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Decomposition and integral representation of Cauchy interactions associated with measures*

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Cauchy interactions between subbodies of a continuous body are introduced in the framework of Measure Theory, extending the class of previously admissible ones. A decomposition theorem into a volume and a surface interaction is proved, as well as characterizations of the single components. Finally, an extension result and a generalized balance law are given.

Key words: Cauchy interactions – balance equations – sets with finite perimeter

1 Introduction

It is well-known that the modelization of interactions in Continuum Physics deals with set functions associated with physical quantities rather than with functions evaluated at single points (see e.g. [7]). Very important examples of these are the stress and the heat flux. This, in turn, implies that the concept of subbody of a material body B has to be taken into account. However, subbodies are not completely physical (although they may be used to describe the situation arising in the body in a very satisfactory way), since the class of subsets which have to represent them is a matter of choice.

For such set functions should satisfy some reasonable additivity condition, it is natural to put the approach into the framework of Measure Theory. An example of how this way of thinking has been developed is given by the Cauchy Stress Theorem, leading in [3] to the notion of Cauchy flux. For further developments, we refer the reader to [3, 8, 5, 1] and the references quoted therein.

In [4], Gurtin, Williams and Ziemer proposed to choose the normalized sets of finite perimeter as subbodies and introduced the concept of *Cauchy interaction* in order to represent an interaction between two disjoint subbodies, possibly having a part of their boundary in common. This is, roughly speaking, a set function I of two variables, the subbodies, which is additive on each variable and which is Lipschitz continuous with respect to the area measure of the common part of the boundaries and the volume measure. In that paper it is proved that:

(a) I can be decomposed as the sum of a "body interaction" I_b and a "contact interaction" I_c, satisfying the bounds

$$|I_b(A,C)| \leqslant K_C \mathscr{L}^n(A)\,, \qquad |I_c(A,C)| \leqslant K\mathscr{H}^{n-1}(\partial_* A \cap \partial_* C)\,;$$

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(*b*) I_b admits the representation

$$I_b(A,C) = \int_{A\times C} b(x,y)\,dx\,dy\,, \quad \text{for a suitable } b \in L^1(B\times B)\,;$$

(*c*) in the balanced case, i.e. when $|I(A,\mathbb{R}^n\setminus A)| \leqslant K\mathscr{L}^n(A)$, the contact part I_c is a Cauchy flux which admits the representation

$$I_c(A,C) = \int_{\partial_* A\cap\partial_* C} \boldsymbol{q}\cdot\boldsymbol{n}_{\partial_* A\cap\partial_* C}\, d\mathscr{H}^{n-1}\,,$$

where $\boldsymbol{q} : B \to \mathbb{R}^n$ is a bounded vector field with bounded divergence.

In our work we extend this definition of Cauchy interaction in order to allow the corresponding densities to be also distributions of order zero. In this way, also interactions which are singular can be considered. To do this, we deal with notions of "almost all subbodies" and "almost every material surface", already introduced by Šilhavý [5] and extended by Degiovanni, Marzocchi and Musesti [1] for the formulation of the Cauchy Stress Theorem. In particular, for almost all subbodies we first show that:

(a') I can be decomposed as the sum of a "body interaction" I_b and a "contact interaction" I_c, satisfying the bounds

$$|I_b(A,C)| \leqslant \eta(A\times C)\,, \qquad |I_c(A,C)| \leqslant \int_{\partial_* A\cap\partial_* C} h(x)\, d\mathscr{H}^{n-1}(x)\,,$$

where η is a Radon measure and h a positive function in L^1_{loc};

(b') I_b admits the representation

$$I_b(A,C) = \int_{A\times C} b(x,y)\, d\mu(x,y)\,,$$

where μ is a Radon measure and $b : B\times B \to \{-1,1\}$ is a Borel function;

(c') in the balanced case, i.e. when $|I(A,\mathbb{R}^n\setminus A)| \leqslant \lambda(A)$ for a Radon measure λ, the contact part I_c is a Cauchy flux which admits the representation

$$I_c(A,C) = \int_{\partial_* A\cap\partial_* C} \boldsymbol{q}\cdot\boldsymbol{n}_{\partial_* A\cap\partial_* C}\, d\mathscr{H}^{n-1}\,,$$

where $\boldsymbol{q} : B \to \mathbb{R}^n$ is a locally integrable vector field with divergence measure.

Next, we show the uniqueness of the decomposition (a') and a correspondence between contact interactions and Cauchy fluxes also in the non-balanced case (Theorem 6.1), which is not treated in [1]. Finally, we derive from the previous theorems a generalized form of the balance equation associated to a balanced Cauchy interaction.

Our last result, in the spirit of [1], is that the choice of all normalized subsets of finite perimeter as subbodies is a natural one: in fact, it is sufficient to verify the existence of a Cauchy interaction on a very small (in comparison with that of the sets of finite perimeter) class of subbodies, to have automatically an essentially unique extension of the interaction on almost every disjoint pair of normalized subbodies of finite perimeter, with the same properties given on the simpler subbodies (see Sect. 8). Since the above class is comprised of parallelepipeds, which anybody would wish to consider as subbodies, our result shows that the class of the normalized subsets of finite perimeter is the smallest (although very wide, since sets of finite perimeter can be quite irregular) "natural" class of subbodies of a material body.

It is worth to point out that our definition of Cauchy interaction, as well as that of [4], is modeled on the situation in which the set function represents the sum of the heat generated in the subbody and the heat transferred through its boundary. This leads to the peculiar choice of the subbodies in Definition 3.1: it is requested that either the subsets lie in the interior of the body, or that their complements have this property.

2 Preliminary lemmas from Geometric Measure Theory

Let $M \subseteq \mathbb{R}^n$. We denote by $\mathrm{cl}\, M$ and $\mathrm{int}\, M$ the closure and the interior of M in $\mathbb{R}^n$, respectively. When M is a Borel set, we also denote by $\mathfrak{B}(M)$ the σ-algebra of Borel subsets of M.

We denote by $\mathscr{L}^n$ the Lebesgue outer measure on $\mathbb{R}^n$ and by $\mathscr{H}^k$ the k-dimensional Hausdorff outer measure. Denoting by $\mathrm{B}_x(r)$ the open ball with center x and radius r, we introduce

$$M_* = \left\{ x \in \mathbb{R}^n : \lim_{r \to 0^+} \frac{\mathscr{L}^n(\mathrm{B}_x(r) \setminus M)}{\mathscr{L}^n(\mathrm{B}_x(r))} = 0 \right\}$$

and

$$\partial_* M = \mathbb{R}^n \setminus \left[M_* \cup (\mathbb{R}^n \setminus M)_* \right] ,$$

(the so called *measure-theoretic interior and measure-theoretic boundary of* M, respectively). It is well-known that M_* and $\partial_* M$ are Borel subsets of $\mathbb{R}^n$. We say that M is *normalized*, if $M_* = M$.

Now let $M \subseteq \mathbb{R}^n$, $x \in \partial_* M$ and $u \in \mathbb{R}^n$ with $|u| = 1$. We say that u is *a unit exterior normal vector* to M at x if

$$\lim_{r \to 0^+} \frac{\mathscr{L}^n\left(\{\xi \in \mathrm{B}_x(r) \cap M : (\xi - x) \cdot u > 0\}\right)}{\mathscr{L}^n(\mathrm{B}_x(r))} = 0 ,$$

$$\lim_{r \to 0^+} \frac{\mathscr{L}^n\left(\{\xi \in \mathrm{B}_x(r) \setminus M : (\xi - x) \cdot u < 0\}\right)}{\mathscr{L}^n(\mathrm{B}_x(r))} = 0 .$$

If u and v are two unit exterior normal vectors to M at x, it turns out that $u = v$, so we can define a map $\boldsymbol{n}^M : \partial_* M \to \mathbb{R}^n$, setting $\boldsymbol{n}^M(x)$ equal to the unit exterior normal vector to M at x, where it exists, and $\boldsymbol{n}^M(x) = 0$ otherwise. Then $\boldsymbol{n}^M$ is a Borel and bounded map, that is called *the unit exterior normal* to M.

We say that M has *finite perimeter* if $\mathscr{H}^{n-1}(\partial_* M) < +\infty$ (this implies the $\mathscr{L}^n$-measurability of M). In such a case, $|\boldsymbol{n}^M(x)| = 1$ for $\mathscr{H}^{n-1}$-a.e. $x \in \partial_* M$ and the Gauss-Green Theorem

$$\int_M \boldsymbol{v} \cdot \nabla f \, d\mathscr{L}^n = \int_{\partial_* M} f \, \boldsymbol{v} \cdot \boldsymbol{n}^M \, d\mathscr{H}^{n-1} - \int_M f \, \mathrm{div}\, \boldsymbol{v} \, d\mathscr{L}^n$$

holds whenever $f : \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{v} : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz continuous with compact support (see e.g. [2, Theorem 4.5.6] or [9, Theorem 5.8.2]).

Let Ω be an open subset of $\mathbb{R}^n$. We denote by $\mathfrak{M}(\Omega)$ the set of Borel measures $\mu : \mathfrak{B}(\Omega) \to [0, +\infty]$ finite on compact subsets of Ω and by $\mathscr{L}^1_{loc,+}(\Omega)$ the set of Borel functions $h : \Omega \to [0, +\infty]$ with $\int_K h \, d\mathscr{L}^n < +\infty$ for every compact subset $K \subseteq \Omega$.

For a finite-dimensional normed space X, we denote by $\mathscr{L}^1_{loc}(\Omega; X)$ the set of Borel maps $\boldsymbol{v} : \Omega \to X$ with $\int_K \|\boldsymbol{v}\| \, d\mathscr{L}^n < +\infty$ for any compact subset K of Ω. We also denote by $L^1_{loc}(\Omega, \mu)$ the quotient set of Borel functions $f : \Omega \to \mathbb{R}$ such that $\int_K f \, d\mu < +\infty$ for every compact subset $K \subseteq \Omega$, where we identify the functions that agree μ-almost everywhere in Ω.

In the remainder of the section we establish some general properties of measure-theoretic boundary which will be used in the sequel.

Proposition 2.1 *Let M, N be two $\mathscr{L}^n$-measurable subsets of $\mathbb{R}^n$.*

Then we have

$$\left[((\partial_* M) \setminus N_*) \cup ((\partial_* N) \setminus M_*)\right] \setminus (\partial_* M \cap \partial_* N) \subseteq \partial_*(M \cup N) \subseteq ((\partial_* M) \setminus N_*) \cup ((\partial_* N) \setminus M_*) ,$$

$$(N_* \cap \partial_* M) \cup (M_* \cap \partial_* N) \subseteq \partial_*(M \cap N) \subseteq (N_* \cap \partial_* M) \cup (M_* \cap \partial_* N) \cup (\partial_* M \cap \partial_* N) ,$$

$$\left[((\partial_* M) \setminus N_*) \cup (M_* \cap \partial_* N)\right] \setminus (\partial_* M \cap \partial_* N) \subseteq \partial_*(M \setminus N) \subseteq ((\partial_* M) \setminus N_*) \cup (M_* \cap \partial_* N) .$$

Proof. It is well-known that if (and only if) M is $\mathscr{L}^n$-measurable, then $\mathscr{L}^n((M \setminus M_*) \cup (M_* \setminus M)) = 0$. In particular, this implies that $\partial_* M = \partial_*(M_*)$ for every $\mathscr{L}^n$-measurable subset $M \subseteq \mathbb{R}^n$. Thus we can suppose that M and N are normalized. The claimed properties follow now from [4, Lemma 3.2] and [5, Proposition 2.1]. □

The following refines Proposition 2.1, establishing decompositions of the measure-theoretic boundary of $M \cup N$, $M \cap N$ and $M \setminus N$ up to sets of zero surface measure.

Proposition 2.2 *Let M, N be two $\mathscr{L}^n$-measurable subsets of $\mathbb{R}^n$ of finite perimeter and let $A = (\partial_* M \setminus (N_* \cup \partial_* N))$, $B = (\partial_* N \setminus (M_* \cup \partial_* M))$, $C = (M_* \cap \partial_* N)$, $D = (N_* \cap \partial_* M)$,*

$$E = \{x \in \partial_* M \cap \partial_* N : \boldsymbol{n}^M(x) \neq 0, \boldsymbol{n}^N(x) \neq 0, \boldsymbol{n}^M(x) \neq -\boldsymbol{n}^N(x)\},$$
$$F = \{x \in \partial_* M \cap \partial_* N : \boldsymbol{n}^M(x) \neq 0, \boldsymbol{n}^N(x) \neq 0, \boldsymbol{n}^M(x) \neq \boldsymbol{n}^N(x)\}.$$

Then there exist three sets $R_k \subseteq \partial_* M \cap \partial_* N$, for $k = 1, 2, 3$, such that $\mathscr{H}^{n-1}(R_k) = 0$ and

$$\begin{aligned} \partial_*(M \cup N) &= A \cup B \cup E \cup R_1, \\ \partial_*(M \cap N) &= C \cup D \cup E \cup R_2, \\ \partial_*(M \setminus N) &= A \cup C \cup F \cup R_3, \end{aligned}$$

where the unions are disjoint.

Proof. As in Proposition 2.1, we can suppose that M and N are normalized. We start from the last equality. From Proposition 2.1 we have that $\partial_* M \setminus (N \cup \partial_* N) \subseteq \partial_*(M \setminus N)$ and $M \cap \partial_* N \subseteq \partial_*(M \setminus N)$. Let $x \in F$ and consider the cone

$$C_e = \{\xi \in \mathbb{R}^n : (\xi - x) \cdot \boldsymbol{n}^N(x) < 0 < (\xi - x) \cdot \boldsymbol{n}^M(x)\};$$

for every $r > 0$ we have that

$$(C_e \cap \mathrm{B}_x(r)) \subseteq \{\xi \in \mathrm{B}_x(r) \cap M : (\xi - x) \cdot \boldsymbol{n}^M(x) > 0\} \cup [\mathrm{B}_x(r) \setminus (M \setminus N)].$$

By the definition of unit exterior normal vector, this implies that

$$\limsup_{r \to 0^+} \frac{\mathscr{L}^n(\mathrm{B}_x(r) \setminus (M \setminus N))}{\mathscr{L}^n(\mathrm{B}_x(r))} \geqslant \lim_{r \to 0^+} \frac{\mathscr{L}^n(C_e \cap \mathrm{B}_x(r))}{\mathscr{L}^n(\mathrm{B}_x(r))} > 0,$$

hence $x \notin (M \setminus N)_*$.

In the same way, setting

$$C_i = \{\xi \in \mathbb{R}^n : (\xi - x) \cdot \boldsymbol{n}^M(x) < 0 < (\xi - x) \cdot \boldsymbol{n}^N(x)\},$$

for every $r > 0$ one can prove that

$$\begin{aligned} &\left[(C_i \cap \mathrm{B}_x(r)) \setminus \{\xi \in \mathrm{B}_x(r) \setminus M : (\xi - x) \cdot \boldsymbol{n}^M(x) < 0\}\right] \\ &\qquad \setminus \{\xi \in \mathrm{B}_x(r) \cap N : (\xi - x) \cdot \boldsymbol{n}^N(x) > 0\} \subseteq \mathrm{B}_x(r) \cap (M \setminus N), \end{aligned}$$

hence $x \notin (\mathbb{R}^n \setminus (M \setminus N))_*$. Thus $F \subseteq \partial_*(M \setminus N)$.

Now let $x \in \partial_* M \cap \partial_* N$ be such that $\boldsymbol{n}^M(x) = \boldsymbol{n}^N(x) \neq 0$; for every $r > 0$ we have that the set

$$\{\xi \in \mathrm{B}_x(r) \cap M : (\xi - x) \cdot \boldsymbol{n}^M(x) > 0\} \cup \{\xi \in \mathrm{B}_x(r) \setminus N : (\xi - x) \cdot \boldsymbol{n}^N(x) \leqslant 0\}$$

contains $\mathrm{B}_x(r) \cap (M \setminus N)$, hence

$$\lim_{r \to 0^+} \frac{\mathscr{L}^n(\mathrm{B}_x(r) \cap (M \setminus N))}{\mathscr{L}^n(\mathrm{B}_x(r))} = 0.$$

This means that $x \in (\mathbb{R}^n \setminus (M \setminus N))_*$, thus $x \notin \partial_*(M \setminus N)$. Setting

$$R_3 = \partial_*(M \setminus N) \setminus [(\partial_* M \setminus (N \cup \partial_* N)) \cup (M \cap \partial_* N) \cup F],$$

it follows that

$$R_3 \subseteq \{\xi \in \partial_* M \cap \partial_* N : \boldsymbol{n}^M(x) = 0 \text{ or } \boldsymbol{n}^N(x) = 0\}$$

and, by the properties of the unit exterior normal, we have $\mathscr{H}^{n-1}(R_3) = 0$. This prove the last equality.

The other two formulas turn out if we write $M \cap N$ as $M \setminus (\mathbb{R}^n \setminus N)$ and $M \cup N$ as $\mathbb{R}^n \setminus ((\mathbb{R}^n \setminus M) \cap (\mathbb{R}^n \setminus N))$. □

Proposition 2.3 *Let M_1, M_2, M_3 be three mutually disjoint subsets of $\mathbb{R}^n$ of finite perimeter. Then*

$$\mathscr{H}^{n-1}(\partial_* M_1 \cap \partial_* M_2 \cap \partial_* M_3) = 0.$$

Proof. It is an easy consequence of the properties of the unit exterior normal. See e.g. [4, Proposition 3.4]. □

3 Main definitions

Throughout the remainder of this work, B will denote a bounded normalized subset of $\mathbb{R}^n$ of finite perimeter, which we call *a body*.

Definition 3.1 *Let $\mathscr{M}$ be the collection of all normalized subsets of B of finite perimeter. We set (as in [4])*

$$\mathscr{N} = \{C \subseteq \mathbb{R}^n : C \text{ is normalized}, C \in \mathscr{M} \text{ or } (\mathbb{R}^n \setminus C)_* \in \mathscr{M}\},$$

$$\mathfrak{D} = \{(A, C) \in \mathscr{M} \times \mathscr{N} : A \cap C = \varnothing\}.$$

Moreover, we define

$$\mathscr{M}^{loc} = \{A \in \mathscr{M} : \operatorname{cl} A \subseteq \operatorname{int} B\},$$

$$\mathscr{N}^{loc} = \mathscr{M}^{loc} \cup \{A \cup (\mathbb{R}^n \setminus B)_* : A \in \mathscr{M}^{loc}\},$$

$$\mathfrak{D}^{loc} = \{(A, C) \in \mathscr{M}^{loc} \times \mathscr{N}^{loc} : A \cap C = \varnothing\}.$$

Let now $h \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$. We set, following the ideas of [5],

$$\mathscr{M}^{loc}_{h\nu} = \left\{A \in \mathscr{M}^{loc} : \int_{\partial_* A} h \, d\mathscr{H}^{n-1} < +\infty, \ \nu(\partial_* A) = 0\right\},$$

$$\mathscr{N}^{loc}_{h\nu} = \left\{C \in \mathscr{N}^{loc} : (C \cap B) \in \mathscr{M}^{loc}_{h\nu}\right\},$$

$$\mathfrak{D}^{loc}_{h\nu} = \mathfrak{D}^{loc} \cap \left(\mathscr{M}^{loc}_{h\nu} \times \mathscr{N}^{loc}_{h\nu}\right).$$

Remark 3.1 In Definition 3.1 we may assume, without loss of generality, that $h : \operatorname{int} B \to [0, +\infty]$ is a Borel function with $\int_{\operatorname{int} B} h \, d\mathscr{L}^n < +\infty$ and $\nu : \mathfrak{B}(\operatorname{int} B) \to [0, +\infty]$ is a positive Borel measure with $\nu(\operatorname{int} B) < +\infty$. In fact, given an increasing sequence (K_m) of compact subsets of $\operatorname{int} B$ with $\operatorname{int} B = \bigcup_{m=1}^{\infty} \operatorname{int} K_m$, we can set

$$\hat{h}(x) = \begin{cases} \dfrac{h(x)}{1 + \int_{K_1} h \, d\mathscr{L}^n} & \text{if } x \in K_1, \\[2ex] \dfrac{h(x)}{2^{m-1}(1 + \int_{K_m} h \, d\mathscr{L}^n)} & \text{if } x \in K_m \setminus K_{m-1}, \ m \geqslant 2, \end{cases}$$

$$\hat{\nu}(M) = \frac{\nu(M \cap K_1)}{1 + \nu(K_1)} + \sum_{m=2}^{\infty} \frac{\nu(M \cap (K_m \setminus K_{m-1}))}{2^{m-1}(1 + \nu(K_m))} \quad (M \in \mathfrak{B}(\operatorname{int} B)).$$

Then $\hat{h}$, $\hat{\nu}$ have the required properties and $\mathscr{M}^{loc}_{\hat{h}\hat{\nu}} = \mathscr{M}^{loc}_{h\nu}$, $\mathscr{N}^{loc}_{\hat{h}\hat{\nu}} = \mathscr{N}^{loc}_{h\nu}$, $\mathfrak{D}^{loc}_{\hat{h}\hat{\nu}} = \mathfrak{D}^{loc}_{h\nu}$.

Remark 3.2 For every $\eta \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$ we can define a measure $\nu \in \mathfrak{M}(\operatorname{int} B)$ such that $\eta \ll \nu \times \nu$. In fact, we can take an increasing sequence (K_m) of compact subsets of $\operatorname{int} B$ with $\operatorname{int} B = \bigcup_{m=1}^{\infty} \operatorname{int} K_m$ and set

$$\forall E \in \mathfrak{B}(\operatorname{int} B) : \nu(E) = \sum_{m=1}^{\infty} \frac{\eta((E \cap K_m) \times K_m) + \eta(K_m \times (E \cap K_m))}{2^{m-1}(1 + \eta(K_m \times K_m))}.$$

In this way, given $h \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$ we have $\eta((\partial_* A) \times \operatorname{int} B) = \eta((\operatorname{int} B) \times \partial_* A) = 0$ for every $A \in \mathscr{M}^{loc}_{h\nu}$.

Remark 3.3 If $(A, C) \in \mathfrak{D}^{loc}$, then $A \cap \partial_* C = C \cap \partial_* A = \varnothing$. In fact, from Proposition 2.1 we have

$$(A \cap \partial_* C) \cup (C \cap \partial_* A) \subseteq \partial_*(A \cap C) = \varnothing.$$

Definition 3.2 *We say that* $\mathscr{D} \subseteq \mathscr{M}^{loc}$ contains almost all of $\mathscr{M}^{loc}$, *if* $\mathscr{M}^{loc}_{h\nu} \subseteq \mathscr{D}$ *for some* $h \in \mathscr{L}^1_{loc,+}(\mathrm{int}\, B)$ *and* $\nu \in \mathfrak{M}(\mathrm{int}\, B)$.

A property π holds almost everywhere in $\mathscr{M}^{loc}$, *if the set*

$$\left\{A \in \mathscr{M}^{loc} : \pi(A) \text{ is defined and } \pi(A) \text{ holds}\right\}$$

contains almost all of $\mathscr{M}^{loc}$.

We say that $\mathscr{D} \subseteq \mathfrak{D}^{loc}$ contains almost all of $\mathfrak{D}^{loc}$, *if* $\mathfrak{D}^{loc}_{h\nu} \subseteq \mathscr{D}$ *for some* $h \in \mathscr{L}^1_{loc,+}(\mathrm{int}\, B)$ *and* $\nu \in \mathfrak{M}(\mathrm{int}\, B)$.

A property π holds almost everywhere in $\mathfrak{D}^{loc}$, *if the set*

$$\left\{(A, C) \in \mathfrak{D}^{loc} : \pi(A, C) \text{ is defined and } \pi(A, C) \text{ holds}\right\}$$

contains almost all of $\mathfrak{D}^{loc}$.

For a discussion about this concept we refer the reader to [1, Section 3].

Proposition 3.1 *The following assertions hold:*

(a) *if* $h \in \mathscr{L}^1_{loc,+}(\mathrm{int}\, B)$, $\nu \in \mathfrak{M}(\mathrm{int}\, B)$ *and* $M_1, M_2 \in \mathscr{M}^{loc}_{h\nu}$, *then* $(M_1 \cup M_2)_*$, $M_1 \cap M_2$, $(M_1 \setminus M_2)_* \in \mathscr{M}^{loc}_{h\nu}$;
(b) *if* (h_m), (ν_m) *are sequences in* $\mathscr{L}^1_{loc,+}(\mathrm{int}\, B)$ *and* $\mathfrak{M}(\mathrm{int}\, B)$ *respectively, then there exist* $h \in \mathscr{L}^1_{loc,+}(\mathrm{int}\, B)$ *and* $\nu \in \mathfrak{M}(\mathrm{int}\, B)$ *such that*

$$\mathscr{M}^{loc}_{h\nu} \subseteq \bigcap_{m=1}^{\infty} \mathscr{M}^{loc}_{h_m \nu_m}.$$

Proof. Assertion (a) is a simple consequence of Proposition 2.1.

To prove (b), we can take an increasing sequence (K_m) of compact subsets of $\mathrm{int}\, B$ with $\mathrm{int}\, B = \bigcup_{m=1}^{\infty} \mathrm{int}\, K_m$. Setting

$$\forall x \in \mathrm{int}\, B : \quad h(x) = \sum_{m=1}^{\infty} \frac{h_m(x)}{2^m \left(1 + \int_{K_m} h_m \, d\mathscr{L}^n\right)},$$

$$\forall E \in \mathfrak{B}(\mathrm{int}\, B) : \quad \nu(E) = \sum_{m=1}^{\infty} \frac{\nu_m(E)}{2^m (1 + \nu_m(K_m))},$$

it is not difficult to see that h and ν have the required properties. □

Remark 3.4 In view of (b) of Proposition 3.1, given a countable set of properties such that each of them holds on almost all of $\mathscr{M}^{loc}$, there exist $h \in \mathscr{L}^1_{loc,+}(\mathrm{int}\, B)$ and $\nu \in \mathfrak{M}(\mathrm{int}\, B)$ such that they hold on $\mathscr{M}^{loc}_{h\nu}$. The same happens for $\mathscr{N}^{loc}$ and $\mathfrak{D}^{loc}$.

Definition 3.3 *An ordered orthonormal basis* $(e_1, \ldots, e_n)$ *in* $\mathbb{R}^n$ *will be called* a frame. *A frame* $(e_1, \ldots, e_n)$ *is said to be* positively oriented, *if the determinant of the matrix with columns* $e_1, \ldots, e_n$ *is positive.*

A grid G *is an ordered triple*

$$G = \left(x_0, (e_1, \ldots, e_n), \widehat{G}\right),$$

where $x_0 \in \mathbb{R}^n$, $(e_1, \ldots, e_n)$ *is a positively oriented frame in* $\mathbb{R}^n$ *and* $\widehat{G}$ *is a Borel subset of* $\mathbb{R}$. *If* G_1, G_2 *are two grids, we write* $G_1 \subseteq G_2$ *if the first two components coincide and* $\widehat{G}_1 \subseteq \widehat{G}_2$. *A grid* G *is said to be* full, *if* $\mathscr{L}^1(\mathbb{R} \setminus \widehat{G}) = 0$.

Let G *be a grid; a subset* I *of* $\mathbb{R}^n$ *is said to be* an open n-dimensional G-interval, *if*

$$I = \left\{x \in \mathbb{R}^n : \quad a^{(j)} < (x - x_0) \cdot e_j < b^{(j)} \quad \forall j = 1, \ldots, n\right\}$$

for some $a^{(1)}, b^{(1)}, \ldots, a^{(n)}, b^{(n)} \in \widehat{G}$. *We set*

$$\mathscr{I}_G = \{I : I \text{ is an open } n\text{-dimensional } G\text{-interval with } \mathrm{cl}\, I \subseteq \mathrm{int}\, B\},$$

$$\mathscr{M}_G = \left\{ Y : Y = \left(\bigcup_{I \in \mathscr{F}} I \right)_* \text{ for some finite family } \mathscr{F} \text{ in } \mathscr{T}_G \right\},$$

$$\begin{aligned} \mathfrak{D}_G \quad = \quad & \{(A, C) \in \mathfrak{D} : A, C \in \mathscr{T}_G,\, A \cap C = \varnothing\} \\ & \cup \{(A, C \cup (\mathbb{R}^n \setminus B)_*) : A, C \in \mathscr{T}_G,\, A \cap C = \varnothing\}. \end{aligned}$$

Proposition 3.2 *Let $x_0 \in \mathbb{R}^n$ and $(e_1, \ldots, e_n)$ be a positively oriented frame in $\mathbb{R}^n$. Then for every $h \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ there exists a full grid G of the form $G = \left(x_0, (e_1, \ldots, e_n), \widehat{G}\right)$ such that $\mathscr{M}_G \subseteq \mathscr{M}^{loc}_{h\nu}$.*

Proof. See [1, Proposition 4.5]. □

Definition 3.4 *Let $\mathscr{A} \subseteq \mathscr{N}$. We say that a function $F : \mathscr{A} \to \mathbb{R}$ is* additive *if for every $A_1, A_2 \in \mathscr{A}$ such that $(A_1 \cup A_2)_* \in \mathscr{A}$ and $A_1 \cap A_2 = \varnothing$, we have*

$$F\left((A_1 \cup A_2)_*\right) = F(A_1) + F(A_2).$$

Let $\mathscr{D} \subseteq \mathfrak{D}$. We say that a function $F : \mathscr{D} \to \mathbb{R}$ is biadditive *if the functions*

$$F(\,\cdot\,, C) : \{A' \in \mathscr{M} : (A', C) \in \mathscr{D}\} \to \mathbb{R},$$

$$F(A,\,\cdot\,) : \{C' \in \mathscr{N} : (A, C') \in \mathscr{D}\} \to \mathbb{R},$$

are additive for every $(A, C) \in \mathscr{D}$.

We are going to introduce the main character of the paper.

Definition 3.5 *Let $\mathscr{D} \subseteq \mathfrak{D}^{loc}$ be a set containing almost all of $\mathfrak{D}^{loc}$ and let $I : \mathscr{D} \to \mathbb{R}$. We say that I is* a Cauchy interaction, *if the following properties hold:*

(*a*) *I is biadditive;*
(*b*) *there exist $h \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$, $\eta \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$ and $\eta_e \in \mathfrak{M}(\operatorname{int} B)$ such that the inequality*

$$|I(A, C)| \leqslant \begin{cases} \displaystyle\int_{\partial_* A \cap \partial_* C} h \, d\mathscr{H}^{n-1} + \eta(A \times C) & \text{if } C \subseteq B, \\ \displaystyle\int_{\partial_* A \cap \partial_* C} h \, d\mathscr{H}^{n-1} + \eta(A \times (C \cap B)) + \eta_e(A) & \text{otherwise,} \end{cases} \tag{3.1}$$

holds almost everywhere in $\mathfrak{D}^{loc}$.

Remark 3.5 The dichotomy in the previous definition arise from the thermodynamical intuition that the exterior of the body is considered regardless to its structure, but it can interact with the body, like e.g. a heat reservoir. Of course, one can forget the exterior setting $\eta_e = 0$.

Definition 3.6 *A Cauchy interaction I is said to be:*

(*a*) a body interaction, *if in the previous definition we can choose $h = 0$;*
(*b*) a contact interaction, *if in the previous definition we can choose $\eta = 0$ and $\eta_e = 0$.*

4 Decomposition of Cauchy interactions

In this section we will show that Cauchy interactions can be decomposed in an essentially unique way into a sum of a body and a contact interaction, in the sense specified below.

Lemma 4.1 *Let G be a full grid and K be a compact subset of* $\operatorname{int} B$*. Then for every $(A,C) \in \mathfrak{D}^{loc}$ with $C \subseteq B$ there exist two sequences (A_k), (C_k) in $\mathscr{M}_G$ such that* $\operatorname{cl} A_k \cap \operatorname{cl} C_k = \varnothing$ *for every $k \in \mathbb{N}$ and*

$$\lim_k \eta((A_k \triangle A) \times K) = 0, \quad \lim_k \eta(K \times (C_k \triangle C)) = 0, \quad \lim_k \eta_e(A_k \triangle A) = 0,$$

where $\triangle$ denotes the symmetric difference of sets.

Proof. For $k \in \mathbb{N}$ let K_1, K_2 be compact subsets of $\operatorname{int} B$ such that $K_1 \subseteq A$, $K_2 \subseteq C$ and

$$\eta\left((A \setminus K_1) \times K\right) < \frac{1}{k}, \qquad \eta\left(K \times (C \setminus K_2)\right) < \frac{1}{k}, \eta \qquad {}_e(A \setminus K_1) < \frac{1}{k}.$$

Let $A_k, C_k \in \mathscr{M}_G$ with $K_1 \subseteq A_k$, $K_2 \subseteq C_k$, $\operatorname{cl} A_k \cap \operatorname{cl} C_k = \varnothing$, such that

$$\eta\left((A_k \setminus K_1) \times K\right) < \frac{1}{k}, \qquad \eta\left(K \times (C_k \setminus K_2)\right) < \frac{1}{k}, \eta \qquad {}_e(A_k \setminus K_1) < \frac{1}{k}.$$

We have therefore that $A \setminus A_k \subseteq A \setminus K_1$ and $A_k \setminus A \subseteq A_k \setminus K_1$, hence

$$\eta((A_k \triangle A) \times K) < \frac{2}{k}, \qquad \eta_e(A_k \triangle A) < \frac{2}{k}.$$

The same happens for $\eta(K \times (C_k \triangle C))$. □

Theorem 4.1 *Let I be a Cauchy interaction. Then there exist a body interaction I_b and a contact interaction I_c such that $I = I_b + I_c$ on almost all of $\mathfrak{D}^{loc}$.*

Moreover, if there exist a body interaction $\widehat{I_b}$ and a contact interaction $\widehat{I_c}$ such that $I = \widehat{I_b} + \widehat{I_c}$ on almost all of $\mathfrak{D}^{loc}$, then

$$I_b = \widehat{I_b}, \qquad I_c = \widehat{I_c}$$

on almost all of $\mathfrak{D}^{loc}$.

Finally, if I_1, I_2 are two Cauchy interactions that agree, for some full grid G, on $\mathfrak{D}_G$, then $(I_1)_b = (I_2)_b$ on almost all of $\mathfrak{D}^{loc}$.

Proof. Let $h \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$, $\eta \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$ and $\eta_e, \nu \in \mathfrak{M}(\operatorname{int} B)$ be such that $\eta \ll \nu \times \nu$, $\eta_e \ll \nu$, the domain of I contains $\mathfrak{D}^{loc}_{h\nu}$ and 3.1 holds for every $(A,C) \in \mathfrak{D}^{loc}_{h\nu}$, as specified in Remark 3.4. Let H be a full grid as in Proposition 3.2. For $(A,C) \in \mathfrak{D}^{loc}_{h\nu}$ with $C \subseteq B$, there are two compact subsets K_A, K_C of $\operatorname{int} B$ such that $\operatorname{cl} A \subseteq \operatorname{int} K_A$ and $\operatorname{cl} C \subseteq \operatorname{int} K_C$. By Lemma 4.1, consider two sequences (A_k), (C_k) in $\mathscr{M}_H$ such that $\operatorname{cl} A_k \cap \operatorname{cl} C_k = \varnothing$ and

$$\lim_k \eta((A_k \triangle A) \times K_C) = 0, \quad \lim_k \eta(K_A \times (C_k \triangle C)) = 0,$$

$$\lim_k \eta_e(A_k \triangle A) = 0;$$

without loss of generality, we can require that $A_k \subseteq K_A$ and $C_k \subseteq K_C$. It follows from the biadditivity of I and the properties of normalized subsets that

$$\begin{aligned}
|I(A_k, C_k) - I(A_i, C_i)| &= |I((A_k \setminus A_i)_*, C_k) + I(A_k \cap A_i, (C_k \setminus C_i)_*) \\
&\quad - I(A_i, (C_i \setminus C_k)_*) - I((A_i \setminus A_k)_*, C_i \cap C_k)| \\
&\leqslant \eta((A_k \triangle A_i) \times K_C) + \eta(K_A \times (C_k \triangle C_i)) \\
&\leqslant \eta((A_k \triangle A) \times K_C) + \eta(K_A \times (C_k \triangle C)) \\
&\quad + \eta((A_i \triangle A) \times K_C) + \eta(K_A \times (C_i \triangle C)),
\end{aligned}$$

therefore $(I(A_k, C_k))$ is a Cauchy sequence in $\mathbb{R}$. Moreover,

$$
\begin{aligned}
|I(A,C)-I(A_k,C_k)| &\leqslant |I((A\setminus A_k)_*,C)+I(A\cap A_k,(C\setminus C_k)_*)\\
&\quad -I((A_k\setminus A)_*,C_k)-I(A_k\cap A,(C_k\setminus C)_*)|\\
&\leqslant \int_{\partial_*(A\setminus A_k)\cap\partial_* C} h\,d\mathscr{H}^{n-1}+\int_{\partial_*(A\cap A_k)\cap\partial_*(C\setminus C_k)} h\,d\mathscr{H}^{n-1}\\
&\quad +\eta((A_k\triangle A)\times K_C)+\eta(K_A\times(C_k\triangle C))\\
&\leqslant 2\int_{\partial_* A\cap\partial_* C} h\,d\mathscr{H}^{n-1}+\eta((A_k\triangle A)\times K_C)+\eta(K_A\times(C_k\triangle C))\,,
\end{aligned}
\tag{4.1}
$$

where the last inequality follows from Remark 3.3. For every $(A,C)\in\mathfrak{D}^{loc}_{h\nu}$ we define

$$
I_b(A,C)=\begin{cases} \lim\limits_k I(A_k,C_k) & \text{if } C\subseteq B,\\ \lim\limits_k \left[I(A_k,(C\cap B)_k)+I(A,(\mathbb{R}^n\setminus B)_*)\right] & \text{otherwise.}\end{cases}
$$

It is easy to see that I_b does not depend on the chosen sequences. Moreover we have

$$
|I_b(A,C)|\leqslant\begin{cases} \eta(A\times C) & \text{if } C\subseteq B,\\ \eta(A\times(C\cap B))+\eta_e(A) & \text{otherwise,}\end{cases}
$$

since $\partial_* A_k\cap\partial_* C_k=\varnothing$ for every $k\in\mathbb{N}$.

We now show the biadditivity of I_b. Let A,A',C be three mutually disjoint subsets of B such that $(A,C),(A',C)\in\mathfrak{D}^{loc}_{h\nu}$ and let $(A_k),(A'_k),(C_k)$ three sequences in $\mathscr{M}_H$ as in Lemma 4.1. We can require that $\mathrm{cl}\,A_k\cap\mathrm{cl}\,A'_k=\varnothing$. Since

$$
(A\cup A')\triangle(A_k\cup A'_k)\subseteq(A\triangle A_k)\cup(A'\triangle A'_k)\,,
$$
$$
A\cup A'\subseteq(A\cup A')_*\subseteq A\cup A'\cup(\partial_* A\cap\partial_* A')\,,
$$

it follows that

$$
\lim_k \eta\big(((A\cup A')_*\triangle(A_k\cup A'_k))\times K\big)=0
$$

for every compact subset $K\subseteq\mathrm{int}\,B$. Hence

$$
\begin{aligned}
I_b((A\cup A')_*,C) &= \lim_k I((A_k\cup A'_k),C_k)=\lim_k(I(A_k,C_k)+I(A'_k,C_k))\\
&= I_b(A,C)+I_b(A',C)\,.
\end{aligned}
$$

The case $C\nsubseteq B$ is similar. In the same way, we can prove the additivity on the second component, therefore I_b is a body interaction.

Setting

$$
\forall (A,C)\in\mathfrak{D}^{loc}_{h\nu}: I_c(A,C)=I(A,C)-I_b(A,C)\,,
$$

it follows that I_c is a biadditive function on $\mathfrak{D}^{loc}_{h\nu}$; by 4.1 it is a contact interaction.

Now take $h\in\mathscr{L}^1_{loc,+}(\mathrm{int}\,B)$ and $\nu\in\mathfrak{M}(\mathrm{int}\,B)$ such that

$$
I=I_b+I_c=\widehat{I}_b+\widehat{I}_c
$$

on $\mathfrak{D}^{loc}_{h\nu}$. Given $(A,C)\in\mathfrak{D}^{loc}_{h\nu}$, let (A_k), (C_k) be two sequences in $\mathscr{M}_H$ as in Lemma 4.1; we have then

$$
I_c(A_k,C_k)=\widehat{I}_c(A_k,C_k)=0\,.
$$

Passing to the limit as $k\to\infty$, it follows

$$
\forall (A,C)\in\mathfrak{D}^{loc}_{h\nu}: I_b(A,C)=\widehat{I}_b(A,C)\,,
$$

and then also $I_c=\widehat{I}_c$ on $\mathfrak{D}^{loc}_{h\nu}$.

Finally, if two Cauchy interactions I_1, I_2 agree on $\mathfrak{D}_G$ for some full grid G, we can choose $h\in\mathscr{L}^1_{loc,+}(\mathrm{int}\,B)$, $\nu\in\mathfrak{M}(\mathrm{int}\,B)$ and the full grid H in the preceding construction such that I_1, I_2 are defined on $\mathfrak{D}^{loc}_{h\nu}$ and $H\subseteq G$. It follows that $(I_1)_b=(I_2)_b$ on $\mathfrak{D}^{loc}_{h\nu}$. □

5 Body interactions

In this section we will denote by D the set $\{(x,x) : x \in \operatorname{int} B\}$.

The following lemma can be checked by a combinatorial technique.

Lemma 5.1 *Let $x_0 \in \mathbb{R}^n$ and $(e_1, \ldots, e_n)$ be a positively oriented frame in $\mathbb{R}^n$. Let*

$$J_1 = \left\{x \in \mathbb{R}^n : \quad a^{(j)} < (x - x_0) \cdot e_j < b^{(j)} \quad \forall j = 1, \ldots, n\right\} ,$$

$$J_2 = \left\{x \in \mathbb{R}^n : \quad c^{(j)} < (x - x_0) \cdot e_j < d^{(j)} \quad \forall j = 1, \ldots, n\right\}$$

be two open n-dimensional G-intervals such that $J_1 \cap J_2 = \varnothing$ and $(J_1 \cup J_2)_*$ is an open n-dimensional G-interval. Then there exists $i \in \{1, \ldots, n\}$ such that:

(i) *either $b^{(i)} = c^{(i)}$ or $a^{(i)} = d^{(i)}$;*
(ii) *$a^{(j)} = c^{(j)}$ and $b^{(j)} = d^{(j)}$ for every $j \neq i$.*

Theorem 5.1 *Let $\mu_1 \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$, $\mu_2 \in \mathfrak{M}(\operatorname{int} B)$ and let $f \in L^1_{loc}(\operatorname{int} B \times \operatorname{int} B, \mu_1)$, $g \in L^1_{loc}(\operatorname{int} B, \mu_2)$. Then f is μ_1-summable on $A \times (C \cap B)$ and g is μ_2-summable on A for every $(A, C) \in \mathfrak{D}^{loc}$; moreover, the formula*

$$I(A, C) = \begin{cases} \displaystyle\int_{A \times C} f \, d\mu_1 & \text{if } C \subseteq B, \\ \displaystyle\int_{A \times (C \cap \operatorname{int} B)} f \, d\mu_1 + \int_A g \, d\mu_2 & \text{otherwise,} \end{cases}$$

defines a body interaction.

Proof. The summability of f and g is clear. Now let $h = 0$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ be such that $\mu_1 \ll \nu \times \nu$ and $\mu_2 \ll \nu$, which is possible by Remark 3.4. Then I is biadditive on $\mathfrak{D}^{loc}_{h\nu}$. Moreover, setting $\eta = |f| \, d\mu_1$ and $\eta_e = |g| \, d\mu_2$, inequality 3.1 is satisfied, hence I is a body interaction. □

The main result of this section is the converse of Theorem 5.1.

Theorem 5.2 *Let I be a body interaction and $\eta \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$, $\eta_e \in \mathfrak{M}(\operatorname{int} B)$ be as in Definition 3.5. Then there exist $\mu \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$, $\mu_e \in \mathfrak{M}(\operatorname{int} B)$ and two Borel functions $b : \operatorname{int} B \times \operatorname{int} B \to \mathbb{R}$, $b_e : \operatorname{int} B \to \mathbb{R}$ such that*

$$\mu(D) = 0 ,$$

$$|b(x,y)| = 1 \quad \text{for } \mu\text{-a.e. } (x,y) \in \operatorname{int} B \times \operatorname{int} B ,$$

$$|b_e(x)| = 1 \qquad \text{for } \mu_e\text{-a.e. } x \in \operatorname{int} B ,$$

$$I(A, C) = \begin{cases} \displaystyle\int_{A \times C} b \, d\mu & \text{if } C \subseteq B, \\ \displaystyle\int_{A \times (C \cap \operatorname{int} B)} b \, d\mu + \int_A b_e \, d\mu_e & \text{otherwise,} \end{cases} \quad \text{on almost all of } \mathfrak{D}^{loc}.$$

Moreover, we have $\mu \leqslant \eta$ and $\mu_e \leqslant \eta_e$.

Proof. Let $\nu \in \mathfrak{M}(\operatorname{int} B)$ such that $\eta \ll \nu \times \nu$ and the domain of I contains $\mathfrak{D}^{loc}_{h\nu}$. Let $G = \left(x_0, (e_1, \ldots, e_n), \widehat{G}\right)$ be a full grid such that $\mathscr{M}_G \subseteq \mathscr{M}^{loc}_{h\nu}$ and consider the open set $\Omega = (\operatorname{int} B \times \operatorname{int} B) \setminus D$, the full grid

$$\tilde{G} = \left((x_0, x_0), \left((e_1, 0), \ldots, (e_n, 0), (0, e_1), \ldots, (0, e_n)\right), \widehat{G}\right)$$

and the set

$$\mathscr{J}_G = \{J \subseteq \mathbb{R}^{2n} : J \text{ is an open } 2n\text{-dimensional } \tilde{G}\text{-interval with } \operatorname{cl} J \subseteq \Omega\} .$$

Since Ω does not contain the pairs (x,x), it is clear that every $J \in \mathscr{J}_G$ is of the form $J = J_1 \times J_2$ with $J_1, J_2 \in \mathscr{I}_G$, $J_1 \cap J_2 = \varnothing$. By means of this decomposition, we define a function $R : \mathscr{J}_G \to \mathbb{R}$ setting

$$R(J) = I(J_1, J_2) .$$

Let $J, J' \in \mathscr{J}_G$ be such that $(J \cup J')_* \in \mathscr{J}_G$; if $J_1, J_2, J'_1, J'_2 \in \mathscr{I}_G$ are such that $J = J_1 \times J_2$, $J' = J'_1 \times J'_2$, then by Lemma 5.1 we have the following alternative:

(i) either $J_1 \cap J_1' = \varnothing$ and $J_2 = J_2'$,
(ii) or $J_2 \cap J_2' = \varnothing$ and $J_1 = J_1'$.

Suppose for instance that (i) holds; it follows

$$R((J \cup J')_*) = I((J_1 \cup J_1')_*, J_2) = I(J_1, J_2) + I(J_1', J_2) = R(J) + R(J').$$

The same happens in the case (ii), hence R is additive. Moreover, $|R(J)| \leqslant \eta(J_1 \times J_2)$ for every $J = J_1 \times J_2$ in $\mathscr{J}_G$, so R is countably additive. By well-known theorems about extensions of additive functions (see e.g. [6, Chap. 12, Sect. 2]), there exists a unique signed measure $\widehat{\mu}$ on $\mathfrak{B}(\Omega)$ such that

$$\forall J \in \mathscr{J}_G : \widehat{\mu}(J) = R(J),$$

$$\forall E \in \mathfrak{B}(\Omega) : |\widehat{\mu}|(E) \leqslant \eta(E).$$

We define a measure $\mu \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$ setting $\mu(E) = |\widehat{\mu}|(E \cap \Omega)$ for every $E \in \mathfrak{B}(\operatorname{int} B \times \operatorname{int} B)$, and a function $b : \operatorname{int} B \times \operatorname{int} B \to \mathbb{R}$ as $\dfrac{d\widehat{\mu}}{d\mu}$. Clearly, $|b(x,y)| = 1$ μ-a.e. in $\operatorname{int} B \times \operatorname{int} B$ and

$$I(A, C) = \int_{A \times C} b \, d\mu$$

for every $(A, C) \in \mathfrak{D}^{loc}_{h\nu}$ with $C \subseteq B$. Modifying the value of b on a μ-negligible set, we can suppose that b is a Borel function on $\operatorname{int} B \times \operatorname{int} B$ as in the assertion.

Now we define an additive function $R_e : \mathscr{T}_G \to \mathbb{R}$, such that $|R_e(J)| \leqslant \eta_e(J)$, by $R_e(J) = I(J, (\mathbb{R}^n \setminus B)_*)$. Then there exists a signed measure $\widehat{\mu}_e$ on $\mathfrak{B}(\operatorname{int} B)$ such that

$$\forall J \in \mathscr{T}_G : \widehat{\mu}_e(J) = R_e(J),$$

$$\forall A \in \mathfrak{B}(\operatorname{int} B) : |\widehat{\mu}_e|(A) \leqslant \eta_e(A).$$

Setting $\mu_e = |\widehat{\mu}_e|$, we define $b_e = \dfrac{d\widehat{\mu}_e}{d\mu_e}$; as we can suppose that b_e is a Borel function, the proof is complete. □

Theorem 5.3 *Let I_1, I_2 be two body interactions and for $j = 1, 2$ let $\mu^{(j)}$, $\mu_e^{(j)}$, $b^{(j)}$, $b_e^{(j)}$ be as in the statement of Theorem 5.2. Then $I_1 = I_2$ on almost all of $\mathfrak{D}^{loc}$ if and only if $\mu^{(1)} = \mu^{(2)}$, $\mu_e^{(1)} = \mu_e^{(2)}$, $b^{(1)}(x) = b^{(2)}(x)$ $\mu^{(1)}$-a.e. in $\operatorname{int} B \times \operatorname{int} B$ and $b_e^{(1)}(x) = b_e^{(2)}(x)$ $\mu_e^{(1)}$-a.e. in $\operatorname{int} B$.*

Proof. Let $h \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ be such that the equality $I_1 = I_2$ holds in $\mathfrak{D}^{loc}_{h\nu}$. Let G be a full grid such that $\mathscr{M}_G \subseteq \mathscr{M}^{loc}_{h\nu}$. Then, denoting by $R^{(1)}, R^{(2)}$ the functions on $\mathscr{J}_G$ in the proof of Theorem 5.2, we have that $R^{(1)} = R^{(2)}$, hence $\mu^{(1)} = \mu^{(2)}$. In the same way, it follows that $\mu_e^{(1)} = \mu_e^{(2)}$. The remainder of the proof is now easy. □

6 Contact interactions

An oriented surface S in $\mathbb{R}^n$ is a pair $(\widehat{S}, \boldsymbol{n}_S)$, where $\widehat{S}$ is a Borel subset of $\mathbb{R}^n$ and $\boldsymbol{n}_S : \widehat{S} \to \mathbb{R}^n$ is a Borel map such that there exists a normalized set $M \subseteq \mathbb{R}^n$ of finite perimeter with $\widehat{S} \subseteq \partial_* M$ and $\boldsymbol{n}_S = \boldsymbol{n}^M|_{\widehat{S}}$. In this case, we say that S is *subordinated* to M. We call $\boldsymbol{n}_S$ *the normal* to the surface S. If S, T are two oriented surfaces, we shall write $S \subseteq T$ if $\widehat{S} \subseteq \widehat{T}$ and $\boldsymbol{n}_T|_{\widehat{S}} = \boldsymbol{n}_S$. Two oriented surfaces S and T are said to be *disjoint*, if $\widehat{S} \cap \widehat{T} = \varnothing$. They are said to be *compatible*, if there exists a normalized set $M \subseteq \mathbb{R}^n$ of finite perimeter such that S and T are subordinated to M. If S and T are two compatible oriented surfaces, we denote by $S \cup T$ the oriented surface $(\widehat{S} \cup \widehat{T}, \boldsymbol{n}_{S \cup T})$ such that

$$\boldsymbol{n}_{S \cup T}(x) = \begin{cases} \boldsymbol{n}_S(x) & \text{if } x \in \widehat{S}, \\ \boldsymbol{n}_T(x) & \text{if } x \in \widehat{T}. \end{cases}$$

In the following, we shall sometimes identify $\widehat{S}$ with S and we shall consider expressions like, e.g., "S is compact", "$\mathscr{H}^{n-1}(S)$" instead of "$\widehat{S}$ is compact", "$\mathscr{H}^{n-1}(\widehat{S})$". In the same spirit, if S is an oriented surface and T is a Borel subset of $\widehat{S}$, we shall denote by T also the oriented surface $\big(T, \boldsymbol{n}_S|_T\big)$, provided that the reference to S is clear.

Definition 6.1 *Let S be an oriented surface. We say that S is* a material surface in the body B, *if S is subordinated to some $A \in \mathscr{M}$.*

We denote by $\mathscr{S}$ the collection of the material surfaces in the body B.

Definition 6.2 *For every $h \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ we set*

$$\mathscr{S}_{h\nu} = \left\{S \in \mathscr{S} : S \text{ is subordinated to some } A \in \mathscr{M}^{loc}_{h\nu}\right\} .$$

Definition 6.3 *Given a set $\mathscr{A} \subseteq \mathscr{S}$, we say that $\mathscr{A}$* contains almost all of $\mathscr{S}$, *if $\mathscr{S}_{h\nu} \subseteq \mathscr{A}$ for some $h \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$; given a property π, we say that π* holds almost everywhere in $\mathscr{S}$, *if the set*

$$\{S \in \mathscr{S} : \pi(S) \text{ is defined and } \pi(S) \text{ holds}\}$$

contains almost all of $\mathscr{S}$.

Definition 6.4 *For a grid $G = \big(x_0, (e_1 \ldots, e_n), \widehat{G}\big)$ and $1 \leqslant j \leqslant n$, we denote by $\mathscr{S}^j_G$ the family of all the oriented surfaces S with $\boldsymbol{n}_S = e_j$,*

$$\widehat{S} = \left\{x \in \mathbb{R}^n : \quad (x - x_0) \cdot e_j = s \,, \quad a^{(i)} < (x - x_0) \cdot e_i < b^{(i)} \quad \forall i \neq j \right\} ,$$

$a^{(1)}, b^{(1)}, \ldots, s, \ldots, a^{(n)}, b^{(n)} \in \widehat{G}$ and $\operatorname{cl} \widehat{S} \subseteq \operatorname{int} B$. We set also

$$\mathscr{S}_G = \bigcup_{j=1}^{n} \mathscr{S}^j_G .$$

Given a positively oriented frame $(e_1, \ldots, e_n)$ and $x_0 \in \mathbb{R}^n$, for every $h \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$ and $\eta \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$ there exists a full grid $G = \big(x_0, (e_1 \ldots, e_n), \widehat{G}\big)$ such that $\mathscr{S}_G \subseteq \mathscr{S}_{h\nu}$ (see [1, Proposition 4.5]).

Definition 6.5 *Let $\mathscr{A} \subseteq \mathscr{S}$ be a set containing almost all of $\mathscr{S}$ and let $Q : \mathscr{A} \to \mathbb{R}$. We say that Q is* a (scalar) Cauchy flux, *if the following properties hold:*

(*a*) *if $S, T \in \mathscr{A}$ are compatible and disjoint with $S \cup T \in \mathscr{A}$, then*

$$Q(S \cup T) = Q(S) + Q(T) ;$$

(*b*) *there exists $h \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$ such that the inequality*

$$|Q(S)| \leqslant \int_S h \, d\mathscr{H}^{n-1}$$

holds almost everywhere in $\mathscr{S}$.

Lemma 6.1 *Let $h \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$, $\nu \in \mathfrak{M}(\operatorname{int} B)$, $A \in \mathscr{M}^{loc}_{h\nu}$, S be a material surface subordinated to A. Then there exists a sequence (C_k) in $\mathscr{M}^{loc}_{h\nu}$ such that $A \cap C_k = \varnothing$ and*

$$\lim_k \mathscr{H}^{n-1}\left((\partial_* A \cap \partial_* C_k) \triangle \widehat{S}\right) = 0 .$$

Proof. Let G be a full grid such that $\mathscr{M}_G \subseteq \mathscr{M}^{loc}_{h\nu}$. Since $\mathscr{H}^{n-1}(\widehat{S}) < +\infty$, it follows that for any fixed $k \in \mathbb{N}$ there exists a compact subset of $\widehat{S}$, say K, such that

$$\mathscr{H}^{n-1}(\widehat{S} \setminus K) < \frac{1}{k} .$$

Let (Y_m) be a decreasing sequence in $\mathscr{M}_G$ such that $K \subseteq Y_m$ and $K = \bigcap_{m=1}^{\infty} \operatorname{cl} Y_m$. It happens that $\mathscr{H}^{n-1}(\partial_* A \cap \operatorname{cl} Y_1) < +\infty$, then there exists an index m_k with

$$\mathscr{H}^{n-1}((\partial_* A \cap \operatorname{cl} Y_{m_k}) \setminus K) < \frac{1}{k} .$$

Set $C_k = (Y_{m_k} \setminus A)_*$; by Proposition 2.1 it follows that $C_k \in \mathscr{M}^{loc}_{h\nu}$, $A \cap C_k = \varnothing$ and

$$(\partial_* A \cap \partial_* C_k) \setminus \widehat{S} \subseteq (\partial_* A \cap \operatorname{cl} Y_{m_k}) \setminus \widehat{S} \subseteq (\partial_* A \cap \operatorname{cl} Y_{m_k}) \setminus K ,$$

$$\widehat{S} \setminus (\partial_* A \cap \partial_* C_k) \subseteq \widehat{S} \setminus K .$$

Then (C_k) is the desired sequence. □

Lemma 6.2 *Let I be a contact interaction whose domain contains $\mathfrak{D}^{loc}_{h\nu}$. Let $A, A' \in \mathscr{M}^{loc}_{h\nu}$ and S be a material surface subordinated to A and to A'. Let (C_k), (C'_k) be two sequences in $\mathscr{M}^{loc}_{h\nu}$ such that*

$$\lim_k \mathscr{H}^{n-1}\left((\partial_* A \cap \partial_* C_k) \triangle \widehat{S}\right) = 0 ,$$

$$\lim_k \mathscr{H}^{n-1}\left((\partial_* A' \cap \partial_* C'_k) \triangle \widehat{S}\right) = 0 .$$

Then we have

$$\lim_k |I(A, C_k) - I(A', C'_k)| = 0 .$$

Proof. We want to prove that each element of the decomposition

$$\begin{aligned} I(A, C_k) - I(A', C'_k) &= I((A \setminus A')_*, C_k) + I(A \cap A', (C_k \setminus C'_k)_*) + \\ &\quad -I((A' \setminus A)_*, C'_k) - I(A \cap A', (C' \setminus C)_*) \end{aligned}$$

vanishes as $k \to \infty$. By Proposition 2.2 we have that $\mathscr{H}^{n-1}(\partial_*(A \setminus A') \cap \widehat{S}) = 0$, since A and A' share the same unit exterior normal on S. Hence

$$\lim_k \mathscr{H}^{n-1}(\partial_*(A \setminus A') \cap \partial_* C_k) \leqslant \lim_k \mathscr{H}^{n-1}((\partial_* A \cap \partial_* C_k) \setminus \widehat{S}) = 0 ,$$

$$\lim_k I((A \setminus A')_*, C_k) = 0 .$$

On the other hand, by Proposition 2.3 we have

$$\begin{aligned} \mathscr{H}^{n-1}(\partial_*(A \cap A') \cap \partial_*(C_k \setminus C'_k)) &= \mathscr{H}^{n-1}((\partial_*(A \cap A') \cap \partial_*(C_k \setminus C'_k)) \setminus \partial_* C'_k) \\ &\leqslant \mathscr{H}^{n-1}((\partial_* A \cap \partial_* C_k) \setminus \partial_* C'_k) \\ &\leqslant \mathscr{H}^{n-1}((\partial_* A \cap \partial_* C_k) \setminus \widehat{S}) + \mathscr{H}^{n-1}(\widehat{S} \setminus \partial_* C'_k) , \end{aligned}$$

hence

$$\lim_k \mathscr{H}^{n-1}(\partial_*(A \cap A') \cap \partial_*(C_k \setminus C'_k)) = 0 ,$$

$$\lim_k I(A \cap A', (C_k \setminus C'_k)_*) = 0 .$$

In the same way we can show that

$$\lim_k I((A' \setminus A)_*, C'_k) = \lim_k I(A \cap A', (C'_k \setminus C_k)_*) = 0 ,$$

and the proof is complete. □

The next theorem shows that there is a strict correspondence between contact interactions and Cauchy fluxes. For $(A,C)\in\mathfrak{D}^{loc}$, with $\partial_*A\cap\partial_*C$ we will denote also the material surface $\left(\partial_*A\cap\partial_*C\,,\boldsymbol{n}^A|_{\partial_*A\cap\partial_*C}\right)$.

Theorem 6.1 *The following facts hold:*

(i) *for every contact interaction I there exists a Cauchy flux Q such that*

$$Q(\partial_*A\cap\partial_*C)=I(A,C)$$

on almost all of $\mathfrak{D}^{loc}$ and

$$|Q(S)|\leqslant\int_S\hat{h}\,d\mathscr{H}^{n-1}$$

for almost all of $\mathscr{S}$, where $\hat{h}\in\mathscr{L}^1_{loc,+}(\mathrm{int}\,B)$ is as in Definition 3.5;

(ii) *for every Cauchy flux Q there exists a contact interaction I such that*

$$Q(\partial_*A\cap\partial_*C)=I(A,C)\,,\qquad|I(A,C)|\leqslant\int_{\partial_*A\cap\partial_*C}\hat{h}\,d\mathscr{H}^{n-1}$$

on almost all of $\mathfrak{D}^{loc}$, where $\hat{h}\in\mathscr{L}^1_{loc,+}(\mathrm{int}\,B)$ is as in Definition 6.5;

(iii) *if I_1, I_2 are two contact interactions and Q_1, Q_2 are two Cauchy fluxes with*

$$\forall j=1,2:Q_j(\partial_*A\cap\partial_*C)=I_j(A,C)\quad\text{on almost all of }\mathfrak{D}^{loc}\,,$$

then we have $Q_1=Q_2$ on almost all of $\mathscr{S}$ if and only if $I_1=I_2$ on almost all of $\mathfrak{D}^{loc}$.

Proof. (i) Let $h\in\mathscr{L}^1_{loc,+}(\mathrm{int}\,B)$ and $\nu\in\mathfrak{M}(\mathrm{int}\,B)$ be such that the domain of I contains $\mathfrak{D}^{loc}_{h\nu}$. Given a set $S\in\mathscr{S}_{h\nu}$, there exists $A\in\mathscr{M}^{loc}_{h\nu}$ such that S is subordinated to A. Let (C_k) be a sequence as in Lemma 6.1 and $k,i\in\mathbb{N}$. Then $(A,C_k),(A,C_i)\in\mathfrak{D}^{loc}_{h\nu}$ and from Proposition 2.3 we have that $\mathscr{H}^{n-1}(\partial_*A\cap\partial_*(C_k\setminus C_i)\cap\partial_*C_i)=0$. Hence

$$\begin{aligned}\mathscr{H}^{n-1}(\partial_*A\cap\partial_*(C_k\setminus C_i))&=\mathscr{H}^{n-1}((\partial_*A\cap\partial_*(C_k\setminus C_i))\setminus\partial_*C_i)\\&\leqslant\mathscr{H}^{n-1}((\partial_*A\cap(C_k\cup\partial_*C_k))\setminus\partial_*C_i)\\&\leqslant\mathscr{H}^{n-1}((\partial_*A\cap\partial_*C_k)\setminus\widehat{S})+\mathscr{H}^{n-1}(\widehat{S}\setminus(\partial_*A\cap\partial_*C_i))\\&\leqslant\mathscr{H}^{n-1}((\partial_*A\cap\partial_*C_k)\triangle\widehat{S})+\mathscr{H}^{n-1}((\partial_*A\cap\partial_*C_i)\triangle\widehat{S})\end{aligned}$$

and, in the same way,

$$\mathscr{H}^{n-1}(\partial_*A\cap\partial_*(C_i\setminus C_k))\leqslant\mathscr{H}^{n-1}((\partial_*A\cap\partial_*C_i)\triangle\widehat{S})+\mathscr{H}^{n-1}((\partial_*A\cap\partial_*C_k)\triangle\widehat{S})\,.$$

Since we have

$$|I(A,C_k)-I(A,C_i)|=|I(A,(C_k\setminus C_i)_*)-I(A,(C_i\setminus C_k)_*)|\leqslant\int_{\partial_*A\cap\partial_*(C_k\setminus C_i)}\hat{h}\,d\mathscr{H}^{n-1}+\int_{\partial_*A\cap\partial_*(C_i\setminus C_k)}\hat{h}\,d\mathscr{H}^{n-1}\,,$$

it follows that $(I(A,C_k))$ is a Cauchy sequence in $\mathbb{R}$. We set $Q(S)=\lim_k I(A,C_k)$; by Lemma 6.2, $Q(S)$ does not depend on the set A and on the sequence (C_k). Moreover, we have

$$|Q(S)|\leqslant\int_S\hat{h}\,d\mathscr{H}^{n-1}$$

on almost all of $\mathscr{S}$.

Now we prove the additivity. Let S, T be two compatible and disjoint surfaces in $\mathscr{S}_{h\nu}$; following Lemma 6.1, we can construct two sequences (C_k^S) and (C_k^T) such that $\mathrm{cl}\,C_k^S\cap\mathrm{cl}\,C_k^T=\varnothing$ and

$$\lim_k\mathscr{H}^{n-1}((\partial_*A\cap\partial_*C_k^S)\triangle\widehat{S})=0\,,$$

$$\lim_k\mathscr{H}^{n-1}((\partial_*A\cap\partial_*C_k^T)\triangle\widehat{T})=0\,.$$

Moreover we have that $\partial_*(C_k^S \cup C_k^T) = \partial_* C_k^S \cup \partial_* C_k^T$ and

$$\lim_k \mathscr{H}^{n-1}((\partial_* A \cap \partial_*(C_k^S \cup C_k^T)) \triangle (\widehat{S} \cup \widehat{T})) = 0\,,$$

hence

$$Q(S) + Q(T) = \lim_k (I(A, C_k^S) + I(A, C_k^T)) = \lim_k I(A, C_k^S \cup C_k^T) = Q(S \cup T)\,.$$

Then $Q : \mathscr{S}_{h\nu} \to \mathbb{R}$ is a Cauchy flux.
(ii) Let $h \in \mathscr{L}^1_{loc,+}(\mathrm{int}\, B)$ and $\nu \in \mathfrak{M}(\mathrm{int}\, B)$ be such that the domain of Q contains $\mathscr{S}_{h\nu}$. For every $(A, C) \in \mathfrak{D}^{loc}_{h\nu}$, we set

$$I(A, C) = Q(\partial_* A \cap \partial_* C)\,.$$

First, it is clear that

$$|I(A, C)| \leqslant \int_{\partial_* A \cap \partial_* C} \hat{h}\, d\mathscr{H}^{n-1}.$$

Now let $(A_1, C), (A_2, C) \in \mathfrak{D}^{loc}_{h\nu}$ with $A_1 \cap A_2 = \varnothing$. By Proposition 2.3 we observe that $\mathscr{H}^{n-1}(\partial_* A_1 \cap \partial_* A_2 \cap \partial_* C) = 0$ and $\boldsymbol{n}^{A_1}(x) = -\boldsymbol{n}^{A_2}(x)$ for $\mathscr{H}^{n-1}$-a.e. $x \in \partial_* A_1 \cap \partial_* A_2$. Since $A_1 \cap \partial_* A_2 = A_2 \cap \partial_* A_1 = \varnothing$, by Proposition 2.2 it follows

$$Q(\partial_*(A_1 \cup A_2) \cap \partial_* C) = Q(\partial_* A_1 \cap \partial_* C) + Q(\partial_* A_2 \cap \partial_* C)\,,$$

hence I is additive on the first component; the additivity on the other component is similar. Then $I : \mathfrak{D}^{loc}_{h\nu} \to \mathbb{R}$ is a contact interaction.
(iii) Let $h \in \mathscr{L}^1_{loc,+}(\mathrm{int}\, B)$ and $\nu \in \mathfrak{M}(\mathrm{int}\, B)$ be such that the domains of I_j and Q_j contain $\mathfrak{D}^{loc}_{h\nu}$ and $\mathscr{S}_{h\nu}$ respectively, and

$$\forall (A, C) \in \mathfrak{D}^{loc}_{h\nu} : Q_j(\partial_* A \cap \partial_* C) = I_j(A, C)\,,$$

$$\forall S \in \mathscr{S}_{h\nu} : Q_1(S) = Q_2(S)\,.$$

Given $(A, C) \in \mathfrak{D}^{loc}_{h\nu}$, we have that $\partial_* A \cap \partial_* C \in \mathscr{S}_{h\nu}$, hence

$$I_1(A, C) = Q_1(\partial_* A \cap \partial_* C) = Q_2(\partial_* A \cap \partial_* C) = I_2(A, C)\,.$$

On the other hand, let $h \in \mathscr{L}^1_{loc,+}(\mathrm{int}\, B)$ and $\nu \in \mathfrak{M}(\mathrm{int}\, B)$ be such that the domains of I_j and Q_j contain $\mathfrak{D}^{loc}_{h\nu}$ and $\mathscr{S}_{h\nu}$ respectively and

$$Q_j(\partial_* A \cap \partial_* C) = I_j(A, C)\,, \qquad I_1(A, C) = I_2(A, C)\,,$$

for every $(A, C) \in \mathfrak{D}^{loc}_{h\nu}$. Let $S \in \mathscr{S}_{h\nu}$; then there exists $A \in \mathscr{M}^{loc}_{h\nu}$ such that S is subordinated to A. Let (C_k) be a sequence with

$$\lim_k \mathscr{H}^{n-1}((\partial_* A \cap \partial_* C_k) \triangle \widehat{S}) = 0\,;$$

for $j = 1, 2$ we have that $\partial_* A \cap \partial_* C_k \in \mathscr{S}_{h\nu}$ and $Q_j(S) = \lim_k Q_j(\partial_* A \cap \partial_* C_k)$. Since $(A, C_k) \in \mathfrak{D}^{loc}_{h\nu}$, it follows that

$$Q_1(S) = \lim_k I_1(A, C_k) = \lim_k I_2(A, C_k) = Q_2(S)\,,$$

and the proof is complete. □

7 Balanced interactions

In this section we will study the case in which I obeys a balance law, as specified below. In [4], such a balance is expressed by the inequality

$$\exists K \geqslant 0 : |I(A, (\mathbb{R}^n \setminus A)_*)| \leqslant K\mathscr{L}^n(A) .$$

In view of the other assumptions of [4], such an inequality is in turn equivalent to

$$\exists K \geqslant 0 : |I(A, C)| \leqslant K\mathscr{L}^n(A)$$

whenever $(A, C) \in \mathfrak{D}$ and $\partial_* A \subseteq \partial_* C$ (so that between A and $(\mathbb{R}^n \setminus (A \cup C))_*$ there is no contact interaction).

The purpose of the next Definition 7.1 is to generalize and adapt such a condition to our setting. However, we will see in Theorem 7.4 that, in the balanced case, also the interaction $I(A, (\mathbb{R}^n \setminus A)_*)$ can be naturally defined and is subjected to an inequality of the form

$$\exists \lambda \in \mathfrak{M}(\operatorname{int} B) : |I(A, (\mathbb{R}^n \setminus A)_*)| \leqslant \lambda(A) .$$

Definition 7.1 *A Cauchy interaction I is said to be* balanced, *if there exists $\lambda \in \mathfrak{M}(\operatorname{int} B)$ such that*

$$\partial_* A \subseteq \partial_* C \Longrightarrow |I(A, C)| \leqslant \lambda(A) \tag{7.1}$$

on almost all of $\mathfrak{D}^{loc}$. A Cauchy flux Q is said to be balanced, *if there exists $\lambda \in \mathfrak{M}(\operatorname{int} B)$ such that*

$$|Q(\partial_* A)| \leqslant \lambda(A)$$

on almost all of $\mathscr{M}^{loc}$.

Theorem 7.1 *The following properties hold:*

(i) *a Cauchy interaction I is balanced if and only if I_b and I_c are both balanced;*
(ii) *a body interaction I is balanced if and only if $\mu(K \times \operatorname{int} B) < +\infty$ for each compact subset $K \subseteq \operatorname{int} B$, where μ is given by Theorem 5.2; if this is the case, one has*

$$|I(A, C)| \leqslant \lambda(A)$$

on almost all of $\mathfrak{D}^{loc}$;
(iii) *a contact interaction I is balanced if and only if the Cauchy flux induced by I is balanced.*

Proof. (i) Let $\lambda \in \mathfrak{M}(\operatorname{int} B)$ be as in Definition 7.1 and let h, ν as in the proof of Theorem 4.1 with $\lambda \leqslant \nu$. Let also H be as in the proof of Theorem 4.1. If $A, C \in \mathscr{M}_H$ and $\operatorname{cl} A \cap \operatorname{cl} C = \varnothing$, let $\widehat{C} \in \mathscr{M}_H$ be such that $(A \cup C) \cap \widehat{C} = \varnothing$ and $\partial_* A \subseteq \partial_* \widehat{C}$. It follows that $\partial_* A \subseteq \partial_*(C \cup \widehat{C})$, hence

$$|I(A, C)| \leqslant |I(A, (C \cup \widehat{C})_*)| + |I(A, \widehat{C})| \leqslant 2\lambda(A) .$$

Let now $(A, C) \in \mathfrak{D}^{loc}_{h\eta}$ with $C \subseteq B$ and let (A_k, C_k) be a sequence as in the proof of Theorem 4.1 such that $\lim \lambda(A_k \triangle A) = 0$. We have that $|I(A_k, C_k)| \leqslant 2\lambda(A_k)$, then

$$|I_b(A, C)| \leqslant 2\lambda(A) . \tag{7.2}$$

If $C \not\subseteq B$, inequality 7.2 still holds, since we can find again a similar $\widehat{C}$.

In particular, I_b and I_c are both balanced. The converse is obvious.

(ii) Let I be a balanced body interaction. From 7.2 it follows that

$$|I(A, C)| \leqslant 2\lambda(A)$$

on almost all of $\mathfrak{D}^{loc}$. Let $h \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ be such that Theorem 5.2 and the preceding inequality hold on $\mathfrak{D}^{loc}_{h\nu}$. Let G be a full grid such that $\mathfrak{D}_G \subseteq \mathfrak{D}^{loc}_{h\nu}$. We denote with P the set on which $b = 1$ and with Q a normalized finite union of $2n$-dimensional G-intervals such that $\mu(P \triangle Q) < 1$. Let K be a compact subset of $\operatorname{int} B$ and let $Y \in \mathscr{M}_G$ be such that $K \subseteq Y$; clearly $\mu(Y \times Y) < +\infty$. Setting $E = (\operatorname{int} B) \setminus Y$, it is enough to prove that $\mu(Y \times E) < +\infty$; we argue by contradiction, supposing $\mu(Y \times E) = +\infty$. For every $m \in \mathbb{N}$ there exists a set $F_m \in \mathscr{M}_G$ with $\mathrm{cl}\, F_m \cap \mathrm{cl}\, Y = \varnothing$ and $\mu(Y \times F_m) > m$. The set $(Y \times F_m) \cap Q$ is a normalized finite union of $2n$-dimensional G-intervals, hence we can find some sets $Y_k \in \mathscr{T}_G$ and $G_k \in \mathscr{M}_G$ such that the Y_k's are mutually disjoint and

$$(Y \times F_m) \cap Q = \left(\bigcup_{k=1}^{q} (Y_k \times G_k) \right)_* .$$

In the same way,

$$((Y \times F_m) \setminus Q)_* = \left(\bigcup_{k=1}^{p} (Y'_k \times G'_k) \right)_* ,$$

where $Y'_k \in \mathscr{T}_G$ are mutually disjoint and $G'_k \in \mathscr{M}_G$. We have

$$2\lambda(Y) \geqslant 2\lambda \left(\left(\bigcup_{k=1}^{q} Y_k \right)_* \right) \geqslant \left| \sum_{k=1}^{q} I(Y_k, G_k) \right| = \left| \int_{(Y \times F_m) \cap Q} b \, d\mu \right| \geqslant \mu((Y \times F_m) \cap Q) - 2 .$$

Acting in the same way, we can prove that

$$2\lambda(Y) \geqslant \mu\big(((Y \times F_m) \setminus Q)_*\big) - 2 .$$

Adding the two inequalities we find that

$$4\lambda(Y) \geqslant \mu(Y \times F_m) - 4 \geqslant m - 4 ;$$

since Y has compact closure in $\mathrm{int}\, B$, letting $m \to +\infty$ we get the contradiction.

Conversely, suppose that $\mu(K \times \mathrm{int}\, B) < +\infty$ for every compact subset $K \subseteq \mathrm{int}\, B$ and consider the measure $\lambda = \mu(\,\cdot\, \times \mathrm{int}\, B) + \mu_e$; it follows immediately that $\lambda \in \mathfrak{M}(\mathrm{int}\, B)$ and

$$|I(A, C)| \leqslant \left| \int_{A \times (C \cap \mathrm{int}\, B)} b \, d\mu \right| + \left| \int_A b_e \, d\mu_e \right| \leqslant \mu(A \times (C \cap B)) + \mu_e(A) \leqslant \lambda(A)$$

on almost all of $\mathfrak{D}^{loc}$, hence I is balanced.
(*iii*) It is obvious. □

Theorem 7.2 *Let I_1, I_2 be two balanced Cauchy interactions that agree on $\mathfrak{D}_G$ for some full grid G. Then $I_1 = I_2$ on almost all of $\mathfrak{D}^{loc}$.*

Proof. Let $I_1 = (I_1)_b + (I_1)_c$, $I_2 = (I_2)_b + (I_2)_c$ where $(I_j)_b$ are body interactions and $(I_j)_c$ contact interactions. From Theorem 4.1, we have that $(I_1)_b = (I_2)_b$ on almost all of $\mathfrak{D}^{loc}$; in particular, there exists a full grid H such that $(I_1)_c = (I_2)_c$ on $\mathscr{T}_H$. Defining two Cauchy fluxes Q_1, Q_2 by the formula

$$Q_j(\partial_* A \cap \partial_* C) = (I_j)_c(A, C)$$

as in (*a*) of Theorem 6.1, it follows that Q_1 and Q_2 are balanced and agree on $\mathscr{S}_H$. Hence they agree on almost all of $\mathscr{S}$ by [1, Theorem 4.9]. By (*c*) of Theorem 6.1, it comes that $(I_1)_c = (I_2)_c$ on almost all of $\mathfrak{D}^{loc}$. □

Theorem 7.3 *Let I be a balanced contact interaction. Then there exists a vector field $\boldsymbol{q} \in \mathscr{L}^1_{loc}(\mathrm{int}\, B; \mathbb{R}^n)$ with divergence measure such that*

$$I(A, C) = \int_{\partial_* A \cap \partial_* C} \boldsymbol{q} \cdot \boldsymbol{n}_{\partial_* A \cap \partial_* C} \, d\mathscr{H}^{n-1}$$

on almost all of $\mathfrak{D}^{loc}$.

Moreover, $\boldsymbol{q}$ is uniquely determined $\mathscr{L}^n$-almost everywhere.

Proof. Let Q be a Cauchy flux such that

$$Q(\partial_* A \cap \partial_* C) = I(A, C)$$

on almost all of $\mathfrak{D}^{loc}$, as in (a) of Theorem 6.1. Since I is balanced, then Q is also balanced. Moreover, Q is uniquely determined on almost all of $\mathscr{S}$.

Now we can apply [1, Theorem 7.1] and obtain the assertion. □

For a balanced Cauchy interaction I we can give the following integral representation.

Theorem 7.4 *Let I be a balanced Cauchy interaction and let b, b_e, μ, μ_e and $\boldsymbol{q}$ as in Theorems 5.2 and 7.3. Then there exist $h \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ such that*

$$I(A,C) = \begin{cases} \displaystyle\int_{A\times C} b\,d\mu + \int_{\partial_* A\cap\partial_* C} \boldsymbol{q}\cdot\boldsymbol{n}_{\partial_* A\cap\partial_* C}\, d\mathscr{H}^{n-1} & \text{if } C \subseteq B, \\ \displaystyle\int_{A\times(C\cap \operatorname{int} B)} b\,d\mu + \int_A b_e\, d\mu_e + \int_{\partial_* A\cap\partial_* C} \boldsymbol{q}\cdot\boldsymbol{n}_{\partial_* A\cap\partial_* C}\, d\mathscr{H}^{n-1} & \text{otherwise,} \end{cases} \tag{7.3}$$

for every $(A, C) \in \mathfrak{D}^{loc}_{h\nu}$ and the same formula admits a natural extension to all

$$\mathfrak{D}^{loc}_{h\nu} \cup \{(A,C) \in \mathscr{M}^{loc}_{h\nu} \times \mathscr{N} : (\mathbb{R}^n \setminus C)_* \in \mathscr{M}^{loc}_{h\nu}, A\cap C = \varnothing\}\,.$$

Moreover, there exists $\lambda \in \mathfrak{M}(\operatorname{int} B)$ such that

$$\forall A \in \mathscr{M}^{loc}_{h\nu} : |I(A, (\mathbb{R}^n \setminus A)_*)| \leqslant \lambda(A)\,.$$

Proof. Let $h_0 \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$, $\nu \in \mathfrak{M}(\operatorname{int} B)$ and $\lambda \in \mathfrak{M}(\operatorname{int} B)$ be such that 7.1 and Theorems 5.2 and 7.3 hold on $\mathfrak{D}^{loc}_{h_0\nu}$. Then it is easy to deduce 7.3. Setting $h = h_0 + |\boldsymbol{q}|$ and remembering that $\mu(K \times \operatorname{int} B) < +\infty$ for every compact subset $K \subseteq \operatorname{int} B$, it is possible to extend the domain of I as stated in the assertion.

Moreover, let G be a full grid with $\mathscr{M}_G \subseteq \mathscr{M}^{loc}_{h\nu}$. For a given $A \in \mathscr{M}^{loc}_{h\nu}$, we can find a sequence (Y_k) in $\mathscr{M}_G$ such that $\operatorname{cl} A \subseteq Y_k$ and $\bigcup_{k=1}^{\infty} Y_k = \operatorname{int} B$. As I is balanced, we have

$$|I(A, (Y_k \setminus A)_* \cup (\mathbb{R}^n \setminus B)_*)| \leqslant \lambda(A)\,,$$

and the left member goes to $|I(A, (\mathbb{R}^n \setminus A)_*)|$ by the Dominated Convergence Theorem. □

Finally, we can state a weak form of the balance equation for a balanced Cauchy interaction.

Theorem 7.5 *Let I be a balanced Cauchy interaction and let $\mu, \mu_e, b, b_e, \boldsymbol{q}$ be as in Theorems 5.2 and 7.3. Then there exist $h \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$, $\nu \in \mathfrak{M}(\operatorname{int} B)$, $\gamma \in \mathfrak{M}(\operatorname{int} B)$ and a Borel function $c : \operatorname{int} B \to \mathbb{R}$ such that $|c(x)| = 1$ for γ-a.e. $x \in \operatorname{int} B$ and*

$$\int_A c\, d\gamma = I(A, (\mathbb{R}^n \setminus A)_*) + \int_{A\times A} b\, d\mu$$

for every $A \in \mathscr{M}^{loc}_{h\nu}$.

Moreover, γ is uniquely determined and c is uniquely determined γ-a.e.

Finally, one has

$$\int_{\operatorname{int} B} f\, c\, d\gamma = -\int_{\operatorname{int} B} \boldsymbol{q}\cdot\nabla f\, d\mathscr{L}^n + \int_{\operatorname{int} B} f\, b_e\, d\mu_e + \int\int_{\operatorname{int} B\times\operatorname{int} B} f(x)\, b(x,y)\, d\mu(x,y)$$

for every $f \in C^\infty_0(\operatorname{int} B)$.

Proof. Let $h \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ be as in Theorem 7.4; then we can define a function $g : \mathscr{M}^{loc}_{h\nu} \to \mathbb{R}$ setting

$$g(A) = \int_A \operatorname{div} \boldsymbol{q} + \int_{A \times \operatorname{int} B} b \, d\mu + \int_A b_e \, d\mu_e \, .$$

Extending g to a (signed) measure on $\mathfrak{B}(\operatorname{int} B)$, we can find $\gamma \in \mathfrak{M}(\operatorname{int} B)$ and a Borel function $c : \operatorname{int} B \to \mathbb{R}$ such that $|c(x)| = 1$ for γ-a.e. $x \in \operatorname{int} B$ and

$$\int_A c \, d\gamma = g(A)$$

for every $A \in \mathscr{M}^{loc}_{h\nu}$. The measure γ is clearly unique and the function c is uniquely determined γ-a.e.

The last assertion follows from the Gauss-Green Theorem. □

8 An extension result

Although the domain of a Cauchy interaction is quite large, in this section we will prove that each function defined only on $\mathfrak{D}_G$, for some full grid G, and satisfying suitable conditions, can be uniquely extended to almost all of $\mathfrak{D}^{loc}$.

Let $G_0 = (x_0, (e_1, \ldots, e_n), \hat{G}_0)$ denote a full grid and $I_0 : \mathfrak{D}_{G_0} \to \mathbb{R}$ a map satisfying the following properties:

(a) I_0 is biadditive;
(b) there exist $h \in \mathscr{L}^1_{loc,+}(\operatorname{int} B)$, $\eta \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$ and $\eta_e \in \mathfrak{M}(\operatorname{int} B)$ such that

$$|I_0(A, C)| \leqslant \begin{cases} \displaystyle\int_{\partial_* A \cap \partial_* C} h \, d\mathscr{H}^{n-1} + \eta(A \times C) & \text{if } C \subseteq B, \\ \displaystyle\int_{\partial_* A \cap \partial_* C} h \, d\mathscr{H}^{n-1} + \eta(A \times (C \cap B)) + \eta_e(A) & \text{otherwise,} \end{cases}$$

for every $(A, C) \in \mathfrak{D}_{G_0}$.

Theorem 8.1 *There exist a full grid $G \subseteq G_0$ and two functions $(I_0)_b, (I_0)_c : \mathfrak{D}_G \to \mathbb{R}$ satisfying properties (a) and (b) for every $(A, C) \in \mathfrak{D}_G$ with $h = 0$ and $\eta = 0, \eta_e = 0$ respectively, such that $I_0 = (I_0)_b + (I_0)_c$ on $\mathfrak{D}_G$.*

Moreover, if $\check{G}$, $(\check{I}_0)_b$ and $(\check{I}_0)_c$ have the same properties, then $(\check{I}_0)_b = (I_0)_b$ and $(\check{I}_0)_c = (I_0)_c$ on $\mathfrak{D}_G \cap \mathfrak{D}_{\check{G}}$.

Proof. Let G be a full grid such that $G \subseteq G_0$ and $\int_{\partial_* A} h \, d\mathscr{H}^{n-1} < +\infty$, $\eta((\partial_* A) \times \operatorname{int} B) = \eta((\operatorname{int} B) \times \partial_* A) = \eta_e(\partial_* A) = 0$ for every $A \in \mathscr{T}_G$. Let $(A, C) \in \mathfrak{D}_G$; then

$$A = \left\{x \in \mathbb{R}^n : \quad a^{(j)} < (x - x_0) \cdot e_j < b^{(j)} \quad \forall j = 1, \ldots, n\right\} ,$$

$$C = \left\{x \in \mathbb{R}^n : \quad c^{(j)} < (x - x_0) \cdot e_j < d^{(j)} \quad \forall j = 1, \ldots, n\right\} ,$$

for some $a^{(j)}, b^{(j)}, c^{(j)}, d^{(j)} \in G$. If $\partial_* A \cap \partial_* C = \varnothing$, then we set $(I_0)_b(A, C) = I_0(A, C)$ and $(I_0)_c(A, C) = 0$. Elsewhere, denote by i the index in $\{1, \ldots, n\}$ such that

$$\partial_* A \cap \partial_* C \subseteq \{x \in \mathbb{R}^n : \quad x \cdot e_i = 0\}$$

and suppose that $b^{(i)} \leqslant c^{(i)}$. Let (s_k) be a sequence in G such that $s_k \downarrow c^{(i)}$ as $k \to \infty$. We set

$$C_k = C \cap \{x \in \mathbb{R}^n : \quad x \cdot e_i > s_k\} \, .$$

Then it is clear that $(A, C_k) \in \mathfrak{D}_G$ for every $k \in \mathbb{N}$, $(I_0(A, C_k))$ is a Cauchy sequence in $\mathbb{R}$ and $|I_0(A, C_k)| \leqslant \eta(A \times C_k)$. Moreover,

$$|I_0(A, C) - I_0(A, C_k)| \leqslant |I_0(A, (C \setminus C_k)_*)| \leqslant \int_{\partial_* A \cap \partial_* (C \setminus C_k)} h \, d\mathscr{H}^{n-1} + \eta(A \times (C \setminus C_k)) \, .$$

We define

$$(I_0)_b(A,C) = \begin{cases} \lim_k I_0(A, C_k) & \text{if } C \subseteq B, \\ I_0(A, (\mathbb{R}^n \setminus B)_*) + \lim_k I_0(A, (C \cap B)_k) & \text{otherwise,} \end{cases}$$

and also

$$(I_0)_c(A,C) = I_0(A,C) - (I_0)_b(A,C)\,.$$

Then $(I_0)_b$ and $(I_0)_c$ satisfy (a) and (b) with $h = 0$ and $\eta = 0, \eta_e = 0$ respectively. The remainder of the proof is now easy. □

Theorem 8.2 *Let $I_0 : \mathfrak{D}_{G_0} \to \mathbb{R}$ be a map satisfying properties (a) and (b) with $h = 0$. Then there exists a body interaction I such that:*

(i) *its domain contains $\mathfrak{D}_{G_0}$;*
(ii) *it coincides with I_0 on $\mathfrak{D}_{G_0}$.*

Moreover, if another body interaction $\check{I}$ shares properties (i) and (ii), then $\check{I} = I$ on almost all of $\mathfrak{D}^{loc}$.

Proof. Following the proof of Theorem 5.2, we find $\mu \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$ and $b \in L^1_{loc}(\operatorname{int} B \times \operatorname{int} B, \mu)$ such that

$$I_0(A,C) = \int_{A\times C} b \, d\mu$$

for every $(A,C) \in \mathfrak{D}_{G_0}$ with $C \subseteq B$. In the same way we find $\mu_e \in \mathfrak{M}(\operatorname{int} B)$ and $b_e \in L^1_{loc}(\operatorname{int} B, \mu_e)$ such that

$$I_0(A, (\mathbb{R}^n \setminus B)_*) = \int_A b_e \, d\mu_e\,.$$

Defining, whenever possible,

$$I(A,C) = \begin{cases} \displaystyle\int_{A\times C} b\,d\mu & \text{if } C \subseteq B, \\ \displaystyle\int_{A\times(C\cap B)} b\,d\mu + \int_A b_e\,d\mu_e & \text{otherwise,} \end{cases}$$

we have that the domain of I contains $\mathfrak{D}_{G_0}$, I is a body interaction by Theorem 5.1 and

$$I_0(A,C) = I(A,C) \quad \text{for every } (A,C) \in \mathfrak{D}_{G_0}\,.$$

If $\check{I}$ is another body interaction that extends I_0, it is obvious that $\check{I}(A,C) = I(A,C)$ for every $(A,C) \in \mathfrak{D}_{G_0}$; then by Theorem 4.1 we have that $\check{I} = I$ on almost all of $\mathfrak{D}^{loc}$. □

Now we require the map I_0 to satisfy also the following balance property:

(c) there exists $\lambda \in \mathfrak{M}(\operatorname{int} B)$ such that

$$\left| \sum_{j=1}^{k} I_0(A, C^{(j)}) \right| \leqslant \lambda(A)$$

whenever $(A, C^{(j)}) \in \mathfrak{D}_{G_0}$ for every $j = 1, \ldots, k$, the sets $C^{(j)}$ are mutually disjoint and $\partial_* A \subseteq \partial_* \left(\bigcup_{j=1}^{k} C^{(j)} \right)$.

Theorem 8.3 *Consider the full grid G and the maps $(I_0)_b$ and $(I_0)_c$ of Theorem 8.1; consider also the extension I_b of $(I_0)_b$, as stated in Theorem 8.2. Then the following facts hold:*

(i) *there exist a balanced contact interaction I_c and a full grid $H \subseteq G$ such that the domain of I_c contains $\mathfrak{D}_H$ and $I_c = (I_0)_c$ on $\mathfrak{D}_H$; moreover, if $\check{H}$ and $\check{I}_c$ have the same properties of H and I_c, then $\check{I}_c = I_c$ on almost all of $\mathfrak{D}^{loc}$;*

(ii) I_b is balanced.

Proof. (i) First of all, we will prove that $(I_0)_c$ satisfies property (c). In fact, for $j = 1, \dots, k$ let $(A, C^{(j)}) \in \mathfrak{D}_G$ be such that the sets $C^{(j)}$ are mutually disjoint and $\partial_* A \subseteq \partial_* \left(\bigcup_{j=1}^{k} C^{(j)} \right)$; then consider the sequences $(C_k^{(j)})$ as in the proof of Theorem 8.1. We have that

$$(I_0)_c(A, C^{(j)}) = \lim_k I_0(A, (C^{(j)} \setminus C_k^{(j)})_*)$$

and $\partial_* A \subseteq \partial_* \left(\bigcup_{j=1}^{k} (C^{(j)} \setminus C_k^{(j)})_*) \right)$; hence

$$\left| \sum_{j=1}^{k} (I_0)_c(A, C_j) \right| = \lim_k \left| \sum_{j=1}^{k} I_0(A, (C^{(j)} \setminus C_k^{(j)})_*) \right| \leqslant \lambda(A) \, .$$

Now let $S \in \mathscr{S}_G$; then there exists $(A, C) \in \mathfrak{D}_G$ such that $(\partial_* A \cap \partial_* C \, , \boldsymbol{n}^A|_{\partial_* A \cap \partial_* C}) = (S \, , \boldsymbol{n}_S)$. If $(\hat{A}, \hat{C})$ has the same property, by biadditivity of $(I_0)_c$ and properties (a) and (b) it is easy to prove that

$$(I_0)_c(A, C) = (I_0)_c(A \cap \hat{A}, C \cap \hat{C}) = (I_0)_c(\hat{A}, \hat{C}) \, .$$

This allows us to define the map

$$\begin{aligned} Q_0 : \qquad & \mathscr{S}_G \longrightarrow \mathbb{R} \\ & S \mapsto (I_0)_c(A, C) \, , \end{aligned}$$

which happens to satisfy (i), (ii) and (iii) of [1, sect. 6].

Combining [1, Theorem 6.1] with Theorems 6.1 and 7.1, it results that there exist a balanced contact interaction I_c and a full grid $H \subseteq G$ such that the domain of I_c contains $\mathfrak{D}_H$ and

$$I_c(A, C) = (I_0)_c(A, C) \quad \text{for every } (A, C) \in \mathfrak{D}_H \, .$$

(ii) This is easily proved noting that, by difference, also $(I_0)_b$ satisfies property (c). □

It is appropriate to summarize Theorems 8.1, 8.2 and 8.3 in the following statement.

Corollary 8.1 *There exists a full grid $G \subseteq G_0$ and a balanced Cauchy interaction I such that the domain of I contains $\mathfrak{D}_G$ and $I = I_0$ on $\mathfrak{D}_G$.*

Moreover, if $\check{G}$ and $\check{I}$ have the same properties of G and I, then $\check{I} = I$ on almost all of $\mathfrak{D}^{loc}$.

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