Omega\times Y) \end{cases} \tag{5.7}$$
for some
$$(u_0, U_0, P_0, Z_0) \in H^1(\Omega;\mathbb{R}^N)\times L^2(\Omega; H^1_{per,0}(Y;\mathbb{R}^N))\times L^2(\Omega; H^1_{per}(Y;\mathbb{R}^N))\times L^2(\Omega\times Y).$$

Lemma 5.6. *The configuration (u_0, U_0, P_0, Z_0) is admissible for $\psi(0)$ and globally stable according to* (5.6).

Proof. The condition $u_0 = \psi(0)$ on $\partial_D\Omega$ comes from the strong convergence for the traces of u^0_ε. The admissibility thus follows if we prove that $(P_0, Z_0) \in \tilde{\mathcal{C}}$. This comes from the inclusion
$$(\mathcal{T}_\varepsilon(p^0_\varepsilon), \mathcal{T}_\varepsilon(z^0_\varepsilon)) \in \tilde{\mathcal{C}},$$
$\mathcal{T}_\varepsilon$ being the unfolding operator (3.1), together with the fact that the convex cone $\tilde{\mathcal{C}}$ is weakly closed.

Let us prove the global stability condition. Given $(v, V, Q, \Xi) \in \tilde{\mathcal{A}}(\psi(0))$, we want to show that
$$\tilde{\mathcal{Q}}(u_0, U_0, P_0, Z_0) - \langle l(0), u_0\rangle \le \tilde{\mathcal{Q}}(v, V, Q, \Xi) - \langle l(0), v\rangle + \tilde{\mathcal{H}}(Q - P_0, \Xi - Z_0). \tag{5.8}$$
We may assume that $\tilde{\mathcal{H}}(Q - P_0, \Xi - Z_0) < +\infty$, so that
$$\sqrt{|Q - P_0|^2 + \ell^2|\nabla_y Q - \nabla_y P_0|^2} \le \Xi - Z_0 \qquad \text{a.e. in } \Omega\times Y.$$
In view of Remark 3.7, we can find $q_\varepsilon \in H^1(\Omega;\mathrm{M}^N_D)$ and $\xi_\varepsilon \in L^2(\Omega)$ such that for $\varepsilon \to 0$
$$q_\varepsilon \overset{s\text{-}2}{\to} Q - P_0 \qquad \text{two-scale strongly in } L^2(\Omega\times Y;\mathrm{M}^N_D)$$
$$\varepsilon\nabla q_\varepsilon \overset{s\text{-}2}{\to} \nabla_y Q - \nabla_y P_0 \qquad \text{two-scale strongly in } L^2(\Omega\times Y;\mathbb{M}^N_D)$$
$$\xi_\varepsilon \overset{s\text{-}2}{\to} \Xi - Z_0 \qquad \text{two-scale strongly in } L^2(\Omega\times Y)$$
and
$$\sqrt{|q_\varepsilon|^2 + \varepsilon^2\ell^2|\nabla q_\varepsilon|^2} \le \xi_\varepsilon \qquad \text{a.e. in } \Omega.$$
This implies (using Proposition 3.2, point (7)) that
$$\lim_{\varepsilon\to 0} \mathcal{H}_\varepsilon(q_\varepsilon, \xi_\varepsilon) = \tilde{\mathcal{H}}(Q - P_0, \Xi - Z_0).$$
By Proposition 3.3 and Remark 3.4, we can find $v_\varepsilon \in H^1(\Omega;\mathbb{R}^N)$ such that for $\varepsilon \to 0$
$$v_\varepsilon \rightharpoonup v - u_0 \qquad \text{weakly in } H^1(\Omega;\mathbb{R}^N),$$
$v_\varepsilon = v - u_0$ on $\partial\Omega$, and
$$Ev_\varepsilon \overset{s\text{-}2}{\to} Ev - Eu_0 + E_y V - E_y U_0 \qquad \text{two-scale strongly in } L^2(\Omega\times Y;\mathrm{M}^N_{\mathrm{sym}}).$$
By comparing $(u^0_\varepsilon, p^0_\varepsilon, z^0_\varepsilon)$ with $(v_\varepsilon + u^0_\varepsilon, q_\varepsilon + p^0_\varepsilon, \xi_\varepsilon + z^0_\varepsilon) \in \mathcal{A}_\varepsilon(\psi(0))$ we get
$$\mathcal{Q}_\varepsilon(u^0_\varepsilon, p^0_\varepsilon, z^0_\varepsilon) - \langle l(0), u^0_\varepsilon\rangle \le \mathcal{Q}_\varepsilon(v_\varepsilon + u^0_\varepsilon, q_\varepsilon + p^0_\varepsilon, \xi_\varepsilon + z^0_\varepsilon) - \langle l(0), v_\varepsilon + u^0_\varepsilon\rangle + \mathcal{H}_\varepsilon(q_\varepsilon, \xi_\varepsilon).$$

Expanding the terms of the free energy and erasing the quadratic terms involving p_ε^0 and z_ε^0 we obtain

$$0 \le \frac{1}{2}\int_\Omega \mathbb{C}\left(\frac{x}{\varepsilon}\right)[Ev_\varepsilon - q_\varepsilon] : [Ev_\varepsilon - q_\varepsilon]\,dx + \int_\Omega \mathbb{C}\left(\frac{x}{\varepsilon}\right)[Ev_\varepsilon - q_\varepsilon] : [Eu_\varepsilon^0 - p_\varepsilon^0]\,dx \\ + \frac{1}{2}\int_\Omega \xi_\varepsilon^2\,dx + \int_\Omega \xi_\varepsilon z_\varepsilon^0\,dx - \langle l(0), v_\varepsilon\rangle + \mathcal{H}_\varepsilon(q_\varepsilon, \xi_\varepsilon).$$

Letting $\varepsilon \to 0$ we obtain (using Proposition 3.2, points (6) and (7))

$$0 \le \frac{1}{2}\int_{\Omega\times Y} \mathbb{C}(y)[Ev - Eu_0 + E_yV - E_yU_0 - Q + P_0] : [Ev - Eu_0 + E_yV - E_yU_0 - Q + P_0]\,dxdy \\ + \int_{\Omega\times Y} \mathbb{C}(y)[Ev - Eu_0 + E_yV - E_yU_0 - Q + P_0] : [Eu_0 + E_yU_0 - P_0]\,dxdy \\ + \frac{1}{2}\int_{\Omega\times Y} (\Xi - Z_0)^2\,dxdy + \int_{\Omega\times Y} (\Xi - Z_0)Z_0\,dxdy \\ - \langle l(0), v - u_0\rangle + \tilde{\mathcal{H}}(Q - P_0, \Xi - Z_0),$$

so that, adding to both sides $\tilde{\mathcal{Q}}(u_0, U_0, P_0, Z_0)$ we get precisely the global stability (5.8). □

We assume moreover that

$$\lim_{\varepsilon\to 0}\left(\mathcal{Q}_\varepsilon(u_\varepsilon^0, p_\varepsilon^0, z_\varepsilon^0) - \langle l(0), u_\varepsilon^0\rangle\right) = \tilde{\mathcal{Q}}(u_0, U_0, P_0, Z_0) - \langle l(0), u_0\rangle. \tag{5.9}$$

Remark 5.7. Notice that the case of purely elastic initial configurations fulfill the global stability condition and our assumptions (5.7) and (5.9).

More precisely, let $u_\varepsilon^0 \in H^1(\Omega;\mathbb{R}^N)$ be the elastic configuration associated to the boundary displacement $\psi(0)$ and the external load $l(0)$, such that the associated Cauchy stress σ_ε^0 satisfies

$$|(\sigma_\varepsilon^0)_D| \le b\left(\frac{x}{\varepsilon}\right) \qquad \text{for a.e. } x \in \Omega \tag{5.10}$$

for every $\varepsilon > 0$. Then the initial configuration

$$(u_\varepsilon^0, 0, 0)$$

satisfies (5.7) and (5.9) with respect to $(u_0, U_0, 0, 0)$, for suitable

$$u_0 \in H^1(\Omega;\mathbb{R}^N) \qquad \text{and} \qquad U_0 \in L^2(\Omega; H^1_{per,0}(Y;\mathbb{R}^N)).$$

The global stability condition follows since $(u_\varepsilon^0, 0, 0)$ is the minimizer of the convex function

$$(u, p, z) \mapsto \mathcal{Q}_\varepsilon(u, p, z) - \langle l(0), u\rangle + \mathcal{H}_\varepsilon(p, z).$$

The minimality is consequence of (5.10) which entails

$$-\partial Q_\varepsilon(u_\varepsilon^0, 0, 0) + l(0) \in \mathcal{C}_\varepsilon.$$

Thanks to Lemma 5.6, $(u_0, U_0, P_0, Z_0) \in \tilde{\mathcal{A}}(\psi(0))$ is a globally stable configuration, so that the associated two-scale quasistatic evolution is well defined. We are now in a position to state the main result of the section.

Theorem 5.8 (**Asymptotic behaviour of a quasistatic evolution**). *Let*

$$t \mapsto (u_\varepsilon(t), p_\varepsilon(t), z_\varepsilon(t))$$

be the quasistatic evolution with initial configuration $(u_\varepsilon^0, p_\varepsilon^0, z_\varepsilon^0)$ *satisfying* (5.7) *and* (5.9). *Let*

$$t \mapsto (u(t), U(t), P(t), Z(t))$$

be the two-scale quasistatic evolution with initial configuration (u_0, U_0, P_0, Z_0).

Then for every $t \in [0,T]$

$$\begin{aligned} u_\varepsilon(t) &\rightharpoonup u(t) && \textit{weakly in } H^1(\Omega;\mathbb{R}^N) \\ Eu_\varepsilon(t) &\overset{w\text{-}2}{\rightharpoonup} Eu(t) + E_yU(t) && \textit{two-scale weakly in } L^2(\Omega\times Y;\mathrm{M}^N_{\mathrm{sym}}) \\ p_\varepsilon(t) &\overset{w\text{-}2}{\rightharpoonup} P(t) && \textit{two-scale weakly in } L^2(\Omega\times Y;\mathrm{M}^N_D) \\ \varepsilon\nabla p_\varepsilon(t) &\overset{w\text{-}2}{\rightharpoonup} \nabla_y P(t) && \textit{two-scale weakly in } L^2(\Omega\times Y;\mathbb{M}^N_D) \\ z_\varepsilon(t) &\overset{s\text{-}2}{\to} Z(t) && \textit{two-scale strongly in } L^2(\Omega\times Y). \end{aligned}$$

Finally, concerning the elastic strain we have for every $t \in [0,T]$

$$Eu_\varepsilon(t) - p_\varepsilon(t) \overset{s\text{-}2}{\to} Eu(t) + E_yU(t) - P(t) \qquad \textit{two-scale strongly in } L^2(\Omega\times Y;\mathrm{M}^N_{\mathrm{sym}}).$$

Proof. We divide the proof in several steps.

Step 1: Compactness for the plastic strain and the hardening variable. From the energy balance

$$E_\varepsilon(t) + \mathcal{D}_\varepsilon(p_\varepsilon, z_\varepsilon; 0, t) = E_\varepsilon(0) - \int_0^t \langle \dot{l}(\tau), u_\varepsilon(\tau)\rangle\, d\tau \tag{5.11}$$

and recalling that by (5.9)

$$E_\varepsilon(0) = \mathcal{Q}_\varepsilon(u^0_\varepsilon, p^0_\varepsilon, z^0_\varepsilon) - \langle l(0), u^0_\varepsilon\rangle \to \tilde{\mathcal{Q}}(u_0, U_0, P_0, Z_0) - \langle l(0), u_0\rangle \qquad \text{as } \varepsilon \to 0,$$

in view of the coercivity assumptions for the elastic and plastic moduli we obtain for every $t \in [0,T]$ and for $\varepsilon > 0$ small enough

$$\|Eu_\varepsilon(t) - p_\varepsilon(t)\|^2_{L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})} + \|z_\varepsilon(t)\|^2_{L^2(\Omega)} \le C(1 + \max_{\tau\in[0,t]} \|u_\varepsilon(\tau)\|_{H^1(\Omega;\mathbb{R}^N)})$$

where $C > 0$ is a suitable constant independent of ε and t. Since $(u_\varepsilon(t), p_\varepsilon(t), z_\varepsilon(t)) \in \mathcal{A}_\varepsilon(\psi(t))$ we get up to changing C

$$\|Eu_\varepsilon(t)\|^2_{L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})} + \|z_\varepsilon(t)\|^2_{L^2(\Omega)} \le C(1 + \max_{\tau\in[0,t]} \|u_\varepsilon(\tau)\|_{H^1(\Omega;\mathbb{R}^N)})$$

so that we infer in view of Korn's inequality

$$\max_{t\in[0,T]} \|Eu_\varepsilon(t)\|^2_{L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})} \le C(1 + \max_{t\in[0,T]} \|Eu_\varepsilon(t)\|_{L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})}).$$

This entails, by (5.11) and using again Korn's inequality, that for ε small enough the quantity

$$E_\varepsilon(t) + \mathcal{D}_\varepsilon(p_\varepsilon, z_\varepsilon; 0, t)$$

is uniformly bounded for $t \in [0,T]$.

Taking into account the definition of $\mathcal{D}_\varepsilon(p_\varepsilon, z_\varepsilon; 0, t)$, and using the coercivity for the yielding function b, we infer that the total variation of

$$t \mapsto z_\varepsilon(t)$$

from $[0,T]$ to $L^2(\Omega)$ is uniformly bounded for ε small. By admissibility of the configurations, we deduce also that the total variation of

$$t \mapsto (p_\varepsilon(t), \varepsilon\nabla p_\varepsilon(t))$$

from $[0,T]$ to $L^2(\Omega;\mathrm{M}^N_D) \times L^2(\Omega;\mathbb{M}^N_D)$ is uniformly bounded for ε small.

From the bound on $E_\varepsilon(t)$, using again Korn's inequality and the admissibility of the configurations, we infer that there exists $\tilde{C} > 0$ such that for ε small enough and for every $t \in [0,T]$

$$\|u_\varepsilon(t)\|_{H^1(\Omega;\mathbb{R}^N)} + \|p_\varepsilon(t)\|_{L^2(\Omega;\mathrm{M}^N_D)} + \|\varepsilon\nabla p_\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^N_D)} + \|z_\varepsilon(t)\|_{L^2(\Omega)} \le \tilde{C}. \tag{5.12}$$

Since the unfolding operator $\mathcal{T}_\varepsilon$ is an isometry, we deduce that the total variation of

$$t \mapsto (\mathcal{T}_\varepsilon(p_\varepsilon(t)), \mathcal{T}_\varepsilon(\varepsilon\nabla p_\varepsilon(t)), \mathcal{T}_\varepsilon(z_\varepsilon(t)))$$

on $[0,T]$ with values in $L^2(\Omega\times Y;\mathrm{M}^N_D)\times L^2(\Omega\times Y;\mathbb{M}^N_D)\times L^2(\Omega\times Y)$ is uniformly bounded for ε small. By the generalized version of Helly's theorem [3, Lemma 7.2], and in view of Theorem 3.5, we deduce that there exist a function of bounded variation

$$t\mapsto (P(t),Z(t))\in H^1_{per}(Y;\mathrm{M}^N_D)\times L^2(\Omega\times Y)$$

and a sequence $\varepsilon_n\to 0$ such that setting

$$(u_n(t),p_n(t),z_n(t)):=(u_{\varepsilon_n}(t),p_{\varepsilon_n}(t),z_{\varepsilon_n}(t)),$$

for every $t\in[0,T]$

$$p_n(t)\xrightharpoonup{w\text{-}2} P(t)\qquad \text{two-scale weakly in } L^2(\Omega\times Y;\mathrm{M}^N_D), \tag{5.13}$$

$$\varepsilon_n p_n(t)\xrightharpoonup{w\text{-}2} \nabla_y P(t)\qquad \text{two-scale weakly in } L^2(\Omega\times Y;\mathbb{M}^N_D), \tag{5.14}$$

and

$$z_n(t)\xrightharpoonup{w\text{-}2} Z(t)\qquad \text{two-scale weakly in } L^2(\Omega\times Y). \tag{5.15}$$

Notice that from the admissibility of $(u_\varepsilon(t),p_\varepsilon(t),z_\varepsilon(t))$ we infer that $(P(t),Z(t))\in\tilde{\mathcal{C}}$ for every $t\in[0,T]$.

Step 2: Compactness for the displacement. Let us fix $t\in[0,T]$. In view of (5.12) and Proposition 3.3, up to a further subsequence we have that

$$u_n(t)\rightharpoonup \tilde u\qquad \text{weakly in } H^1(\Omega;\mathbb{R}^N)$$

$$Eu_n(t)\xrightharpoonup{w\text{-}2} E\tilde u+E_y\tilde U\qquad \text{two-scale weakly in } L^2(\Omega\times Y;\mathrm{M}^N_{\mathrm{sym}})$$

for some

$$\tilde u\in H^1(\Omega;\mathbb{R}^N)\qquad\text{and}\qquad \tilde U\in L^2(\Omega;H^1_{per,0}(Y;\mathbb{R}^N)).$$

Clearly, $\tilde u=\psi(t)$ on $\partial_D\Omega$, so that $(\tilde u,\tilde U,P(t),Z(t))\in\tilde{\mathcal{A}}(\psi(t))$.

We claim that the pair $(\tilde u,\tilde U)$ is uniquely determined. Indeed, let $(v,V)\in H^1(\Omega;\mathbb{R}^N)\times L^2(\Omega;H^1_{per,0}(Y;\mathbb{R}^N))$, and $v_n\in H^1(\Omega;\mathbb{R}^N)$ such that, according to Remark 3.4, $v_n=v-\tilde u$ on $\partial\Omega$,

$$v_n\rightharpoonup v-\tilde u\qquad \text{weakly in } H^1(\Omega;\mathbb{R}^N)$$

and

$$Ev_n\xrightarrow{s\text{-}2} Ev-E\tilde u+E_yV-E_y\tilde U\qquad \text{two-scale strongly in } L^2(\Omega\times Y;\mathrm{M}^N_{\mathrm{sym}}).$$

The global stability of $(u_n(t),p_n(t),z_n(t))$ yields by comparison with $(u_n(t)+v_n,p_n(t),z_n(t))\in\mathcal{A}_{\varepsilon_n}(\psi(t))$

$$\mathcal{Q}_{\varepsilon_n}(u_n(t),p_n(t),z_n(t))-\langle l(t),u_n(t)\rangle\le \mathcal{Q}_{\varepsilon_n}(u_n(t)+v_n,p_n(t),z_n(t))-\langle l(t),u_n(t)+v_n\rangle. \tag{5.16}$$

Since

$$\begin{aligned}\mathcal{Q}_{\varepsilon_n}(u_n(t)+v_n,p_n(t),z_n(t))&=\mathcal{Q}_{\varepsilon_n}(u_n(t),p_n(t),z_n(t))\\&\quad+\frac12\int_\Omega \mathbb{C}\left(\frac{x}{\varepsilon_n}\right)Ev_n:Ev_n\,dx+\int_\Omega \mathbb{C}\left(\frac{x}{\varepsilon_n}\right)Ev_n:(Eu_n(t)-p_n(t))\,dx,\end{aligned}$$

taking the limit in (5.16) (using Proposition 3.2, points (6) and (7)) we get

$$\begin{aligned}0\le\frac12\int_{\Omega\times Y}&\mathbb{C}(y)(Ev-E\tilde u+E_yV-E_y\tilde U):(Ev-E\tilde u+E_yV-E_y\tilde U)\,dxdy\\&+\int_{\Omega\times Y}\mathbb{C}(y)(Ev-E\tilde u+E_yV-E_y\tilde U):(E\tilde u+E_y\tilde U-P(t))\,dxdy-\langle l(t),v-\tilde u\rangle.\end{aligned}$$

Adding to both sides the quantity

$$\frac12\int_{\Omega\times Y}\mathbb{C}(y)(E\tilde u+E_y\tilde U-P(t)):(E\tilde u+E_y\tilde U-P(t))\,dxdy-\langle l(t),\tilde u\rangle,$$

we deduce that the pair $(\tilde{u}, \tilde{U})$ is a minimizer (under the boundary condition for the displacement) of the map

$$(v, V) \mapsto \frac{1}{2} \int_{\Omega \times Y} \mathbb{C}\,(y)\,(Ev + E_y V) : (Ev + E_y V)\,dxdy - \int_{\Omega \times Y} \mathbb{C}\,(y)\,(Ev + E_y V) : P(t)\,dxdy - \langle l(t), v\rangle.$$

By strict convexity we conclude that $(\tilde{u}, \tilde{U})$ is uniquely determined, so that we denote it by $(u(t), U(t))$. We infer that (without passing to a subsequence since the limit point is uniquely determined)

$$u_n(t) \rightharpoonup u(t) \qquad \text{weakly in } H^1(\Omega; \mathbb{R}^N) \tag{5.17}$$

and

$$Eu_n(t) \stackrel{w\text{-}2}{\rightharpoonup} Eu(t) + E_y U(t) \qquad \text{two-scale weakly in } L^2(\Omega \times Y; \mathrm{M}^N_{\mathrm{sym}}). \tag{5.18}$$

Step 3: The limit trajectory is a quasistatic evolution. Let us prove that the limit trajectory

$$t \mapsto (u(t), U(t), P(t), Z(t)) \in \tilde{\mathcal{A}}(\psi(t))$$

given by the previous steps satisfies the global stability and the energy balance of Definition 5.4.

Global stability follows by the same arguments of Lemma 5.6 by replacing $(u^0_\varepsilon, p^0_\varepsilon, z^0_\varepsilon)$ with $(u_n(t), p_n(t), z_n(t))$, and (u_0, U_0, P_0, Z_0) with $(u(t), U(t), P(t), Z(t))$.

Concerning the energy balance, let us write $\mathcal{Q}_n$ and $\mathcal{D}_n$ for $\mathcal{Q}_{\varepsilon_n}$ and $\mathcal{D}_{\varepsilon_n}$ respectively. Since

$$\mathcal{Q}_n(u_n(t), p_n(t), z_n(t)) = \frac{1}{2} \int_{\Omega \times Y} \mathbb{C}(y)(\mathcal{T}_{\varepsilon_n}(Eu_n(t)) - \mathcal{T}_{\varepsilon_n}(P_n(t))) : (\mathcal{T}_{\varepsilon_n}(Eu_n(t)) - \mathcal{T}_{\varepsilon_n}(P_n(t))) + |\mathcal{T}_{\varepsilon_n}(z_n(t))|^2\,dxdy$$

and

$$\mathcal{D}_n(p_n, z_n; 0, t) = \tilde{\mathcal{D}}(\mathcal{T}_{\varepsilon_n}(p_n), \mathcal{T}_{\varepsilon_n}(z_n); 0, t),$$

we obtain for every $t \in [0, T]$

$$\tilde{\mathcal{Q}}(u(t), U(t), P(t), Z(t)) \le \liminf_{n\to\infty} \mathcal{Q}_n(u_n(t), p_n(t), z_n(t))$$

and (since $\tilde{\mathcal{D}}$ is a sort of total variation in time)

$$\tilde{\mathcal{D}}(P, Z; 0, t) \le \liminf_{n\to\infty} \mathcal{D}_n(p_n, z_n; 0, t).$$

Taking the limit for $n \to \infty$ in

$$E_n(t) + \mathcal{D}_n(p_n, z_n; 0, t) = E_n(0) - \int_0^t \langle \dot{l}(\tau), u_n(\tau)\rangle\,d\tau,$$

in view of (5.9) we deduce that

$$\tilde{E}(t) + \tilde{\mathcal{D}}(P, Z; 0, t) \le \lim_{n\to\infty} [E_n(t) + \mathcal{D}_n(p_n, z_n; 0, t)] = \tilde{E}(0) - \int_0^t \langle \dot{l}(\tau), u(\tau)\rangle\,d\tau. \tag{5.19}$$

On the other hand, the global stability implies that for every $t \in [0, T]$ (see for example [12, Theorem 4.4])

$$\tilde{E}(t) + \tilde{\mathcal{D}}(P, Z; 0, t) \ge \tilde{E}(0) - \int_0^t \langle \dot{l}(\tau), u(\tau)\rangle\,d\tau$$

so that the energy balance condition holds. The map $t \mapsto (u(t), U(t), P(t), Z(t))$ is thus a quasistatic evolution with initial configuration (u_0, U_0, P_0, Z_0). Since the evolution is uniquely determined, we conclude that the convergences (5.13)-(5.18) hold indeed along the entire family for $\varepsilon \to 0$.

Finally from (5.19) (which is indeed an equality) we infer that for every $t \in [0, T]$

$$\lim_{\varepsilon\to 0} E_\varepsilon(t) = \tilde{E}(t) \qquad \text{and} \qquad \lim_{\varepsilon\to 0} \mathcal{D}_\varepsilon(p_\varepsilon, z_\varepsilon; 0, t) = \tilde{\mathcal{D}}(P, Z; 0, t).$$

This entails that for every $t \in [0,T]$

$$Eu_\varepsilon(t) - p_\varepsilon(t) \overset{s\text{-}2}{\to} Eu(t) + E_yU(t) - P(t) \qquad \text{two-scale strongly in } L^2(\Omega \times Y; \mathrm{M}^N_{\mathrm{sym}})$$

and

$$z_\varepsilon(t) \overset{s\text{-}2}{\to} Z(t) \qquad \text{two-scale strongly in } L^2(\Omega \times Y).$$

The proof is thus concluded. □

Remark 5.9. Reformulating the two-scale quasistatic evolution $t \mapsto (u(t), U(t), P(t), Z(t))$ in a single scale setting demands for an integration with respect to the microstructural variable y, so that usual weak limits $\hat{P}(t)$ and $\hat{Z}(t)$ of $p_\varepsilon(t)$ and $z_\varepsilon(t)$ are obtained (the displacement $u(t)$ is already the weak limit of $u_\varepsilon(t)$). Unfortunately, in view of the nonlinearities appearing in the global stability and the energy balance conditions, the mean with respect to y cannot be performed preserving the structure of the two properties. In other words, it is not clear if the evolution $t \mapsto (u(t), \hat{P}(t), \hat{Z}(t))$ can be interpreted as a quasistatic evolution for a homogenized standard plasticity model. A hint regarding such a difficulty was given by the analysis of the cell problem of Fleck and Willis considered in Theorem 4.7 (see also Remark 4.8), where the effective plastic potential in a single scale setting depends also on the elastic behavior of the material.

The loss of information entailed by taking the mean with respect to the microstructural variable y could require a description of the evolution in terms of *nonlocal* properties, such as memory effects, as pointed out by Tartar [17, 18]. For example, in the case of linear thermoviscoelasticity Francfort and Suquet [7] showed that homogenization can induce memory effects of fading type.

Recently Visintin [19, 20, 21] dealt with the problem of formulating a single scale description for the two-scale homogenization of nonlinear problems arising in viscoelasticity and elastoplasticity. His arguments are of a variational nature, and are nonlocal in time. The case of elastoplasticity presents technical difficulties due to the linear growth of the dissipation, so that the regularity in time which can be used in the minimum problems is only that of function of bounded variation (and not Sobolev regularity as for other problems in viscoelasticity). Concerning our problem of strain gradient plasticity, Visintin's ideas amount, loosely speaking, in manipulating the energy balance by taking the minimum of the left-hand side along trajectories $t \mapsto (v(t), V(t), Q(t), \Xi(t))$ such that $\hat{v}(t) = u(t)$, $\hat{Q}(t) = \hat{P}(t)$ and $\hat{\Xi}(t) = \hat{Z}(t)$ (the average with respect to y provide the weak limits of the evolution), and which satisfy the global stability condition. Unfortunately, such a formulation seems not to provide any further physical insight into the problem.

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