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On the regularity of solutions in the Pucci–Serrin identity

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Abstract. We extend a celebrated identity by P. Pucci and J. Serrin, concerning C^2 solutions of Euler equations of functionals of the calculus of variations, to the case of C^1 solutions under the only additional assumption of strict convexity in the gradient. Some particular cases in which the mere convexity is sufficient are also considered.

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1. Introduction and main result

Let Ω be a bounded open subset of $\mathbb{R}^n$ with boundary of class C^1 and outer normal ν. Assume that $\mathcal{L}(x, s, \xi)$ is a real function of class C^1 defined on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ and $f(x)$ a continuous real function defined on $\overline{\Omega}$.

Let us consider the problem

$$\begin{cases} -\mathrm{div}\{\nabla_\xi \mathcal{L}(x, u, \nabla u)\} + D_s\mathcal{L}(x, u, \nabla u) = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{$\mathcal{P}$}$$

and recall the celebrated identity proved by Pucci and Serrin [7].

Theorem 1. *Assume that the vector valued function* $\nabla_\xi \mathcal{L}$ *is of class* C^1 *on* $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ *and that* $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ *is a solution of* $(\mathcal{P})$.

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Then

$$\int_{\partial\Omega}\big[\mathcal{L}(x,0,\nabla u)-\nabla_\xi\mathcal{L}(x,0,\nabla u)\cdot\nabla u\big](h\cdot\nu)\,d\mathcal{H}^{n-1}$$
$$=\int_\Omega\big[(\operatorname{div}h)\,\mathcal{L}(x,u,\nabla u)+h\cdot\nabla_x\mathcal{L}(x,u,\nabla u)\big]\,dx$$
$$-\sum_{i,j=1}^{n}\int_\Omega\big[D_juD_ih_j+uD_ia\big]D_{\xi_i}\mathcal{L}(x,u,\nabla u)\,dx \tag{1}$$
$$-\int_\Omega a\big[\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla u+uD_s\mathcal{L}(x,u,\nabla u)\big]\,dx$$
$$+\int_\Omega\big[h\cdot\nabla u+au\big]f\,dx$$

for each $a\in C^1(\overline{\Omega})$ *and* $h\in C^1(\overline{\Omega};\mathbb{R}^n)$.

Theorem 1 generalizes a well-known identity of Pohožaev [6], which has turned out to be a powerful tool in proving non-existence of solutions for problem $(\mathcal{P})$. On the other hand, in some cases the requirement that u is of class $C^2(\Omega)$ seems too restrictive, while $C^1(\overline{\Omega})$ is not (cf. [11] and the problems in which the p–Laplacian operator is involved [4]). Also the assumption that $\nabla_\xi\mathcal{L}$ is of class C^1 excludes the case of the p–Laplacian, when $1<p<2$.

The aim of this paper is to remove the C^2 assumption on u and the C^1 assumption on $\nabla_\xi\mathcal{L}$, by imposing the strict convexity of $\mathcal{L}(x,s,\cdot)$. Actually, the difficult point is to drop the condition on the C^2 regularity of u. On the contrary, if $u\in C^1(\overline{\Omega})\cap C^2(\Omega)$, it is easy to see that the C^1 regularity of $\nabla_\xi\mathcal{L}$ is not necessary (see Remark 2) and no convexity assumption needs to be required.

Our main result is the following:

Theorem 2. *Assume that* $u\in C^1(\overline{\Omega})$ *is a weak solution of* $(\mathcal{P})$ *and that the function* $\{\xi\mapsto\mathcal{L}(x,s,\xi)\}$ *is strictly convex for each* $(x,s)\in\overline{\Omega}\times\mathbb{R}$.

Then identity (1) *holds for each* $a\in C^1(\overline{\Omega})$ *and* $h\in C^1(\overline{\Omega};\mathbb{R}^n)$.

The technique of the proof is based on a suitable approximation of problem $(\mathcal{P})$ with a sequence of problems for which Theorem 1 can be applied.

In more particular situations, the fact that the $C^1(\overline{\Omega})$-regularity of u is enough has been already observed. By a different approximation technique, Guedda and Véron [4] have considered the case $\mathcal{L}(x,s,\xi)=\frac{1}{p}|\xi|^p+\gamma(x,s), p>1$, while Pucci and Serrin [8] have treated by a direct approach the case $\mathcal{L}(x,s,\xi)=\alpha(x)\beta(\xi)+\gamma(x,s)$ when $n=1$.

Let us observe that the strict convexity of $\mathcal{L}(x,s,\cdot)$ is indeed usually assumed in the applications and it is also natural, if one expects the solution u to be of class $C^1(\overline{\Omega})$. In some particular situations (see Theorems 5 and 7), we are also able to relax the strict convexity assumption on $\mathcal{L}(x,s,\cdot)$ to the mere convexity. This is the case if one takes

$$\mathcal{L}(x,s,\xi)=\alpha(x,s)\beta(\xi)+\gamma(x,s)$$

or if $n=1$.

Note that, if the test functions a and h have compact support in Ω, we obtain the variational identity also when u is only locally Lipschitz in Ω. This seems to be useful in particular when $\mathcal{L}(x, s, \cdot)$ is merely convex, as a C^1 regularity of u cannot be expected.

Finally, we refer the reader to [2,4,6–10] for various applications of the variational identity to the qualitative study of nonlinear differential equations.

2. The approximation argument

Let Ω be an open subset of $\mathbb{R}^n$, not necessarily bounded, $\mathcal{L} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ a function of class C^1 and let $f \in L^\infty_{loc}(\Omega)$. Assume also that the function

$$\Big\{\xi \mapsto \mathcal{L}(x, s, \xi)\Big\}$$

is strictly convex for each $(x, s) \in \Omega \times \mathbb{R}$.

Lemma 1. *Let $u : \Omega \to \mathbb{R}$ be a locally Lipschitz solution of*

$$-\mathrm{div}\{\nabla_\xi \mathcal{L}(x, u, \nabla u)\} + D_s\mathcal{L}(x, u, \nabla u) = f \quad \text{in } \mathcal{D}'(\Omega). \tag{2}$$

Then

$$\begin{aligned}\sum_{i,j=1}^{n} \int_\Omega & D_i h_j D_{\xi_i}\mathcal{L}(x, u, \nabla u) D_j u \, dx \\ & - \int_\Omega \big[(\mathrm{div}\, h)\, \mathcal{L}(x, u, \nabla u) + h \cdot \nabla_x \mathcal{L}(x, u, \nabla u)\big]\, dx \\ & = \int_\Omega (h \cdot \nabla u) f \, dx\end{aligned} \tag{3}$$

for every $h \in C^1_c(\Omega; \mathbb{R}^n)$.

Proof. Since h has compact support in Ω, there exists a bounded open set Ω_0 with boundary of class C^∞ such that h has compact support in Ω_0 and Ω_0 has compact closure in Ω. Let $R > 0$ be such that $|\nabla u(x)| \le R$ for a.e. $x \in \Omega_0$.

Let $g = f - D_s\mathcal{L}(x, u, \nabla u)$. Since Ω is a uniform neighbourhood of Ω_0, we can regularize $\mathcal{L}$, g and u by convolution, obtaining sequences of functions $\mathcal{L}_k : \overline{\Omega_0} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $g_k : \overline{\Omega_0} \to \mathbb{R}$ and $u_k : \overline{\Omega_0} \to \mathbb{R}$ of class C^∞ such that $\mathcal{L}_k(x, s, \cdot)$ is convex and

$$\mathcal{L}_k \to \mathcal{L} \qquad \text{in } C^1(K) \text{ for every compact } K \text{ in } \overline{\Omega_0} \times \mathbb{R} \times \mathbb{R}^n\,, \tag{4}$$

$$g_k \to g \qquad \text{a.e. in } \Omega_0 \text{ with } \sup_k \|g_k\|_\infty < +\infty\,, \tag{5}$$

$$u_k \to u \qquad \text{uniformly on } \overline{\Omega_0}\,, \tag{6}$$

$$\nabla u_k \to \nabla u \qquad \text{a.e. in } \Omega_0 \text{ with } \sup_k \|\nabla u_k\|_\infty < +\infty. \tag{7}$$

Given h, it is clearly equivalent to prove the assertion with Ω substituted by Ω_0. Therefore, for the sake of simplicity, in the sequel of the proof we call Ω such an Ω_0.

Let $\vartheta : \mathbb{R}^n \to [0,1]$ be a function of class C^∞, with $\vartheta(\xi) = 1$ for $|\xi| \leq R+2$ and $\vartheta(\xi) = 0$ for $|\xi| \geq R+3$, and define $\overline{\mathcal{L}}_k : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\overline{\mathcal{L}}_k(x,\xi) = \vartheta(\xi)\mathcal{L}_k(x, u_k(x), \xi)\,.$$

Since

$$\nabla^2_{\xi\xi}\overline{\mathcal{L}}_k(x,\xi) = \vartheta(\xi)\nabla^2_{\xi\xi}\mathcal{L}_k(x,u_k(x),\xi) + 2\nabla\vartheta(\xi)\cdot\nabla_\xi\mathcal{L}_k(x,u_k(x),\xi) + \mathcal{L}_k(x,u_k(x),\xi)\nabla^2\vartheta(\xi)\,,$$

from (4), (6) and the convexity of $\mathcal{L}_k(x,s,\cdot)$ it follows that there exists $\omega > 0$ such that

$$\sum_{i,j=1}^n D^2_{\xi_i\xi_j}\overline{\mathcal{L}}_k(x,\xi)\eta_i\eta_j \geq -\omega|\eta|^2$$

for every $k \in \mathbb{N}$, $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^n$ and $\eta \in \mathbb{R}^n$.

Consider now a convex function $\Lambda : \mathbb{R}^n \to [0,+\infty[$ of class C^∞ with $\Lambda(\xi) = 0$ for $|\xi| \leq R+1$, $\nabla^2\Lambda$ bounded and

$$\sum_{i,j=1}^n D^2_{\xi_i\xi_j}\Lambda(\xi)\eta_i\eta_j \geq (\omega+1)|\eta|^2$$

for every $\xi, \eta \in \mathbb{R}^n$ with $|\xi| \geq R+2$.

Finally, define $\widetilde{\mathcal{L}}_k : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\widetilde{\mathcal{L}}_k(x,\xi) = \overline{\mathcal{L}}_k(x,\xi) + \Lambda(\xi) + \frac{1}{k}|\xi|^2\,.$$

Then $\widetilde{\mathcal{L}}_k$ is of class C^∞ and satisfies

$$\widetilde{\mathcal{L}}_k(x,\xi) \geq \frac{\omega}{4}|\xi|^2 - C\,, \tag{8}$$

$$|\xi| \geq R+3 \implies \nabla_x\widetilde{\mathcal{L}}_k(x,\xi) = 0\,, \tag{9}$$

$$\frac{1}{k}|\eta|^2 \leq \sum_{i,j=1}^n D^2_{\xi_i\xi_j}\widetilde{\mathcal{L}}_k(x,\xi)\eta_i\eta_j \leq C_k|\eta|^2 \tag{10}$$

for some $C, C_k > 0$ with C independent of k.

If we define $\widetilde{\mathcal{L}} : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\widetilde{\mathcal{L}}(x,\xi) = \vartheta(\xi)\mathcal{L}(x,u(x),\xi) + \Lambda(\xi)\,,$$

we have that $\widetilde{\mathcal{L}}$ is locally Lipschitz, $\widetilde{\mathcal{L}}(x,\cdot)$ is strictly convex and of class C^1 with $\nabla_\xi \widetilde{\mathcal{L}}$ continuous, $\nabla_x \widetilde{\mathcal{L}}$ is a Carathéodory function and we have

$$\left|\nabla_x \widetilde{\mathcal{L}}(x,\xi)\right| \le \widehat{C}\,, \tag{11}$$

$$\left|\nabla_\xi \widetilde{\mathcal{L}}(x,\xi)\right| \le \widehat{C}(1+|\xi|)\,, \tag{12}$$

$$\left(\widetilde{\mathcal{L}}_k(x,\xi) - \frac{1}{k}|\xi|^2\right) \to \widetilde{\mathcal{L}}(x,\xi) \qquad \text{uniformly on } \overline{\Omega}\times\mathbb{R}^n\,, \tag{13}$$

$$\left(\nabla_\xi \widetilde{\mathcal{L}}_k(x,\xi) - \frac{2}{k}\xi\right) \to \nabla_\xi \widetilde{\mathcal{L}}(x,\xi) \qquad \text{uniformly on } \overline{\Omega}\times\mathbb{R}^n\,, \tag{14}$$

$$\left(\nabla_x \widetilde{\mathcal{L}}_k(x,v_k) - \nabla_x \widetilde{\mathcal{L}}(x,v_k)\right) \to 0 \qquad \text{strongly in } L^1(\Omega)\text{, for every sequence } (v_k) \text{ in } L^2(\Omega;\mathbb{R}^n)\,. \tag{15}$$

Moreover, it is $\widetilde{\mathcal{L}}(x,\xi) = \mathcal{L}(x,u(x),\xi)$ for $|\xi| \le R+1$.

In particular, since u solves (2), then it is the unique minimum of the functional $\mathcal{I} : u + H_0^1(\Omega) \to \mathbb{R}$ given by

$$\mathcal{I}(w) = \int_\Omega \widetilde{\mathcal{L}}(x,\nabla w)\,dx - \int_\Omega gw\,dx.$$

On the other hand, if $\tilde{u}_k$ denotes the minimum of the functional $\mathcal{I}_k : u_k + H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{I}_k(w) = \int_\Omega \widetilde{\mathcal{L}}_k(x,\nabla w)\,dx - \int_\Omega g_k w\,dx\,,$$

then $\tilde{u}_k$ is a solution of the associated Euler equation whence, by standard regularity arguments (see e.g. [5]), $\tilde{u}_k \in C^2(\overline{\Omega})$. From Theorem 1 (see also Remark 2) it follows that

$$\begin{aligned}\sum_{i,j=1}^n \int_\Omega D_i h_j D_{\xi_i} \widetilde{\mathcal{L}}_k(x,\nabla\tilde{u}_k) D_j \tilde{u}_k\,dx \\ - \int_\Omega \left[(\operatorname{div} h)\,\widetilde{\mathcal{L}}_k(x,\nabla\tilde{u}_k) + h\cdot\nabla_x \widetilde{\mathcal{L}}_k(x,\nabla\tilde{u}_k)\right]dx \\ = \int_\Omega (h\cdot\nabla\tilde{u}_k) g_k\,dx\,.\end{aligned} \tag{16}$$

Moreover (5), (6), (7) and (8) imply that $(\tilde{u}_k - u_k)$ is bounded in $H_0^1(\Omega)$, hence, up to a subsequence, weakly convergent to a function that we write as $\tilde{u} - u$. Because of (5), (13), (14) and (15), from (16) and the minimality of $\tilde{u}_k$ we deduce that

$$\begin{aligned}\sum_{i,j=1}^n \int_\Omega D_i h_j D_{\xi_i} \widetilde{\mathcal{L}}(x,\nabla\tilde{u}_k) D_j \tilde{u}_k\,dx \\ - \int_\Omega \left[(\operatorname{div} h)\,\widetilde{\mathcal{L}}(x,\nabla\tilde{u}_k) + h\cdot\nabla_x \widetilde{\mathcal{L}}(x,\nabla\tilde{u}_k)\right]dx \\ = \int_\Omega (h\cdot\nabla\tilde{u}_k) g\,dx + o(1)\,,\end{aligned} \tag{17}$$

$$\int_\Omega \widetilde{\mathcal{L}}(x, \nabla \tilde{u}_k)\, dx - \int_\Omega g\tilde{u}_k\, dx \le \int_\Omega \widetilde{\mathcal{L}}(x, \nabla u)\, dx - \int_\Omega gu\, dx + o(1) \qquad (18)$$

as $k \to \infty$. The convexity of $\widetilde{\mathcal{L}}(x, \cdot)$ then yields

$$\int_\Omega \widetilde{\mathcal{L}}(x, \nabla \tilde{u})\, dx - \int_\Omega g\tilde{u}\, dx \le \int_\Omega \widetilde{\mathcal{L}}(x, \nabla u)\, dx - \int_\Omega gu\, dx .$$

Since u is the unique minimum point of $\mathcal{I}$, we have $\tilde{u} = u$, namely $(\tilde{u}_k)$ is weakly convergent to u in $H^1(\Omega)$. Then (18) also gives

$$\lim_k \int_\Omega \widetilde{\mathcal{L}}(x, \nabla \tilde{u}_k)\, dx = \int_\Omega \widetilde{\mathcal{L}}(x, \nabla u)\, dx .$$

Taking again into account the strict convexity of $\widetilde{\mathcal{L}}(x, \cdot)$, we infer from [12, Theorem 3] that $(\tilde{u}_k)$ is strongly convergent to u in $H^1(\Omega)$.

From (11) and (12) we deduce that

$$\begin{aligned} &\widetilde{\mathcal{L}}(x, \nabla \tilde{u}_k) \to \widetilde{\mathcal{L}}(x, \nabla u) && \text{in } L^1(\Omega)\,, \\ &\nabla_\xi \widetilde{\mathcal{L}}(x, \nabla \tilde{u}_k) \to \nabla_\xi \widetilde{\mathcal{L}}(x, \nabla u) && \text{in } L^2(\Omega; \mathbb{R}^n)\,, \\ &\nabla_x \widetilde{\mathcal{L}}(x, \nabla \tilde{u}_k) \to \nabla_x \widetilde{\mathcal{L}}(x, \nabla u) && \text{in } L^1(\Omega; \mathbb{R}^n)\,. \end{aligned}$$

Then we can pass to the limit in (17) as $k \to \infty$. From the definition of $\widetilde{\mathcal{L}}$ and g the assertion easily follows. □

Theorem 3. *Let* $u : \Omega \to \mathbb{R}$ *be a locally Lipschitz solution of* (2).

Then

$$\begin{aligned} &\int_\Omega \big[(\operatorname{div} h)\, \mathcal{L}(x, u, \nabla u) + h \cdot \nabla_x \mathcal{L}(x, u, \nabla u)\big]\, dx \\ &\qquad - \sum_{i,j=1}^n \int_\Omega \big[D_j u D_i h_j + u D_i a\big] D_{\xi_i} \mathcal{L}(x, u, \nabla u)\, dx \\ &\qquad - \int_\Omega a \big[\nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u + u D_s \mathcal{L}(x, u, \nabla u)\big]\, dx \\ &\qquad + \int_\Omega \big[h \cdot \nabla u + au\big] f\, dx = 0 \end{aligned} \qquad (19)$$

for each $a \in C^1_c(\Omega)$ *and* $h \in C^1_c(\Omega; \mathbb{R}^n)$.

Proof. First of all it is readily seen that Lipschitz test functions with compact support in Ω are allowed in the integral formulation of (2). Choosing au as test function, we get

$$\begin{aligned} &\int_\Omega u \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla a\, dx \\ &\qquad + \int_\Omega a \big[\nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u + u D_s \mathcal{L}(x, u, \nabla u)\big]\, dx \\ &\qquad = \int_\Omega auf\, dx . \end{aligned} \qquad (20)$$

The assertion follows by combining (20) with Lemma 1. □

Let us now assume that Ω is a bounded open subset of $\mathbb{R}^n$ with boundary of class C^1, $\mathcal{L} : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is of class C^1 and $f : \overline{\Omega} \to \mathbb{R}$ is continuous. Suppose also that $\mathcal{L}(x, s, \cdot)$ is strictly convex for each $(x, s) \in \overline{\Omega} \times \mathbb{R}$.

Lemma 2. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of $(\mathcal{P})$. Then it holds*

$$\begin{aligned}\int_{\partial\Omega} &\big[\mathcal{L}(x,0,\nabla u) - \nabla_\xi \mathcal{L}(x,0,\nabla u)\cdot\nabla u\big](h\cdot\nu)\, d\mathcal{H}^{n-1} \\ &= \int_\Omega \big[(\operatorname{div} h)\,\mathcal{L}(x,u,\nabla u) + h\cdot\nabla_x\mathcal{L}(x,u,\nabla u)\big]\, dx \\ &\quad - \sum_{i,j=1}^n \int_\Omega D_i h_j D_{\xi_i}\mathcal{L}(x,u,\nabla u) D_j u\, dx + \int_\Omega (h\cdot\nabla u) f\, dx\end{aligned}$$

for every $h \in C^1(\overline{\Omega};\mathbb{R}^n)$.

Proof. Let $k \geq 1$ and $\varphi_k : \mathbb{R} \to [0,1]$ be given by

$$\varphi_k(s) = \begin{cases} 0 & \text{if } s \leq \frac{1}{k}\,, \\ ks - 1 & \text{if } \frac{1}{k} < s < \frac{2}{k}\,, \\ 1 & \text{if } s \geq \frac{2}{k}\,. \end{cases}$$

Then define a Lipschitz function $\psi_k : \Omega \to [0,1]$ with compact support in Ω by setting

$$\psi_k(x) = \varphi_k(d(x, \mathbb{R}^n \setminus \Omega)).$$

Of course we have $\psi_k(x) \to 1$ for every $x \in \Omega$. It is also well known (see e.g. [3, Sect. 7]) that $-\nabla\psi_k \to \nu\mathcal{H}^{n-1} \llcorner \partial\Omega$ weakly* in the sense of measures on $\overline{\Omega}$. This means that

$$\forall v \in C(\overline{\Omega};\mathbb{R}^n): \quad \lim_k \int_\Omega v\cdot\nabla\psi_k\, dx = -\int_{\partial\Omega} v\cdot\nu\, d\mathcal{H}^{n-1}\,. \tag{21}$$

A simple approximation procedure shows that Lemma 1 holds also when h is Lipschitz continuous with compact support in Ω. If we substitute $\psi_k h$ in place of h in (3), we get

$$\begin{aligned}\sum_{i,j=1}^n &\int_\Omega h_j D_{\xi_i}\mathcal{L}(x,u,\nabla u) D_j u D_i\psi_k\, dx \\ &- \int_\Omega \mathcal{L}(x,u,\nabla u)\,(h\cdot\nabla\psi_k)\, dx \\ &= \int_\Omega \psi_k \big[(\operatorname{div} h)\,\mathcal{L}(x,u,\nabla u) + h\cdot\nabla_x\mathcal{L}(x,u,\nabla u)\big]\, dx \\ &- \sum_{i,j=1}^n \int_\Omega \psi_k D_i h_j D_{\xi_i}\mathcal{L}(x,u,\nabla u) D_j u\, dx \\ &+ \int_\Omega \psi_k (h\cdot\nabla u) f\, dx\,.\end{aligned} \tag{22}$$

On the other hand, by (21) we have

$$\lim_k \int_\Omega \mathcal{L}(x,u,\nabla u)\,(h\cdot\nabla\psi_k)\,dx = -\int_{\partial\Omega} \mathcal{L}(x,0,\nabla u)(h\cdot\nu)\,d\mathcal{H}^{n-1}\,,$$

$$\lim_k \sum_{i,j=1}^n \int_\Omega h_j D_{\xi_i}\mathcal{L}(x,u,\nabla u) D_j u D_i\psi_k\,dx$$
$$= -\sum_{i,j=1}^n \int_{\partial\Omega} h_j D_{\xi_i}\mathcal{L}(x,0,\nabla u) D_j u\nu_i\,d\mathcal{H}^{n-1}\,.$$

As observed in [7], from $u=0$ on $\partial\Omega$ it follows $\nabla u(x) = \lambda(x)\nu(x)$, hence

$$D_j u\nu_i = \lambda\nu_j\nu_i = \nu_j D_i u\,.$$

Therefore we have

$$\sum_{i,j=1}^n h_j D_{\xi_i}\mathcal{L}(x,0,\nabla u) D_j u\nu_i = [\nabla_\xi \mathcal{L}(x,0,\nabla u)\cdot\nabla u]\;(h\cdot\nu) \quad \text{on } \partial\Omega$$

and the assertion follows passing to the limit in (22) as $k\to\infty$. □

Now we can prove our main result.

Proof of Theorem 2. Clearly, in the integral formulation of $(\mathcal{P})$ it is possible to choose any test function in $C^1(\overline{\Omega})$ vanishing on $\partial\Omega$. In particular, the choice of au yields again (20). The assertion follows by combining (20) with Lemma 2. □

Remark 1. Let $N\geq 2$. It is easily seen that Theorem 2 has a vectorial counterpart for solutions $u\in C^1(\overline{\Omega};\mathbb{R}^N)$ of the system

$$\begin{cases} -\mathrm{div}(\nabla_{\xi_k}\mathcal{L}(x,u,\nabla u)) + D_{s_k}\mathcal{L}(x,u,\nabla u) = f_k & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ k = 1,\ldots,N. \end{cases}$$

See also [7, Proposition 3].

Remark 2. If $u\in C^1(\overline{\Omega})$ is a weak solution of $(\mathcal{P})$ with $\nabla u\in BV_{loc}(\Omega;\mathbb{R}^n)$, then the assertion of Theorem 1 holds without any convexity assumption on $\mathcal{L}$ nor regularity hypothesis on $\nabla_\xi\mathcal{L}$.

Moreover, if $u\in C^1(\Omega)$ is a weak solution of (2) with $\nabla u\in BV_{loc}(\Omega;\mathbb{R}^n)$, then (19) holds for any $a\in C^1_c(\Omega)$ and $h\in C^1_c(\Omega;\mathbb{R}^n)$.

Proof. We will see that Lemma 1 holds without any convexity assumption, provided that $u\in C^1(\Omega)$ and $\nabla u\in BV_{loc}(\Omega;\mathbb{R}^n)$. First of all, it is easy to see that the integral formulation of (2) holds for any test function in $BV(\Omega)$ with compact support in Ω. In particular, if $h \in C_c^1(\Omega;\mathbb{R}^n)$, we can choose $h \cdot \nabla u$ as test function, obtaining

$$\begin{aligned}\sum_{i,j=1}^{n} \int_\Omega D_{\xi_i}\mathcal{L}(x,u,\nabla u) D_i h_j D_j u\,dx \\ + \sum_{i,j=1}^{n} \int_\Omega D_{\xi_i}\mathcal{L}(x,u,\nabla u) h_j\, d(D_{ij}^2 u)(x) \\ + \int_\Omega D_s\mathcal{L}(x,u,\nabla u)(h\cdot\nabla u)\,dx = \int_\Omega (h\cdot\nabla u) f\,dx\,.\end{aligned} \tag{23}$$

On the other hand, according to [1], for every $j = 1,\dots,n$ we have

$$\begin{aligned}-\int_\Omega \mathcal{L}(x,u,\nabla u) D_j h_j\,dx = \int_\Omega h_j D_{x_j}\mathcal{L}(x,u,\nabla u)\,dx \\ + \int_\Omega D_s\mathcal{L}(x,u,\nabla u) h_j D_j u\,dx \\ + \sum_{i=1}^{n} \int_\Omega D_{\xi_i}\mathcal{L}(x,u,\nabla u) h_j\, d(D_{ij}^2 u)(x)\,.\end{aligned} \tag{24}$$

By combining (23) with (24), we get (3).

After establishing this variant of Lemma 1, we can go on as before, as the strict convexity of $\mathcal{L}(x,s,\cdot)$ is no longer used. □

3. Nonstrict convexity in some particular cases

In this section we will see that, in some particular cases, the assumption of strict convexity of $\mathcal{L}(x,s,\cdot)$ can be relaxed to the assumption of mere convexity. Let Ω be an open subset of $\mathbb{R}^n$.

Lemma 3. *Let $N \geq 1$ and let $\mathcal{F} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a function with $\mathcal{F}(x,\cdot)$ convex and C^1 for a.e. $x \in \Omega$ and $\mathcal{F}(\cdot,\xi)$ measurable for every $\xi \in \mathbb{R}^N$. Let $1 < p < \infty$ and assume that there exist $a_0 \in L^1(\Omega)$, $a_1 \in L^{p'}(\Omega)$ and $b > 0$ with*

$$|\mathcal{F}(x,\xi)| \leq a_0(x) + b|\xi|^p\,, \tag{25}$$

$$|\nabla_\xi \mathcal{F}(x,\xi)| \leq a_1(x) + b|\xi|^{p-1}\,, \tag{26}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$. Let (w_k) be a sequence weakly convergent to w in $L^p(\Omega;\mathbb{R}^N)$ with

$$\lim_k \int_\Omega \mathcal{F}(x,w_k)\,dx = \int_\Omega \mathcal{F}(x,w)\,dx\,.$$

Then

$$\lim_k \mathcal{F}(x,w_k) = \mathcal{F}(x,w) \qquad \text{weakly in } L^1(\Omega)\,. \tag{27}$$

Moreover, if there exists $d > 0$ *with*

$$\mathcal{F}(x,\xi) \geq d|\xi|^p - a_0(x) \tag{28}$$

for a.e. $x \in \Omega$ *and all* $\xi \in \mathbb{R}^N$*, we have*

$$\lim_k \nabla_\xi \mathcal{F}(x, w_k) = \nabla_\xi \mathcal{F}(x, w) \qquad \textit{strongly in } L^{p'}(\Omega;\mathbb{R}^N) \tag{29}$$

and, up to a subsequence, $|w_k|^p \leq \psi$ *for some* $\psi \in L^1(\Omega)$*.*

Proof. Let us define $\widetilde{\mathcal{F}} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ by setting

$$\widetilde{\mathcal{F}}(x,\xi) = \mathcal{F}(x, w(x)+\xi) - \mathcal{F}(x,w(x)) - \nabla_\xi \mathcal{F}(x,w(x)) \cdot \xi .$$

Note that $\widetilde{\mathcal{F}}(x,\xi) \geq 0$, $\widetilde{\mathcal{F}}(x,0) = 0$, $\nabla_\xi \widetilde{\mathcal{F}}(x,0) = 0$ and

$$\lim_k \int_\Omega \widetilde{\mathcal{F}}(x, w_k - w)\, dx = 0\,,$$

so that

$$\lim_k \widetilde{\mathcal{F}}(x, w_k - w) = 0 \qquad \text{strongly in } L^1(\Omega)\,. \tag{30}$$

On the other hand, for each $\varphi \in L^\infty(\Omega)$ we have

$$\lim_k \int_\Omega \varphi \nabla_\xi \mathcal{F}(x,w) \cdot (w_k - w)\, dx = 0\,.$$

It follows

$$\lim_k \int_\Omega \varphi \big[\mathcal{F}(x,w_k) - \mathcal{F}(x,w)\big]\, dx = 0\,,$$

which proves (27).

Note that, in view of (30), up to a subsequence one has $\widetilde{\mathcal{F}}(x, w_k(x) - w(x)) \to 0$ for a.e. $x \in \Omega$. Fix now such an x; then by (28) up to a subsequence $w_k(x) \to y$ for some $y \in \mathbb{R}^N$, which yields $\widetilde{\mathcal{F}}(x, y - w(x)) = 0$. In particular, $y - w(x)$ is a minimum for $\widetilde{\mathcal{F}}(x,\cdot)$, so that $\nabla_\xi \widetilde{\mathcal{F}}(x, y - w(x)) = 0$, namely $\nabla_\xi \mathcal{F}(x,y) = \nabla_\xi \mathcal{F}(x, w(x))$. Hence we conclude that

$$\lim_k \nabla_\xi \mathcal{F}(x, w_k(x)) = \nabla_\xi \mathcal{F}(x, w(x)) \quad \text{a.e. in } \Omega\,. \tag{31}$$

Up to a further subsequence, by (30) there exists $\widetilde{\psi} \in L^1(\Omega)$ such that

$$\mathcal{F}(x,w_k) - \mathcal{F}(x,w) - \nabla_\xi \mathcal{F}(x,w) \cdot (w_k - w) \leq \widetilde{\psi}\,.$$

By (28) and Young's inequality one finds $C > 0$ such that

$$\frac{d}{2}|w_k|^p \leq a_0 + \mathcal{F}(x,w) - \nabla_\xi \mathcal{F}(x,w) \cdot w + \widetilde{\psi} + C|\nabla_\xi \mathcal{F}(x,w)|^{p'}\,,$$

whence the last assertion. In particular, in view of (26) one deduces that $|\nabla_\xi \mathcal{F}(x, w_k)| \leq \eta$ for some $\eta \in L^{p'}(\Omega)$, which combined with (31) yields (29). □

3.1. The splitting case

In this subsection we will consider the case in which $\mathcal{L}(x,s,\xi)$ is of the form $\alpha(x,s)\beta(\xi)+\gamma(x,s)$.

Lemma 4. *Let $\mathcal{F}(x,\xi)=\alpha(x)\beta(\xi)$, with $\alpha:\Omega\to[0,+\infty[$ locally Lipschitz and $\beta:\mathbb{R}^N\to\mathbb{R}$ convex and of class C^1. Let $1<p<\infty$ and assume that there exist $a_0\in L^1(\Omega)$, $a_1\in L^{p'}(\Omega)$ and $b>0$ satisfying* (25), (26) *and*

$$|\nabla_x\mathcal{F}(x,\xi)|\le a_0(x)+b|\xi|^p \tag{32}$$

for a.e. $x\in\Omega$ and all $\xi\in\mathbb{R}^N$.

Let (w_k) be a sequence weakly convergent to w in $L^p(\Omega;\mathbb{R}^N)$ with

$$|w_k|^p\le\psi \qquad \text{for some } \psi\in L^1(\Omega)\,,$$

$$\lim_k\int_\Omega\mathcal{F}(x,w_k)\,dx=\int_\Omega\mathcal{F}(x,w)\,dx\,.$$

Then

$$\lim_k\nabla_x\mathcal{F}(x,w_k)=\nabla_x\mathcal{F}(x,w)\quad \text{weakly in } L^1(\Omega;\mathbb{R}^n)\,.$$

Proof. Let

$$\Omega^0=\{x\in\Omega:\ \alpha(x)=0\}\ ,$$

$$\forall m\ge1:\ \Omega_m=\left\{x\in\Omega:\ \alpha(x)\ge\frac{1}{m},\ |\nabla\alpha(x)|\le m\right\}\,.$$

Since $\nabla\alpha=0$ a.e. in Ω^0, it is clear that

$$\lim_k\nabla_x\mathcal{F}(x,w_k)=\nabla_x\mathcal{F}(x,w)\quad\text{weakly in } L^1(\Omega^0;\mathbb{R}^n)\,.$$

Given $\varepsilon>0$, there exists $m\ge1$ such that

$$\int_{\Omega\setminus(\Omega^0\cup\Omega_m)}(a_0+b\psi)\,dx<\varepsilon\,.$$

From (32) it follows

$$\forall k\in\mathbb{N}:\ \int_{\Omega\setminus(\Omega^0\cup\Omega_m)}|\nabla_x\mathcal{F}(x,w_k)-\nabla_x\mathcal{F}(x,w)|\ dx<2\varepsilon\,.$$

Therefore, we have only to show that, for any $m\ge1$, it holds

$$\lim_k\nabla_x\mathcal{F}(x,w_k)=\nabla_x\mathcal{F}(x,w)\quad\text{weakly in } L^1(\Omega_m;\mathbb{R}^n)\,. \tag{33}$$

From Lemma 3 we deduce that

$$\lim_k\mathcal{F}(x,w_k)=\mathcal{F}(x,w)\quad\text{weakly in } L^1(\Omega)\,,$$

hence

$$\lim_k\mathcal{F}(x,w_k)=\mathcal{F}(x,w)\quad\text{weakly in } L^1(\Omega_m)\,.$$

Since $(\nabla\alpha)/\alpha\in L^\infty(\Omega_m;\mathbb{R}^n)$, (33) holds and the assertion follows. □

Let now Ω be an open subset of $\mathbb{R}^n$, let

$$\mathcal{L}(x,s,\xi) = \alpha(x,s)\beta(\xi) + \gamma(x,s)\,, \tag{34}$$

with $\alpha : \Omega \times \mathbb{R} \to [0,+\infty[$, $\gamma : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\beta : \mathbb{R}^n \to \mathbb{R}$ of class C^1, and let $f \in L^\infty_{loc}(\Omega)$. Assume also that β is convex.

Lemma 5. *Let $u : \Omega \to \mathbb{R}$ be a locally Lipschitz solution of* (2).
Then (3) *holds for every $h \in C^1_c(\Omega;\mathbb{R}^n)$.*

Proof. Let Ω_0, g_k, u_k, ϑ, Λ, $\widetilde{\mathcal{L}}_k$ and $\widetilde{\mathcal{L}}$ be as in the proof of Lemma 1. The only difference is that now $\widetilde{\mathcal{L}}(x,\cdot)$ is merely convex.

Let $M > 0$ be such that

$$\forall x \in \Omega:\ \alpha(x,u(x)) + |\gamma(x,u(x))| \le M$$

(after substituting Ω with Ω_0). Without loss of generality, we may also assume that the functions

$$\frac{1}{M}\Lambda + \vartheta\beta\,, \qquad \frac{1}{M}\Lambda + \vartheta\,, \qquad \frac{1}{M}\Lambda - \vartheta$$

are all convex.

Since u solves (2), then it is the unique minimum of the functional $\widehat{\mathcal{I}} : u + H^1_0(\Omega) \to \mathbb{R}$ given by

$$\widehat{\mathcal{I}}(w) = \int_\Omega \left(\widetilde{\mathcal{L}}(x,\nabla w) + (w-u)^2\right)\,dx - \int_\Omega gw\,dx\,.$$

On the other hand, if $\tilde{u}_k$ denotes the minimum of the functional $\widehat{\mathcal{I}}_k : u_k + H^1_0(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\mathcal{I}}_k(w) = \int_\Omega \left(\widetilde{\mathcal{L}}_k(x,\nabla w) + (w-u_k)^2\right)\,dx - \int_\Omega g_k w\,dx\,,$$

then $\tilde{u}_k$ is a $C^2(\overline{\Omega})$-solution of the associated Euler equation, whence

$$\begin{aligned}
&\sum_{i,j=1}^n \int_\Omega D_i h_j D_{\xi_i}\widetilde{\mathcal{L}}_k(x,\nabla\tilde{u}_k) D_j\tilde{u}_k\,dx \\
&\qquad - \int_\Omega \left[(\operatorname{div} h)\,\widetilde{\mathcal{L}}_k(x,\nabla\tilde{u}_k) + h\cdot\nabla_x\widetilde{\mathcal{L}}_k(x,\nabla\tilde{u}_k)\right]dx \\
&\qquad - \int_\Omega \left[(\operatorname{div} h)(\tilde{u}_k-u_k)^2 - 2(h\cdot\nabla u_k)(\tilde{u}_k-u_k)\right]\,dx \\
&\qquad = \int_\Omega (h\cdot\nabla\tilde{u}_k)g_k\,dx\,.
\end{aligned} \tag{35}$$

Again we have that $(\tilde{u}_k)$ is weakly convergent, up to a subsequence, to some $\tilde{u}$ in $H^1(\Omega)$. From

$$\int_\Omega \left(\widetilde{\mathcal{L}}_k(x, \nabla\tilde{u}_k) + (\tilde{u}_k - u_k)^2\right) dx - \int_\Omega g_k\tilde{u}_k\, dx \\ \leq \int_\Omega \widetilde{\mathcal{L}}_k(x, \nabla u_k)\, dx - \int_\Omega g_k u_k\, dx\,,$$

it follows that

$$\int_\Omega \left(\widetilde{\mathcal{L}}(x, \nabla\tilde{u}) + (\tilde{u} - u)^2\right) dx - \int_\Omega g\tilde{u}\, dx \leq \int_\Omega \widetilde{\mathcal{L}}(x, \nabla u)\, dx - \int_\Omega gu\, dx\,.$$

Since u is the unique minimum point of the functional $\widehat{\mathcal{I}}$, we can still deduce that $\tilde{u} = u$, namely $(\tilde{u}_k)$ is weakly convergent to u in $H^1(\Omega)$. Again we have

$$\lim_k \int_\Omega \widetilde{\mathcal{L}}(x, \nabla\tilde{u}_k)\, dx = \int_\Omega \widetilde{\mathcal{L}}(x, \nabla u)\, dx \tag{36}$$

and, from (35),

$$\begin{aligned} \sum_{i,j=1}^n \int_\Omega & D_i h_j D_{\xi_i}\widetilde{\mathcal{L}}(x, \nabla\tilde{u}_k) D_j\tilde{u}_k\, dx \\ & - \int_\Omega \left[(\operatorname{div} h)\, \widetilde{\mathcal{L}}(x, \nabla\tilde{u}_k) + h\cdot\nabla_x\widetilde{\mathcal{L}}(x, \nabla\tilde{u}_k)\right] dx \\ & = \int_\Omega (h\cdot\nabla\tilde{u}_k)g + o(1) \end{aligned} \tag{37}$$

as $k \to \infty$. However, because of the lack of strict convexity of $\mathcal{L}(x, s, \cdot)$, we cannot say that $(\tilde{u}_k)$ is strongly convergent to u in $H^1(\Omega)$.

On the other hand, Lemma 3 allows us to deduce that

$$\begin{aligned} &\widetilde{\mathcal{L}}(x, \nabla\tilde{u}_k) \to \widetilde{\mathcal{L}}(x, \nabla u) && \text{weakly in } L^1(\Omega)\,, \\ &\nabla_\xi\widetilde{\mathcal{L}}(x, \nabla\tilde{u}_k) \to \nabla_\xi\widetilde{\mathcal{L}}(x, \nabla u) && \text{strongly in } L^2(\Omega; \mathbb{R}^n) \end{aligned}$$

and that $|\nabla\tilde{u}_k|^2 \leq \psi$ for some $\psi \in L^1(\Omega)$. In order to pass to the limit in (37) and conclude the proof, it is therefore enough to show that

$$\nabla_x\widetilde{\mathcal{L}}(x, \nabla\tilde{u}_k) \to \nabla_x\widetilde{\mathcal{L}}(x, \nabla u) \qquad \text{weakly in } L^1(\Omega; \mathbb{R}^n)\,. \tag{38}$$

Only at this point the particular structure given by (34) will play a role. We have

$$\begin{aligned} \widetilde{\mathcal{L}}(x, \xi) &= \vartheta(\xi)\alpha(x, u(x))\beta(\xi) + \vartheta(\xi)\gamma(x, u(x)) + \Lambda(\xi) \\ &= \mathcal{F}_1(x, \xi) + \mathcal{F}_2(x, \xi) + \mathcal{F}_3(x, \xi) + \mathcal{F}_4(x, \xi)\,, \end{aligned}$$

where

$$\begin{aligned}
\mathcal{F}_1(x,\xi) &= \alpha(x,u(x))\left(\vartheta(\xi)\beta(\xi)+\frac{1}{M}\Lambda(\xi)\right),\\
\mathcal{F}_2(x,\xi) &= \gamma^+(x,u(x))\left(\frac{1}{M}\Lambda(\xi)+\vartheta(\xi)\right),\\
\mathcal{F}_3(x,\xi) &= \gamma^-(x,u(x))\left(\frac{1}{M}\Lambda(\xi)-\vartheta(\xi)\right),\\
\mathcal{F}_4(x,\xi) &= \frac{1}{M}\left(M-\alpha(x,u(x))-|\gamma(x,u(x))|\right)\Lambda(\xi)
\end{aligned}$$

satisfy the assumptions of Lemma 4. Since

$$\forall j=1,\dots,4:\ \liminf_k \int_\Omega \mathcal{F}_j(x,\nabla\tilde{u}_k)\,dx \ge \int_\Omega \mathcal{F}_j(x,\nabla u)\,dx\,,$$

from (36) we get

$$\forall j=1,\dots,4:\ \lim_k \int_\Omega \mathcal{F}_j(x,\nabla\tilde{u}_k)\,dx = \int_\Omega \mathcal{F}_j(x,\nabla u)\,dx\,.$$

By Lemma 4 we deduce that

$$\forall j=1,\dots,4:\ \nabla_x\mathcal{F}_j(x,\nabla\tilde{u}_k)\to\nabla_x\mathcal{F}_j(x,\nabla u) \qquad \text{weakly in } L^1(\Omega;\mathbb{R}^n)\,.$$

Therefore (38) follows and the proof is complete. □

Theorem 4. *Let $u:\Omega\to\mathbb{R}$ be a locally Lipschitz solution of* (2). *Then* (19) *holds for each $a\in C^1_c(\Omega)$ and $h\in C^1_c(\Omega;\mathbb{R}^n)$.*

Proof. It is enough to argue as in the proof of Theorem 3, taking into account Lemma 5 instead of Lemma 1. □

Assume now that Ω is a bounded open subset of $\mathbb{R}^n$ with boundary of class C^1, that $\mathcal{L}(x,s,\xi)=\alpha(x,s)\beta(\xi)+\gamma(x,s)$, with $\alpha:\overline{\Omega}\times\mathbb{R}\to[0,+\infty[$, $\gamma:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ and $\beta:\mathbb{R}^n\to\mathbb{R}$ of class C^1, and that $f\in C(\overline{\Omega})$. Suppose also that β is convex.

Theorem 5. *Let $u\in C^1(\overline{\Omega})$ be a weak solution of* ($\mathcal{P}$). *Then* (1) *holds for each $a\in C^1(\overline{\Omega})$ and $h\in C^1(\overline{\Omega};\mathbb{R}^n)$.*

Proof. After establishing Theorem 4 instead of Theorem 3, we can go on as in the proof of Theorem 2, as the strict convexity of $\mathcal{L}(x,s,\cdot)$ is no longer used. □

3.2. The one-dimensional case

In this subsection we assume that $\Omega \subseteq \mathbb{R}$ is a bounded open interval and $\mathcal{L} : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of class C^1 with $\mathcal{L}(x, s, \cdot)$ convex for any $(x, s) \in \overline{\Omega} \times \mathbb{R}$.

Theorem 6. *Let $f \in L^\infty_{loc}(\Omega)$ and let $u : \Omega \to \mathbb{R}$ be a locally Lipschitz solution of* (2)*. Then* (19) *holds for each $a \in C^1_c(\Omega)$ and $h \in C^1_c(\Omega)$.*

Theorem 7. *Let $f \in C(\overline{\Omega})$ and let $u \in C^1(\overline{\Omega})$ be a weak solution of* $(\mathcal{P})$*. Then* (1) *holds for each $a \in C^1(\overline{\Omega})$ and $h \in C^1(\overline{\Omega})$.*

The proof follows the same lines of that of Theorems 4 and 5. The key point is that the assertion of Lemma 5 holds also in this case. To see it, one has to follow the same argument and appeal, in the final part, to the next Lemma 6 instead of Lemma 4.

Lemma 6. *Let $\mathcal{F} : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function. Assume that there exists a negligible set $N \subseteq \Omega$ such that:*

(a) *for every $(x, \xi) \in (\Omega \setminus N) \times \mathbb{R}$, the function $\mathcal{F}(\cdot, \xi)$ is differentiable at x;*
(b) *for every $x \in \Omega \setminus N$, the function $\mathcal{F}(x, \cdot)$ is convex and of class C^1;*
(c) *for every $x \in \Omega \setminus N$, the function $D_x\mathcal{F}(x, \cdot)$ is continuous.*

Moreover, suppose that there exist $a_0 \in L^1(\Omega)$, $a_1 \in L^{p'}(\Omega)$ and $b, d > 0$ such that (25)*,* (26)*,* (28) *and* (32) *hold.*

Let (w_k) be a sequence weakly convergent to w in $L^p(\Omega)$ with

$$\lim_k \int_\Omega \mathcal{F}(x, w_k)\, dx = \int_\Omega \mathcal{F}(x, w)\, dx\,.$$

Then

$$\lim_k D_x\mathcal{F}(x, w_k) = D_x\mathcal{F}(x, w) \quad \textit{weakly in } L^1(\Omega)\,.$$

Proof. As in the proof of Lemma 3, up to a subsequence one has $|w_k|^p \le \psi \in L^1(\Omega)$ and

$$\lim_k \widetilde{\mathcal{F}}(x, w_k(x) - w(x)) = 0 \quad \text{a.e. in } \Omega\,.$$

Let us set for a.e. $x \in \Omega$

$$y_\flat(x) = \liminf_k w_k(x)\,, \qquad y_\sharp(x) = \limsup_k w_k(x)\,.$$

Notice that $y_\flat, y_\sharp \in L^p(\Omega)$ and

$$y_\flat(x) \le w(x) \le y_\sharp(x) \quad \text{a.e. in } \Omega\,. \tag{39}$$

If $\widetilde{w}_k(x)$ denotes the projection of $w_k(x)$ onto $[y_\flat(x), y_\sharp(x)]$, one has $(\widetilde{w}_k - w_k) \to 0$ in $L^p(\Omega)$. Then, up to substituting w_k with $\widetilde{w}_k$, one can suppose that

$$y_\flat(x) \le w_k(x) \le y_\sharp(x) \quad \text{a.e. in } \Omega\,. \tag{40}$$

Arguing as in the proof of Lemma 3, one obtains

$$\widetilde{\mathcal{F}}(x, y_\flat(x) - w(x)) = 0, \quad \widetilde{\mathcal{F}}(x, y_\sharp(x) - w(x)) = 0 \quad \text{a.e. in } \Omega\,.$$

Since $\widetilde{\mathcal{F}}(x,\xi) \geq 0$ and $\widetilde{\mathcal{F}}(x,\,\cdot\,)$ is convex, it follows

$$\widetilde{\mathcal{F}}(x, (1-\vartheta)y_\flat(x) + \vartheta y_\sharp(x) - w(x)) = 0$$

for a.e. $x \in \Omega$ and every $\vartheta \in [0,1]$, whence

$$\mathcal{F}\big(x, (1-\vartheta)y_\flat(x) + \vartheta y_\sharp(x)\big) = (1-\vartheta)\mathcal{F}(x, y_\flat(x)) + \vartheta\mathcal{F}(x, y_\sharp(x)) \tag{41}$$

for a.e. $x \in \Omega$ and every $\vartheta \in [0,1]$.

For each $m \geq 1$ let us set

$$\Omega_m = \left\{ x \in \Omega \setminus N : \; y_\sharp(x) - y_\flat(x) \geq \frac{1}{m}, \right.$$
$$\left. |D_x\mathcal{F}(x, y_\flat(x))| \leq m,\; |D_x\mathcal{F}(x, y_\sharp(x))| \leq m \right\}.$$

By Lusin's theorem, for each $\varepsilon > 0$ there exists a measurable subset $C_{m,\varepsilon} \subseteq \Omega_m$ such that

$$y_\flat\big|_{C_{m,\varepsilon}},\; y_\sharp\big|_{C_{m,\varepsilon}} \quad \text{are continuous,} \qquad \mathcal{L}^1(\Omega_m \setminus C_{m,\varepsilon}) < \varepsilon\,,$$

where $\mathcal{L}^1$ denotes the one-dimensional Lebesgue measure. Without loss of generality, we may assume that $C_{m,\varepsilon}$ has no isolated points. Let us now take $x \in C_{m,\varepsilon}$ and $\delta > 0$ with

$$y_\flat(x) + \delta < y_\sharp(x) - \delta\,.$$

If (x_k) is a sequence in $C_{m,\varepsilon}$ converging to x, we have

$$y_\flat(x_k) \leq y_\flat(x) + \delta < y_\sharp(x) - \delta \leq y_\sharp(x_k) \tag{42}$$

eventually as $k \to \infty$. By (41), for each $\vartheta \in [0,1]$ one obtains

$$\mathcal{F}(x, (1-\vartheta)(y_\flat(x) + \delta) + \vartheta(y_\sharp(x) - \delta))$$
$$= (1-\vartheta)\mathcal{F}(x, y_\flat(x) + \delta) + \vartheta\mathcal{F}(x, y_\sharp(x) - \delta)\,.$$

Moreover, (42) implies

$$\mathcal{F}(x_k, (1-\vartheta)(y_\flat(x) + \delta) + \vartheta(y_\sharp(x) - \delta))$$
$$= (1-\vartheta)\mathcal{F}(x_k, y_\flat(x) + \delta) + \vartheta\mathcal{F}(x_k, y_\sharp(x) - \delta)\,.$$

Therefore, combining the previous identities yields

$$D_x\mathcal{F}(x, (1-\vartheta)(y_\flat(x) + \delta) + \vartheta(y_\sharp(x) - \delta))$$
$$= (1-\vartheta)D_x\mathcal{F}(x, y_\flat(x) + \delta) + \vartheta D_x\mathcal{F}(x, y_\sharp(x) - \delta)$$

for each $\vartheta \in [0,1]$. Letting $\delta \to 0$ one obtains

$$\forall x \in C_{m,\varepsilon},\ \forall \vartheta \in [0,1]: \quad D_x\mathcal{F}(x,(1-\vartheta)y_\flat(x)+\vartheta y_\sharp(x)) \\ = (1-\vartheta)D_x\mathcal{F}(x,y_\flat(x)) + \vartheta D_x\mathcal{F}(x,y_\sharp(x))\,.$$

By (39) and (40) we can choose

$$\overline{\vartheta} = \frac{w(x)-y_\flat(x)}{y_\sharp(x)-y_\flat(x)}\,, \qquad \overline{\vartheta}_k = \frac{w_k(x)-y_\flat(x)}{y_\sharp(x)-y_\flat(x)}\,.$$

Then one gets

$$D_x\mathcal{F}(x,w(x)) = \frac{y_\sharp(x)-w(x)}{y_\sharp(x)-y_\flat(x)} D_x\mathcal{F}(x,y_\flat(x)) \\ + \frac{w(x)-y_\flat(x)}{y_\sharp(x)-y_\flat(x)} D_x\mathcal{F}(x,y_\sharp(x))$$

and

$$D_x\mathcal{F}(x,w_k(x)) = \frac{y_\sharp(x)-w_k(x)}{y_\sharp(x)-y_\flat(x)} D_x\mathcal{F}(x,y_\flat(x)) \\ + \frac{w_k(x)-y_\flat(x)}{y_\sharp(x)-y_\flat(x)} D_x\mathcal{F}(x,y_\sharp(x))\,.$$

In particular, one concludes that

$$D_x\mathcal{F}(x,w_k(x)) = D_x\mathcal{F}(x,w(x)) \\ + (w_k(x)-w(x))\frac{D_x\mathcal{F}(x,y_\sharp(x)) - D_x\mathcal{F}(x,y_\flat(x))}{y_\sharp(x)-y_\flat(x)}$$

for all $x \in C_{m,\varepsilon}$, which implies that

$$\forall \varphi \in L^\infty(C_{m,\varepsilon}): \quad \lim_k \int_{C_{m,\varepsilon}} D_x\mathcal{F}(x,w_k)\varphi\,dx = \int_{C_{m,\varepsilon}} D_x\mathcal{F}(x,w)\varphi\,dx\,.$$

On the other hand, by (32) one has

$$|D_x\mathcal{F}(x,w_k(x))\varphi(x)| \le \|\varphi\|_\infty\big(a_0(x)+b\psi(x)\big)\,. \tag{43}$$

It follows that

$$\forall \varphi \in L^\infty(\Omega_m): \quad \lim_k \int_{\Omega_m} D_x\mathcal{F}(x,w_k)\varphi\,dx = \int_{\Omega_m} D_x\mathcal{F}(x,w)\varphi\,dx\,.$$

Moreover, since on the set

$$\Omega_\infty = \big\{x \in \Omega: \ \ y_\sharp(x) = y_\flat(x)\big\}$$

one has $\lim\limits_k w_k = w$ a.e., then

$$\forall \varphi \in L^\infty(\Omega_\infty): \quad \lim_k \int_{\Omega_\infty} D_x\mathcal{F}(x,w_k)\varphi\,dx = \int_{\Omega_\infty} D_x\mathcal{F}(x,w)\varphi\,dx\,.$$

Being $\mathcal{L}^1(\Omega \setminus (\Omega_\infty \cup \Omega_m)) \to 0$ as $m \to +\infty$, by (43) one concludes the proof. □

Remark 3. We do not know whether Lemma 6 holds true when $\mathcal{F} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ and Ω is an open subset of $\mathbb{R}^n$, $n \geq 2$. In the affirmative case, the strict convexity assumption on $\mathcal{L}(x, s, \cdot)$ could be relaxed to the mere convexity in general.

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