Stationary and evolutionary flows of nonsimple second order fluids: existence and uniqueness

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Abstract. In this paper we introduce a particular class of nonsimple fluids of second order. Within the general framework of virtual powers, we deduce the dynamical equation for linear isotropic incompressible fluids. Afterwards, existence and uniqueness results are obtained for both the stationary flow and the evolutionary one, with non homogeneous boundary conditions.

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1. Introduction

Nonsimple second order fluids (not to be confused with *second-grade* fluids, which are simple fluids with a quadratic dependence of the stress tensor on the velocity gradient) constitute a class of non-Newtonian fluids which has been introduced quite recently. They are characterized by a dependence of the internal stress tensor up to the second order spatial derivatives of the velocity. Even in the simplest case of a linear isotropic incompressible fluid, the model gives rise to some new parameters, which can incorporate a description of phenomena at different scales, being associated with a characteristic length. Moreover, when dealing with higher order continuous media, further boundary conditions have to be specified: as in the paper of Fried and Gurtin [4], we are led to introduce an *adherence length*, which is useful to encode turbulence effects in the small-scale interaction between the fluid and the boundary. As another point of interest, the addition of second-gradient terms in the flow equation is often employed in numerical analysis to stabilize approximation methods.

Flow equations with higher order derivatives of the velocity have been widely studied in the literature, for instance as a perturbation to the Navier-Stokes equation. However, to our knowledge, the remarkable paper [4] is the first attempt to deduce them from an axiomatic model in Continuum Mechanics; very recently, it has become apparent that the virtual power approach used there can suggest a classification of the continuous media, settled before the usual constitutive prescriptions, by means of some *dynamical prescriptions*. We will adopt the terminology of Podio-Guidugli [9], which proposed to call *simple* the media of first order (that is, when the corresponding internal power depends only on the first gradient of the velocity), and *nonsimple* all the others.

In the present paper, we face again the issue of setting our dynamical problem by the method of virtual powers, which proved to be a powerful tool in Continuum Mechanics. This method naturally allows to deduce the structure of the system of forces in a nonsimple continuous medium. The balance laws are directly given in the weak form requested by the modern functional methods one uses to prove well-posedness results; moreover, the same functional framework allows to construct finite element approximation schemes. As pointed out in [7], the method of virtual powers also provides a relatively simple treatment of concentrated stresses and external loads, and can be easily widened to the case of a manifold as continuous body, since the virtual power is a scalar (stresses on a manifold are, on the contrary, more complicated objects).

The main result of the paper is the existence and uniqueness of the three-dimensional flow of a general linear isotropic fluid of second order, with non homogeneous boundary condition, both in the stationary and in the evolutionary case. The uniqueness result for the stationary problem holds provided that a smallness condition on the norm of the boundary datum and the external force is fulfilled. We deduce the flow equations directly from the principle of virtual powers and impose the boundary conditions in a quite general form. In addition, we study the consequences of the free energy imbalance for a general isotropic second order fluid, improving some results of [4].

The plan of the paper is the following: in Section 2 we introduce the principle of virtual powers, together with the general dynamical and thermodynamical prescriptions needed to obtain the balance equations for second order continuous media. In Section 3 we employ the constitutive prescriptions for the linear isotropic incompressible fluids and study suitable boundary conditions and the consequences of the thermodynamical constraint on the sign of the fluid parameters. Finally, in Section 4 we introduce the functional setting and prove the main results.

2. Modelling nonsimple continuous media

A general way to set down a thermomechanical model for continuous media is to apply the principle of virtual powers as an axiom. As pointed out by Podio-Guidugli in [10] (to which we refer for an extended bibliography), local thermomechanical balances and imbalances can be deduced from the balance of properly defined internal and external powers. In this paper we will restrict our attention to a pure mechanical theory; this choice will result in neglecting heat contributions to the free energy variation (see Section 2.2 below) and in interpreting test functions as velocities in the definition of power.

To begin with, we introduce the main notions of continuous body and of power of order k.

Definition 1. Let $B \subseteq \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and let $\mathcal{M}$ be a collection of sufficiently smooth subsets of B. We call $\mathcal{M}$ the *system of subbodies* of the body B and the pair $\mathcal{B} = (B, \mathcal{M})$ is named *continuous body.*

The choice of B as bounded with Lipschitz boundary is essentially motivated by the use of Sobolev's theorem and Korn's inequality we will make in the following. Any choice of $\mathcal{M}$ should keep into account the possibility of giving a useful definition of outer normal vector and the validity of the Gauss-Green formula (see [1] and the references quoted therein). In a number of cases, a good choice could be that of normalized Caccioppoli sets$^{(1)}$, but when dealing with nonsimple materials some additional features are needed. For instance, in paper [2] the class of *sets with curvature measure* has been proposed.

Definition 2. Given a continuous body $\mathcal{B}$ and a vector space V of test functions $\mathsf{v} : B \to \mathbb{R}^N$, we call *power of order* k a functional

$$P : \begin{cases} \mathcal{M} \times V \to \mathbb{R} \\ (M, \mathsf{v}) \mapsto \displaystyle\sum_{j=0}^{k} \int_M \mathsf{A}^{(j)}(x) \cdot \nabla^j \mathsf{v}(x)\, d\mathcal{L}^n(x) \end{cases} \tag{1}$$

where $\mathsf{A}^{(j)}(x)$ belongs to the same linear space of $\nabla^j \mathsf{v}(x)$, namely $\mathsf{A}^{(0)} : B \to \mathbb{R}^N$, $\mathsf{A}^{(1)} : B \to \mathrm{Lin}(\mathbb{R}^n; \mathbb{R}^N)$, and so on.

In order to give to the above definition a distributional meaning, it is customary to choose a space V which contains $C_0^\infty(B; \mathbb{R}^N)$.

Remark 1. The notion of virtual power, in its full generality, needs at least two classes of "tests", which we denoted by $\mathcal{M}$ and V. One can wonder if these tests are somewhat redundant, in analogy with Classical Mechanics, where only the elements of V come into the definition of the mechanical power. However, in this general framework the tests of V alone are not sufficient to single out the tensor fields $\mathsf{A}^{(j)}$ in a unique way, unless for a power of order zero.

Indeed, suppose for the sake of simplicity to have a first order power

$$P(B, \mathsf{v}) = \int_B \mathsf{A} \cdot \nabla \mathsf{v}\, d\mathcal{L}^n\,, \quad \mathsf{v} \in V\,,$$

$^{(1)}$If $M \subseteq \mathbb{R}^n$ is measurable, we set

$$M_* = \left\{ x \in \mathbb{R}^n \; : \; \lim_{r \to 0^+} \frac{\mathcal{L}^n(M \cap B_r(x))}{\mathcal{L}^n(B_r(x))} = 1 \right\}\,,$$
$$\partial_* M = \mathbb{R}^n \setminus (M_* \cup (\mathbb{R}^n \setminus M)_*)\,,$$

where $B_r(x)$ denotes the open ball of center x and radius r and $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$. The set $M \subseteq \mathbb{R}^n$ is *normalized* if $M_* = M$; it is a *Caccioppoli set* (or a *finite perimeter* set) if $\mathcal{H}^{n-1}(\partial_* M) < +\infty$, where $\mathcal{H}^{n-1}$ denotes the surface Hausdorff measure.

defined only on the whole body B. If $\mathsf{Y} : B \to \mathrm{Lin}(\mathbb{R}^n;\mathbb{R}^N)$ is one of the many solutions of the problem

$$\begin{cases} \operatorname{div}\mathsf{Y} = 0 & \text{in } B \\ \mathsf{Y}\mathsf{n} = 0 & \text{on } \partial B\,, \end{cases} \tag{2}$$

then it is readily seen that, setting $\widehat{\mathsf{A}} := \mathsf{A} + \mathsf{Y}$, one has

$$P(B,\mathsf{v}) = \int_B \mathsf{A}\cdot\nabla\mathsf{v}\, d\mathcal{L}^n = \int_B \widehat{\mathsf{A}}\cdot\nabla\mathsf{v}\, d\mathcal{L}^n\,;$$

hence we could not distinguish between A and $\widehat{\mathsf{A}}$.

On the contrary, once the constitutive prescriptions are assumed, it may happen that such an arbitrariness is overcome and one can avoid to deal with the class of subbodies $\mathcal{M}$. For instance, if one assumes that Y is a gradient, then (2) has only the trivial solution and the tests in V are enough to find a unique representative. As we shall see, this conclusion holds also for the constitutive prescriptions we will assume in the sequel.

Since our aim is to apply the virtual powers framework to Fluid Mechanics, we will take $B \subseteq \mathbb{R}^3$ as the portion of space occupied by the fluid, and V will be a space of *virtual velocities*, so that both n and N will be set equal to 3, while we still have to comment on the order k of the power.

Definition 3. Given a power $P_{\mathrm{int}}(M,\mathsf{v})$ as in (1) with $n = N = 3$, we say that it is an *internal power* if it satisfies the *frame indifference* condition, that is $P_{\mathrm{int}}(M,\cdot)$ should be a scalar under the Galilean group on $\mathbb{R}^3$. Notice that this requirement, which is likely to be interpreted as physical significance, easily implies $\mathsf{A}^{(0)} = \mathsf{0}$ and $\mathsf{A}^{(1)}$, usually denoted by T, symmetric.

2.1. Dynamical prescriptions

So far no assumptions have been made about the model to describe any particular material. Usually, the first step in this direction is to state some *constitutive prescriptions*, which should specify the continuous medium whose mathematical properties one wants to investigate. On the contrary, within the general framework of virtual powers (even restricted to pure mechanical theories), we are forced to state some *dynamical prescriptions*. The first of these ones is the choice of the order of P_{int}: *simple* media correspond to first order powers (a requirement equivalent to the classical Cauchy stress condition), while higher order powers enter the definition of *nonsimple* media.

In this paper we will deal with *second order* media, that is we take $k = 2$ in Definition 2. In such a case, for an internal power only two terms of (1) survive:

$$P_{\mathrm{int}}(M,\mathsf{v}) = \int_M \mathsf{T}\cdot\nabla\mathsf{v} + \int_M \mathbf{G}\cdot\nabla\nabla\mathsf{v}\,, \qquad \mathsf{T}\ \text{symmetric.}$$

Further steps involve choices about what we call *external power* P_{ext}. Applying D'Alembert's principle in order to state evolution problems as equilibrium ones, we include in the external power the inertial term

$$-\int_M \rho\,\dot{\mathsf{u}}\cdot\mathsf{v} := -\int_M \rho\left(\frac{\partial\mathsf{u}}{\partial t} + (\mathsf{u}\cdot\nabla)\mathsf{u}\right)\cdot\mathsf{v}\ ,$$

which makes the problem a nonlinear one.

The second prescription follows from the virtual powers framework, which forces the internal and the external power to be of the same order. We hence assume the external power to be of the form

$$P_{\rm ext}(M,\mathsf{v}) = -\int_M \rho\,\dot{\mathsf{u}}\cdot\mathsf{v} + \int_M \mathsf{a}\cdot\mathsf{v} + \int_M \mathsf{A}\cdot\nabla\mathsf{v} + \int_M \boldsymbol{\mathsf{A}}\cdot\nabla\nabla\mathsf{v}\ . \qquad (3)$$

It is useful to separate (3) into a volume and a surface part. In order to do this we need some notions about the differential geometry of surfaces. We define the projection on the tangent plane to a surface $\mathcal{S}$ as

$$\mathsf{P} := \mathsf{I} - \mathsf{n}\otimes\mathsf{n}\ ,$$

where n is the unit outer normal to $\mathcal{S}$; moreover, for any 1-tensor a and for any 2-tensor A, we set

$$\begin{aligned}\nabla_{\mathcal{S}}\,\mathsf{a} &= (\nabla\mathsf{a})\mathsf{P}\ ,\\ \mathrm{div}_{\mathcal{S}}\,\mathsf{a} = \mathrm{tr}(\nabla_{\mathcal{S}}\,\mathsf{a}) &= \mathsf{P}\cdot\nabla\mathsf{a} = \mathrm{div}\,\mathsf{a} - \mathsf{n}\cdot(\nabla\mathsf{a})\mathsf{n}\ ,\\ (\mathrm{div}_{\mathcal{S}}\,\mathsf{A})_i &= \mathsf{A}_{ij,k}\mathsf{P}_{kj}\ .\end{aligned}$$

We also introduce the normal derivative, the curvature tensor and the mean curvature of $\mathcal{S}$:

$$\begin{aligned}\frac{\partial\mathsf{a}}{\partial n} &= (\nabla\mathsf{a})\mathsf{n}\ ,\\ \mathsf{K} &= -\nabla_{\mathcal{S}}\,\mathsf{n} = -(\nabla\mathsf{n})\mathsf{P}\ ,\\ K &= \frac{1}{2}\mathrm{tr}\,\mathsf{K} = -\frac{1}{2}\mathrm{div}_{\mathcal{S}}\,\mathsf{n}\ .\end{aligned}$$

The following well-known identity, which holds under suitable assumptions on the regularity of the fields and the surfaces, is a direct consequence of Stokes' theorem in Differential Geometry.

Theorem (Surface Divergence). *Let $\boldsymbol{\tau}$ be a tangent vector field to the surface $\mathcal{S}$, $\mathcal{T}$ a subsurface of $\mathcal{S}$ and $\boldsymbol{\nu}$ the unit outer normal to its boundary $\partial\mathcal{T}$; then*

$$\int_{\partial\mathcal{T}}\boldsymbol{\tau}\cdot\boldsymbol{\nu} = \int_{\mathcal{T}}\mathrm{div}_{\mathcal{S}}\,\boldsymbol{\tau}\ .$$

In particular, if X and v are regular 2- and 1-tensor fields, by choosing $\boldsymbol{\tau} = \mathsf{P}\mathsf{X}^\top\mathsf{v}$ and $\mathcal{T} = \mathcal{S} = \partial M$, the following identity holds:

$$\int_{\partial M}\mathsf{X}\cdot\nabla_{\mathcal{S}}\,\mathsf{v} = -\int_{\partial M}(\mathrm{div}_{\mathcal{S}}\,\mathsf{X} + 2K\mathsf{X}\mathsf{n})\cdot\mathsf{v}\ . \qquad (4)$$

We can now start manipulating expression (3). Integrating by parts and using the usual Divergence theorem, one obtains

$$\int_M \mathsf{A}\cdot\nabla\mathsf{v} = -\int_M \operatorname{div}\mathsf{A}\cdot\mathsf{v} + \int_{\partial M}\mathsf{An}\cdot\mathsf{v}\ ,$$

$$\begin{aligned}\int_M \mathbf{A}\cdot\nabla\nabla\mathsf{v} &= -\int_M \operatorname{div}\mathbf{A}\cdot\nabla\mathsf{v} + \int_{\partial M}\mathbf{A}\mathsf{n}\cdot\nabla\mathsf{v} = \\ &= \int_M(\operatorname{div}\operatorname{div}\mathbf{A})\cdot\mathsf{v} + \int_{\partial M}\left(\underline{\mathbf{A}\mathsf{n}\cdot\nabla_{\mathcal{S}}\mathsf{v}} + (\mathbf{A}\mathsf{n})\mathsf{n}\cdot\frac{\partial\mathsf{v}}{\partial n} - (\operatorname{div}\mathbf{A})\mathsf{n}\cdot\mathsf{v}\right)\ .\end{aligned}$$

Applying identity (4) to the underlined term, we finally get

$$\begin{aligned}P_{\text{ext}}(M,\mathsf{v}) = -\int_M \rho\,\dot{\mathsf{u}}\cdot\mathsf{v} + \int_M[\,\mathsf{a} - \operatorname{div}(\mathsf{A} - \operatorname{div}\mathbf{A})\,]\cdot\mathsf{v} + \int_{\partial M}(\mathbf{A}\mathsf{n})\mathsf{n}\cdot\frac{\partial\mathsf{v}}{\partial n} \\ + \int_{\partial M}[\,\mathsf{An} - (\operatorname{div}\mathbf{A})\mathsf{n} - \operatorname{div}_{\mathcal{S}}(\mathbf{A}\mathsf{n}) - 2K(\mathbf{A}\mathsf{n})\mathsf{n}\,]\cdot\mathsf{v}\ .\end{aligned}$$

Defining the vector fields

$$\begin{aligned}\rho\,\mathsf{b} &:= \mathsf{a} - \operatorname{div}(\mathsf{A} - \operatorname{div}\mathbf{A})\ ,\\ \mathsf{t}(\mathsf{n},K) &:= \mathsf{An} - (\operatorname{div}\mathbf{A})\mathsf{n} - \operatorname{div}_{\mathcal{S}}(\mathbf{A}\mathsf{n}) - 2K(\mathbf{A}\mathsf{n})\mathsf{n}\ ,\\ \mathsf{m}(\mathsf{n}) &:= (\mathbf{A}\mathsf{n})\mathsf{n}\ ,\end{aligned}$$

where n and K are related to ∂M, we introduce the volume and the surface part of the external power:

$$P_{\text{vol}}(M,\mathsf{v}) = -\int_M \rho\,\dot{\mathsf{u}}\cdot\mathsf{v} + \int_M \rho\,\mathsf{b}\cdot\mathsf{v}\ ,$$

$$P_{\text{surf}}(\partial M,\mathsf{v}) = \int_{\partial M}\mathsf{t}\cdot\mathsf{v} + \int_{\partial M}\mathsf{m}\cdot\frac{\partial\mathsf{v}}{\partial n}\ .$$

Finally, the following fundamental principle provides the equation to be solved in order to describe the dynamics of the system.

Principle of virtual powers. The body B is at equilibrium if and only if for every $M\in\mathcal{M}$ and every virtual velocity v we have

$$P_{\text{int}}(M,\mathsf{v}) = P_{\text{ext}}(M,\mathsf{v})\ , \tag{5}$$

for any instant in a time interval.

Obviously, we will also need constitutive prescriptions to single out a model fitting a specific material; however, we again emphasize that the dynamical prescriptions about the order of the power and the form of the inertial terms are a matter of choice of the model, and do not enter the general assumptions of Continuum Mechanics.

2.2. Thermodynamical prescriptions

It is customary to introduce also thermodynamical assumptions which contribute to select meaningful constitutive prescriptions among the possible ones. We call ψ the *free energy density* (per unit mass). Every constitutive prescription has to imply that the time increment of the free energy in any region M be less or equal than the power $P_{\rm ext}(M, \mathsf{v})$ expended on that region, which is equal to $P_{\rm int}(M, \mathsf{v})$ by (5); this is the free energy imbalance

$$\frac{d}{dt}\int_M \rho\,\psi \leq P_{\rm ext}(M, \mathsf{v}) = P_{\rm int}(M, \mathsf{v})\ .$$

Since we restrict our attention to a pure mechanical theory, we neglect any change in the free energy, hence $\dot{\psi} = 0$. The free energy imbalance then reads

$$P_{\rm int}(M, \mathsf{v}) \geq 0$$

for every $M \in \mathcal{M}$ and every virtual velocity v. By the arbitrariness of M, any constitutive prescription for T and $\mathbf{G}$ shall imply the dissipation inequality

$$\mathsf{T}\cdot\nabla\mathsf{v} + \mathbf{G}\cdot\nabla\nabla\mathsf{v} \geq 0 \tag{6}$$

for every v.

2.3. Local balances

In this section we will deduce the local balances implied by the principle of virtual powers, which, for second order media, reads

$$\int_M \mathsf{T}\cdot\nabla\mathsf{v} + \int_M \mathbf{G}\cdot\nabla\nabla\mathsf{v} = \int_M \rho(\mathsf{b}-\dot{\mathsf{u}})\cdot\mathsf{v} + \int_{\partial M}\mathsf{t}\cdot\mathsf{v} + \int_{\partial M}\mathsf{m}\cdot\frac{\partial\mathsf{v}}{\partial n}\ , \tag{7}$$

for every $M \in \mathcal{M}$ and every virtual velocity v.

Splitting also the internal power into volume and surface contributions, equation (7) becomes

$$\int_M [-\mathrm{div}\,(\mathsf{T}-\mathrm{div}\,\mathbf{G}) - \rho(\mathsf{b}-\dot{\mathsf{u}})]\cdot\mathsf{v} + \int_{\partial M}[(\mathbf{G}\mathsf{n})\mathsf{n}-\mathsf{m}]\cdot\frac{\partial\mathsf{v}}{\partial n}$$
$$+\int_{\partial M}[\,\mathsf{T}\mathsf{n} - (\mathrm{div}\,\mathbf{G})\mathsf{n} - \mathrm{div}_{\mathcal{S}}\,(\mathbf{G}\mathsf{n}) - 2K(\mathbf{G}\mathsf{n})\mathsf{n} - \mathsf{t}\,]\cdot\mathsf{v} = 0\ ;$$

by the arbitrariness of M and v we obtain the local balances

$$\rho\mathsf{b} - \rho\dot{\mathsf{u}} = -\,\mathrm{div}\,(\mathsf{T}-\mathrm{div}\,\mathbf{G})\ , \tag{8}$$

$$\mathsf{t}(\mathsf{n}, K) = \mathsf{T}\mathsf{n} - (\mathrm{div}\,\mathbf{G})\mathsf{n} - \mathrm{div}_{\mathcal{S}}\,(\mathbf{G}\mathsf{n}) - 2K(\mathbf{G}\mathsf{n})\mathsf{n}\ , \tag{9}$$

$$\mathsf{m}(\mathsf{n}) = (\mathbf{G}\mathsf{n})\mathsf{n}\ , \tag{10}$$

the first of which corresponds to the linear momentum balance, while the second and the third one hold locally for any possible n and K.

3. Linear isotropic incompressible fluids

We now turn to the task of specifying a model by means of constitutive prescriptions. We want to describe fluids, so that the state descriptor at any instant t in the time interval $[0,T] \subset \mathbb{R}$ is the Eulerian velocity field $\mathsf{u}(t,x)$, and the incompressibility condition allows us to set the mass density $\rho = 1$ identically, giving the first constraint on the velocity:

$$\forall t \in [0,T]: \ \operatorname{div} \mathsf{u} = 0 \ . \tag{11}$$

Further prescriptions are related to internal and external power.

3.1. Constitutive prescriptions for the internal power

Linearity and isotropy (*i.e.* covariance under the full orthogonal group on $\mathbb{R}^3$) of the fluid are encoded in the dependence of the tensor fields T and $\mathbf{G}$ on u. In a recent paper, Musesti [8] proved that the most general linear isotropic tensor fields, endowed with the symmetries due to frame indifference and Schwarz's theorem, take the form

$$\mathsf{T}_{ij} = \mu(\mathsf{u}_{i,j} + \mathsf{u}_{j,i}) - p\,\delta_{ij} \ ,$$

$$\begin{aligned}\mathbf{G}_{ij} =& \eta_1 \mathsf{u}_{i,jk} + \eta_2(\mathsf{u}_{j,ki} + \mathsf{u}_{k,ij} - \mathsf{u}_{i,ss}\delta_{jk}) \\ &+ \eta_3(\mathsf{u}_{j,ss}\delta_{ki} + \mathsf{u}_{k,ss}\delta_{ij} - 4\mathsf{u}_{i,ss}\delta_{jk}) - \mathsf{p}_k\delta_{ij} \ ,\end{aligned}$$

where $\mu, \eta_1, \eta_2, \eta_3 \in \mathbb{R}$. The fields p and p, respectively a scalar and a vector one, enter the definition of the *pressure*

$$P := p - \operatorname{div} \mathsf{p} \ ,$$

whose rôle in incompressible theories reduces to that of a Lagrange multiplier of the constraint (11).

Defining the *symmetric part* of an m-tensor X as

$$\operatorname{Sym} \mathsf{X} := \frac{1}{m!} \sum_{\sigma} \mathsf{X}_{\sigma(i_1 \ldots i_m)} \ ,$$

where σ runs over the group of permutations of m elements, the previous relations can be written in intrinsic notation as

$$\mathsf{T} = 2\mu \operatorname{Sym} \nabla \mathsf{u} - p\,\mathsf{I} \ ,$$

$$\begin{aligned}\mathbf{G} =& (\eta_1 - \eta_2)\nabla\nabla\mathsf{u} + 3\eta_2 \operatorname{Sym} \nabla\nabla\mathsf{u} \\ &- (\eta_2 + 5\eta_3)\Delta\mathsf{u} \otimes \mathsf{I} + 3\eta_3 \operatorname{Sym}(\Delta\mathsf{u} \otimes \mathsf{I}) - \mathsf{I} \otimes \mathsf{p} \ .\end{aligned}$$

Following these definitions and imposing also on the virtual velocities the constraint (11), we can write the internal power for a linear isotropic incompressible fluid as

$$P_{\text{int}}(M, \mathsf{v}) = 2\mu \int_M \operatorname{Sym} \nabla \mathsf{u} \cdot \nabla \mathsf{v} + (\eta_1 - \eta_2) \int_M \nabla\nabla \mathsf{u} \cdot \nabla\nabla \mathsf{v} \\ + 3\eta_2 \int_M \operatorname{Sym} \nabla\nabla \mathsf{u} \cdot \nabla\nabla \mathsf{v} - (\eta_2 + 4\eta_3) \int_M \Delta \mathsf{u} \cdot \Delta \mathsf{v} \ .$$

3.2. Constitutive prescriptions for the external power

Constitutive prescriptions for the external power are usually given writing the fields b, t and m in terms of some known (or measurable) external parameters and also in terms of the unknowns of the problem.

As long as the field b is concerned, it is useful to single out a *conservative* contribution, which will play the rôle of a pressure term in an incompressible theory. Hence we write

$$\mathsf{b} = \mathsf{d} + \nabla f \ ,$$

where f is a scalar field and d includes all non conservative contributions to the volume (*i.e.* long-distance) interactions of the medium.

In view of Remark 1, the formulation of our dynamical problem will make use of the balance (7) only on the body B, hence we need a prescription of t and m only on the boundary ∂B. This suggested us to discuss such prescriptions in the next section.

3.3. Boundary conditions

We can divide the realm of boundary conditions into requests of *kinematical admissibility* for the velocity field u and constitutive prescriptions for the fields t and m on the boundary ∂B.

From a kinematical point of view we can partition the boundary into four (possibly empty) regions:

- on $\mathcal{S}_D$ we fix the value of both u and its normal derivative $\frac{\partial \mathsf{u}}{\partial n}$: this is called a *Dirichlet adherence* condition;
- on $\mathcal{S}_W$ we only fix the value of u, giving a *weak adherence* condition;
- on $\mathcal{S}_N$ we do not fix anything, obtaining *Neumann* conditions;
- on $\mathcal{S}_S$ we just set $\mathsf{u} \cdot \mathsf{n} = 0$, which represents a *slip* condition with pure geometrical constraints.

We will say that a velocity field u is *kinematically admissible* if it satisfies the proper condition on each of the previous regions.

We choose $\mathsf{t} = -\alpha \mathsf{u}$ on $\mathcal{S}_S$, with $\alpha \in L^\infty(\mathcal{S}_S; \mathbb{R}^+)$, as a traction which opposes the motion, with a strength proportional to the tangential velocity of the fluid. On $\mathcal{S}_N$ we set $\mathsf{t} = q\mathsf{n}$, where $q \in L^\infty(\mathcal{S}_N; \mathbb{R})$: this term represents the pressure along $\mathcal{S}_N$, which can be used to drive the flow (we could consider for instance a portion of pipe with pressure fixed at the ends).

In a similar way, on $\mathcal{S}_W$ and $\mathcal{S}_S$ we prescribe

$$\mathsf{m} = -\mu\,\ell\,\frac{\partial \mathsf{u}}{\partial n} \ ,$$

where the constitutive modulus $\ell \in L^\infty(\mathcal{S}_W \cup \mathcal{S}_S; \mathbb{R}^+)$ is called *adherence length*(2), since it is a length parameter which measures a kind of slow down of the fluid due to an interaction with the boundary. On $\mathcal{S}_N$ we set $\mathsf{m} = 0$.

Notice that prescriptions of m on $\mathcal{S}_D$ and t on $\mathcal{S}_D \cup \mathcal{S}_W$ should be regarded as compatibility constraints for equations (8)–(10), since their values can be computed after solving the momentum balance equation; therefore we will not need such prescriptions.

3.4. Thermodynamical constraint

The dissipation inequality (6) specialized for our model reads

$$2\mu|\mathrm{Sym}\,\nabla\mathsf{v}|^2 + (\eta_1 - \eta_2)|\nabla\nabla\mathsf{v}|^2 + 3\eta_2|\mathrm{Sym}\,\nabla\nabla\mathsf{v}|^2 - (\eta_2 + 4\eta_3)|\Delta\mathsf{v}|^2 \geq 0$$

for every virtual velocity v.

Since the first and second order derivatives of v can be independently set equal to zero, the dissipation inequality will be satisfied if and only if $\mu \geq 0$ and

$$\Gamma := \nabla\nabla\mathsf{v} \cdot \mathbf{G}[\nabla\nabla\mathsf{v}] = \eta_1 \mathsf{v}_{i,jk}\mathsf{v}_{i,jk} + \eta_2(2\mathsf{v}_{k,ij}\mathsf{v}_{i,jk} - \mathsf{v}_{i,rr}\mathsf{v}_{i,ss}) - 4\eta_3\mathsf{v}_{i,rr}\mathsf{v}_{i,ss} \geq 0$$

for every virtual velocity v. This last requirement is equivalent to the following conditions on the coefficients η_1, η_2 and η_3.

Proposition 1. *We have $\Gamma \geq 0$ for every virtual velocity v if and only if*

$$\eta_1 + 2\eta_2 \geq 0\,, \quad \eta_1 - \eta_2 \geq 0\,, \quad \eta_1 - \eta_2 - 6\eta_3 - 2\sqrt{\eta_2^2 + 4\eta_2\eta_3 + 9\eta_3^2} \geq 0\,.^{(3)} \tag{12}$$

Proof. Let us identify the 18 independent components of $\nabla\nabla\mathsf{v}$ with an element $\mathsf{x} \in \mathbb{R}^{18}$ according to the following table:

$$\begin{array}{lllll}
x_1 = \mathsf{v}_{1,11} & x_2 = \mathsf{v}_{1,22} & x_3 = \mathsf{v}_{1,33} & x_4 = \mathsf{v}_{2,12} & x_5 = \mathsf{v}_{3,13} \\
x_6 = \mathsf{v}_{2,22} & x_7 = \mathsf{v}_{2,33} & x_8 = \mathsf{v}_{2,11} & x_9 = \mathsf{v}_{3,23} & x_{10} = \mathsf{v}_{1,12} \\
x_{11} = \mathsf{v}_{3,33} & x_{12} = \mathsf{v}_{3,11} & x_{13} = \mathsf{v}_{3,22} & x_{14} = \mathsf{v}_{1,13} & x_{15} = \mathsf{v}_{2,23} \\
x_{16} = \mathsf{v}_{1,23} & x_{17} = \mathsf{v}_{2,13} & x_{18} = \mathsf{v}_{3,12}\,. & &
\end{array}$$

Then we can write

$$\Gamma = \mathsf{x} \cdot (\eta_1\mathsf{A} + \mathsf{B})\mathsf{x}\,, \tag{13}$$

where $\mathsf{A} = \mathrm{diag}(A_5, A_5, A_5, A_3)$, $\mathsf{B} = \mathrm{diag}(B_5, B_5, B_5, B_3)$ and

$$A_5 = \mathrm{diag}(1,1,1,2,2), \quad A_3 = \mathrm{diag}(2,2,2),$$

$$B_5 = \begin{bmatrix} \eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & 0 & 0 \\ -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & 2\eta_2 & 0 \\ -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & 0 & 2\eta_2 \\ 0 & 2\eta_2 & 0 & 2\eta_2 & 0 \\ 0 & 0 & 2\eta_2 & 0 & 2\eta_2 \end{bmatrix}, \quad B_3 = \eta_2 \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(2) See [4, Sec. 8.2], where similar assumptions are made.

(3) With the choice $\eta_3 = 0$ one finds $\eta_1 + 2\eta_2 \geq 0$ and $\eta_1 - 3\eta_2 \geq 0$. The reader should compare such inequalities with [4, Eq. (75)], which are only sufficient conditions.

The quadratic form (13) is positive definite if and only if its eigenvalues are all positive. Since A is positive definite, this is tantamount to say that the eigenvalues of $\eta_1 \mathsf{I} + \mathsf{A}^{-1}\mathsf{B}$ are positive definite.

We have $\mathsf{A}^{-1}\mathsf{B} = \operatorname{diag}(A_5^{-1}B_5, A_5^{-1}B_5, A_5^{-1}B_5, A_3^{-1}B_3)$, where

$$A_5^{-1}B_5 = \begin{bmatrix} \eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & 0 & 0 \\ -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & 2\eta_2 & 0 \\ -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & -\eta_2 - 4\eta_3 & 0 & 2\eta_2 \\ 0 & \eta_2 & 0 & \eta_2 & 0 \\ 0 & 0 & \eta_2 & 0 & \eta_2 \end{bmatrix},$$

$$A_3^{-1}B_3 = \eta_2 \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

A straightforward calculation shows that the eigenvalues of $\mathsf{A}^{-1}\mathsf{B}$ are

$$\lambda_{1,2} = -\eta_2 - 6\eta_3 \pm 2\sqrt{\eta_2^2 + 4\eta_2\eta_3 + 9\eta_3^2}\,,\ \lambda_{3,4} = \pm\eta_2\,,\ \lambda_5 = 2\eta_2\,,\ \lambda_6 = -\frac{\eta_2}{2}\,,$$

hence $\Gamma \geq 0$ for every velocity field if and only if $\eta_1 + \lambda_{\min} \geq 0$, where $\lambda_{\min}$ is the minimal eigenvalue. Since

$$\begin{cases} \lambda_{\min} = -\eta_2 - 6\eta_3 - 2\sqrt{\eta_2^2 + 4\eta_2\eta_3 + 9\eta_3^2} & \text{if } \eta_2 + 4\eta_3 \geq 0 \\ \lambda_{\min} = -\eta_2 & \text{if } \eta_2 + 4\eta_3 \leq 0 \text{ and } \eta_2 \geq 0 \\ \lambda_{\min} = 2\eta_2 & \text{if } \eta_2 + 4\eta_3 \leq 0 \text{ and } \eta_2 \leq 0\,, \end{cases}$$

one has the global conditions (12). □

4. Existence and uniqueness

We now want to investigate existence and uniqueness of solutions for both the stationary and the evolutionary motion of a second order incompressible fluid, with boundary conditions as described in Section 3.3. In particular we will focus our attention on the case $\mathcal{H}^2(\mathcal{S}_D) = 0$ and $\mathcal{H}^2(\mathcal{S}_W) > 0$, since, within conditions of an adherence kind, it seems more natural to fix only the value of the velocity field on a part of the boundary; moreover, a *Dirichlet adherence* condition would require minor changes in what follows.

We encode the kinematical admissibility together with the divergence-free condition of the virtual velocities in a suitable linear space X, whose definition will be given in the next section. We assume

$$\mathsf{u} = \mathsf{g}_0 \qquad \text{on } \mathcal{S}_W$$

in the sense of traces in Sobolev spaces, with $\mathsf{g}_0 \in H^{\frac{3}{2}}(\overline{\mathcal{S}_W}; \mathbb{R}^3)$. Moreover, we choose $\mathsf{a} \in H^3(B)$ and set $\mathsf{g} = \operatorname{rot} \mathsf{a}$ in such a way that $\mathsf{g} = \mathsf{g}_0$ on $\mathcal{S}_W$ in the sense of traces (we call g an *interpolator* for the boundary value).

We will now apply the principle of virtual powers in order to obtain a formulation of the dynamical problem for the fluids introduced in the previous sections.

As an application of the uniqueness results proved in the sequel, we are allowed to assume the balance (7) only for $M = B$, since the arbitrariness mentioned in Remark 1 vanishes and we can deduce the balances on subbodies by the choice of suitable virtual velocities v.

Problem 1. *Find* $\mathsf{u} \in X + \mathsf{g}$ *such that for every* $t \in [0, T]$

$$\begin{aligned} 2\mu \int_B \operatorname{Sym} \nabla \mathsf{u} \cdot \nabla \mathsf{v} + (\eta_1 - \eta_2) \int_B \nabla\nabla \mathsf{u} \cdot \nabla\nabla \mathsf{v} \\ + 3\eta_2 \int_B \operatorname{Sym} \nabla\nabla \mathsf{u} \cdot \nabla\nabla \mathsf{v} - (\eta_2 + 4\eta_3) \int_B \Delta \mathsf{u} \cdot \Delta \mathsf{v} \\ + \int_{\mathcal{S}_W \cup \mathcal{S}_S} \mu\, \ell\, \frac{\partial \mathsf{u}}{\partial n} \cdot \frac{\partial \mathsf{v}}{\partial n} + \int_{\mathcal{S}_S} \alpha\, \mathsf{u} \cdot \mathsf{v} + \\ \int_B \left(\frac{\partial \mathsf{u}}{\partial t} + (\mathsf{u} \cdot \nabla) \mathsf{u} \right) \cdot \mathsf{v} = \int_{\mathcal{S}_N} (f + q) \mathsf{v} \cdot \mathsf{n} + \int_B \mathsf{d} \cdot \mathsf{v} \,, \end{aligned} \tag{14}$$

for every $\mathsf{v} \in X$.

4.1. Virtual velocities and Hilbert spaces

The assumption of a balance principle in integral form leads directly to an interpretation of the functions as defined up to negligible sets (with zero Lebesgue measure). We will take the set of test functions (virtual velocities) as a linear space with the useful topological structure of a Hilbert space, endowing the principle of virtual powers with a natural interpretation as equality of linear forms.

First of all the virtual velocities should be regular enough and kinematically admissible; we then consider the set

$$\left\{ \mathsf{v} \in C^\infty(B) : \ \operatorname{div} \mathsf{v} = 0 \,,\ \mathsf{v} \text{ is kinematically admissible} \right\} ;$$

but notice that this is not an $\mathbb{R}$-linear space in general, since kinematical admissibility can be lost by summation of functions or multiplication by scalars, unless all the boundary values be set equal to the zero function.

We will then set, for any surface $\mathcal{S}$ and any $\varepsilon > 0$,

$$B_\varepsilon(\mathcal{S}) := \left\{ x \in B : \ \operatorname{dist}(x, \mathcal{S}) < \varepsilon \right\} ,$$

and define the $\mathbb{R}$-linear space

$$V := \left\{ \mathsf{v} \in C^\infty(B) : \begin{array}{c} \operatorname{div} \mathsf{v} = 0 \,, \\ \exists \varepsilon_1 > 0 : \forall x \in B_{\varepsilon_1}(\mathcal{S}_W) : \mathsf{v}(x) = 0 \\ \exists \varepsilon_2 > 0 : \forall x \in B_{\varepsilon_2}(\mathcal{S}_S) : (\mathsf{v} \cdot \mathsf{n})(x) = 0 \end{array} \right\} ,$$

which turns out to be an $\mathbb{R}$-linear subspace of $C^\infty(B)$.

We denote with H the completion of V in $L^2(B)$, with H^1_V the completion of V with respect to the $H^1(B)$ norm and set $X := H^1_V \cap H^2(B)$ endowed with the $H^2(B)$ norm

$$\|\mathsf{v}\|_X^2 := \int_B |\mathsf{v}|^2 + \int_B |\nabla \mathsf{v}|^2 + \int_B |\nabla\nabla \mathsf{v}|^2 ,$$

that encodes the natural regularity requested by the problem.

Remark 2. The choice of a Hilbert space for the set of virtual velocities v forces indeed a distinction between them and kinematically admissible fields, but the solutions will meet all the kinematical constraints, since they will belong to the coset $X + \mathsf{g}$ of X in $H^2(B)$.

It remains to specify the rôle of the time variable t, which clearly enters the problem *via* the time derivative of u. As a first step we could take $\mathsf{u} \in L^2([0,T];X)$ so that $\mathsf{u}(t,\cdot) \in X$ for almost every $t \in [0,T]$; but we will see that, if u is a solution of our problem, then

$$\mathsf{u} \in L^2([0,T];X) \cap C([0,T];H) \cap H^1([0,T];X') .$$

4.2. Compactness of the Navier operator

In equation (14) the only nonlinearity is the convective term

$$\int_B (\mathsf{u} \cdot \nabla)\mathsf{u} ;$$

clearly it is the source of the difficulties and also of the interest in solving our problem. We will work it out *via* a topological method in which compactness is the key tool; therefore we introduce some considerations about the compactness properties of that term.

Consider the bilinear function

$$F : \begin{cases} H^2 \times H^2 & \to \quad L^2 \\ \quad (\mathsf{u},\mathsf{v}) & \mapsto \ (\mathsf{u}\cdot\nabla)\mathsf{v} \end{cases} ;$$

by Hölder's inequality, since $H^2(B)$ is compactly embedded in $L^\infty(B)$, we have

$$\|F(\mathsf{u},\mathsf{v})\|_{L^2} = \|(\mathsf{u}\cdot\nabla)\mathsf{v}\|_{L^2} \le \|\mathsf{u}\|_{L^\infty} \|\nabla\mathsf{v}\|_{L^2} \le c_0 \|\mathsf{u}\|_{H^2} \|\mathsf{v}\|_{H^2} < +\infty \tag{15}$$

for any $\mathsf{u},\mathsf{v} \in H^2(B)$; hence F is continuous.

For every $\mathsf{g} \in H^2(B)$, we define the nonlinear *Navier operator* K_g as follows:

$$K_\mathsf{g} : \begin{cases} X & \to \quad L^2 \\ \mathsf{u} & \mapsto \quad F(\mathsf{u},\mathsf{u}) + F(\mathsf{u},\mathsf{g}) + F(\mathsf{g},\mathsf{u}) + F(\mathsf{g},\mathsf{g}) . \end{cases}$$

Theorem 1. *The Navier operator K_g is compact from X to X'.*

Proof. Clearly, for fixed $\mathsf{g} \in H^2(B)$, the sum

$$F(\mathsf{u},\mathsf{v}) + F(\mathsf{u},\mathsf{g}) + F(\mathsf{g},\mathsf{v}) + F(\mathsf{g},\mathsf{g})$$

is continuous and so is K_g, by composition with the function $\{\, \mathsf{u} \mapsto (\mathsf{u},\mathsf{u}) \,\}$. Moreover, by virtue of (15), it is bounded on bounded subsets of X.

Since $X \subseteq H^2(B)$ is compactly embedded in $L^2(B)$, we can identify $L^2(B)$ with its dual space and apply Schauder's theorem to obtain $L^2(B)$ compactly embedded in X'. □

Remark 3. The operator K_{g} is also compact from any $W \subseteq H^1(B)$ to W'; in fact the immersion $H^1(B) \to L^q(B)$ is compact for $q \in [1, 6[$ and such it is the dual $L^{q'}(B) \to (H^1(B))'$. We can take $q = 4$, $q' = \frac{4}{3}$, make the extension

$$F : \begin{cases} H^1 \times H^1 & \to \quad L^{\frac{4}{3}} \\ (\mathsf{u}, \mathsf{v}) & \mapsto \quad (\mathsf{u} \cdot \nabla)\mathsf{v} \end{cases}$$

and accordingly define K_{g} on W. We have

$$\|(\mathsf{u} \cdot \nabla)\mathsf{v}\|_{L^{\frac{4}{3}}} \leq \|\mathsf{u}\|_{L^4} \|\nabla \mathsf{v}\|_{L^2} \leq \tilde{c} \|\nabla \mathsf{u}\|_{L^2} \|\nabla \mathsf{v}\|_{L^2} < +\infty$$

for any $\mathsf{u}, \mathsf{v} \in W$ and, following the arguments of the previous theorem, we obtain the compactness of K_{g} from W to W'.

When considering the evolutionary problem we will need K_{g} to be a compact operator from $L^2([0,T]; X)$ to $L^2([0,T]; X')$. We will use the following lemma proved in [6, Chap. 1, Sec. 5.2].

Lemma 1. *Given three Banach spaces $\mathcal{B}_0 \subset \mathcal{B} \subset \mathcal{B}_1$ with $\mathcal{B}_0$ and $\mathcal{B}_1$ reflexive and $\mathcal{B}_0$ compactly embedded in $\mathcal{B}$, we set for $T \in (0, +\infty)$ and $p_0, p_1 \in (1, +\infty)$*

$$W := \left\{ \mathsf{v} : \ \mathsf{v} \in L^{p_0}([0,T]; \mathcal{B}_0) \, , \quad \frac{\partial \mathsf{v}}{\partial t} \in L^{p_1}([0,T]; \mathcal{B}_1) \right\} \, ,$$

which turns out to be a Banach space contained in $L^{p_0}([0,T]; \mathcal{B})$.

Then W is compactly embedded in $L^{p_0}([0,T]; \mathcal{B})$.

Theorem 2. *The Navier operator K_{g} is compact from the space*

$$L^2([0,T]; X) \cap C([0,T]; H) \cap H^1([0,T]; X')$$

into $L^2([0,T]; X')$.

Proof. Since X is compactly embedded in $L^3(B)$ and so is $L^{\frac{3}{2}}(B)$ into X', we can apply Lemma 1 with $X \subset L^3(B) \subset X'$ and $p_0 = p_1 = 2$. It remains to show that K_{g}, as an operator with range in $L^2([0,T]; L^{\frac{3}{2}}(B))$, is bounded on bounded subsets of its domain. The estimate

$$\int_0^T \|F(\mathsf{u}, \mathsf{v})\|^2_{L^{\frac{3}{2}}} \, dt \leq \int_0^T \|\nabla \mathsf{v}\|^2_{L^6} \|\mathsf{u}\|^2_{L^2} \leq c_0^2 \int_0^T \|\mathsf{v}\|^2_{H^2} \|\mathsf{u}\|^2_{L^2}$$

gives the needed property since u belongs to $L^\infty([0,T]; H)$. □

4.3. The stationary problem

We will first solve the stationary version of Problem 1, in which there is no dependence on time of the Eulerian velocity field and of any other term in equation (14). Stationary solutions represent an important class in Fluid Mechanics for many applications; moreover, as we will see later, the solution of the evolutionary problem goes along the same way as the stationary one.

Let us define the bilinear form $a : X \times X \to \mathbb{R}$ as

$$\begin{aligned} a(\mathsf{u},\mathsf{v}) := 2\mu \int_B \mathrm{Sym}\,\nabla\mathsf{u}\cdot\nabla\mathsf{v} + (\eta_1-\eta_2)\int_B \nabla\nabla\mathsf{u}\cdot\nabla\nabla\mathsf{v} \\ + 3\eta_2\int_B \mathrm{Sym}\,\nabla\nabla\mathsf{u}\cdot\nabla\nabla\mathsf{v} - (\eta_2+4\eta_3)\int_B \Delta\mathsf{u}\cdot\Delta\mathsf{v} \\ + \int_{\mathcal{S}_W\cup\mathcal{S}_S} \mu\,\ell\,\frac{\partial\mathsf{u}}{\partial n}\cdot\frac{\partial\mathsf{v}}{\partial n} + \int_{\mathcal{S}_S} \alpha\,\mathsf{u}\cdot\mathsf{v}\,. \end{aligned}$$

In view of the thermodynamical constraint and Proposition 1, we will assume a slightly stronger hypothesis on the sign of the coefficients.

Proposition 2. *Provided that*

$$\mu>0\,,\quad \eta_1+2\eta_2>0\,,\quad \eta_1-\eta_2>0\,,\quad \eta_1-\eta_2-6\eta_3-2\sqrt{\eta_2^2+4\eta_2\eta_3+9\eta_3^2}>0\,,$$

the bilinear form $a(\mathsf{u},\mathsf{v})$ is continuous and coercive on X.

Proof. The continuity of a is apparent. On the other hand, with the notation of Proposition 1 we have

$$\begin{aligned} a(\mathsf{u},\mathsf{u}) = 2\mu\int_B |\mathrm{Sym}\,\nabla\mathsf{u}|^2 + \int_B \Gamma + \int_{\mathcal{S}_W\cup\mathcal{S}_S} \mu\,\ell\left|\frac{\partial\mathsf{u}}{\partial n}\right|^2 + \int_{\mathcal{S}_S}\alpha\,|\mathsf{u}|^2 \geq \\ \geq 2\mu\int_B |\mathrm{Sym}\,\nabla\mathsf{u}|^2 + \int_B\Gamma \geq 2\mu\,\|\mathrm{Sym}\,\nabla\mathsf{u}\|_{L^2}^2 + (\eta_1+\lambda_{\min})\,\|\nabla\nabla\mathsf{u}\|_{L^2}^2\ ; \end{aligned}$$

by an application of Korn's inequality there exists $\kappa>0$ such that

$$a(\mathsf{u},\mathsf{u}) \geq \kappa(\|\mathsf{u}\|_{L^2}^2 + \|\nabla\mathsf{u}\|_{L^2}^2) + (\eta_1+\lambda_{\min})\,\|\nabla\nabla\mathsf{u}\|_{L^2}^2\ .$$

Setting

$$\nu := \min\{\,\kappa\,,\,\eta_1+\lambda_{\min}\,\} > 0\,,$$

we have $a(\mathsf{u},\mathsf{u}) \geq \nu\,\|\mathsf{u}\|_X^2$. □

Consider now the trilinear form $b : H^2\times H^2\times H^2 \to \mathbb{R}$ given by

$$b(\mathsf{u},\mathsf{v},\mathsf{w}) := \int_B F(\mathsf{u},\mathsf{v})\cdot\mathsf{w} = \int_B (\mathsf{u}\cdot\nabla)\mathsf{v}\cdot\mathsf{w}\,,$$

which is indeed continuous since

$$|b(\mathsf{u},\mathsf{v},\mathsf{w})| \leq \|F(\mathsf{u},\mathsf{v})\|_{L^2}\,\|\mathsf{w}\|_{L^2} \leq c_0\,\|\mathsf{u}\|_{H^2}\,\|\mathsf{v}\|_{H^2}\,\|\mathsf{w}\|_{H^2}$$

for every $\mathsf{u},\mathsf{v},\mathsf{w}\in H^2(B)$.

Given the interpolator g for the boundary value, we define the set

$$Y_{\mathsf{g}} := \left\{ \mathsf{h} \in H^2(B) : \operatorname{div} \mathsf{h} = 0 \;,\; \mathsf{h} - \mathsf{g} \in H^1_0(B) \right\} .$$

Lemma 2. *For every* $\mathsf{u} \in X$ *and* $\mathsf{h} \in Y_{\mathsf{g}}$ *we have*

$$b(\mathsf{u},\mathsf{u},\mathsf{u}) = 0 \quad \textit{and} \quad b(\mathsf{h},\mathsf{u},\mathsf{u}) = 0 \;.$$

Proof. By standard formulae in tensor calculus we get the assertion for $\mathsf{u} \in V$ and we can extend it by a density argument. □

Lemma 3. *For every* $\beta > 0$ *there exists* $\mathsf{h} \in Y_{\mathsf{g}}$ *such that*

$$|b(\mathsf{u},\mathsf{h},\mathsf{u})| \le \beta \, \|\mathsf{u}\|_X^2 \;.$$

Proof. See J. L. Lions [6, Chap. 1, Sec. 7.2]. □

We can now prove an existence result for solutions of the non homogeneous stationary problem related to the datum g.

Theorem 3. *There exists* $\mathsf{u} \in X$ *such that, for every* $\mathsf{v} \in X$*,*

$$a(\mathsf{u},\mathsf{v}) + b(\mathsf{u}+\mathsf{g},\mathsf{u}+\mathsf{g},\mathsf{v}) = \langle \varphi, \mathsf{v} \rangle \;, \tag{16}$$

where $\varphi \in X'$ *is the linear form defined by*

$$\langle \varphi, \mathsf{v} \rangle = \int_{\mathcal{S}_N} (f+q)\mathsf{v} \cdot \mathsf{n} + \int_B \mathsf{d} \cdot \mathsf{v} - a(\mathsf{g},\mathsf{v}) \;.$$

Proof. By the Lax-Milgram theorem, the function $L : X \to X'$ defined by

$$\forall \mathsf{v} \in X : \; \langle L(\mathsf{u}), \mathsf{v} \rangle = a(\mathsf{u},\mathsf{v})$$

is a homeomorphism. We have

$$L(\mathsf{u}) + K_{\mathsf{g}}(\mathsf{u}) = \varphi \tag{17}$$

in X', and then

$$\mathsf{u} = L^{-1}\left(\varphi - K_{\mathsf{g}}(\mathsf{u})\right) =: \Phi(\mathsf{u}) \;.$$

Our aim is to apply the following result [3, Corollary 8.1].

Theorem (Fixed Point). *Let* X *be a Banach space and* $\Phi : X \to X$ *a compact operator. Then either* $\Phi(\mathsf{u}) = \mathsf{u}$ *has a solution, or the set*

$$S = \left\{ \mathsf{u} \in X : \; \Phi(\mathsf{u}) = \lambda \mathsf{u} \quad \textit{for some } \lambda > 1 \right\}$$

is unbounded.

Assume that $\mathsf{u} \in X$ is a solution of (17); it means that for every $\mathsf{v} \in X$

$$\langle L(\mathsf{u}), \mathsf{v} \rangle + \langle K_{\mathsf{g}}(\mathsf{u}), \mathsf{v} \rangle = \langle \varphi, \mathsf{v} \rangle \;.$$

Taking $\mathsf{v} = \mathsf{u}$ and applying Lemmas 2 and 3, there exists $\mathsf{h} \in Y_{\mathsf{g}}$ such that

$$\begin{aligned} \nu \, \|\mathsf{u}\|_X^2 \le a(\mathsf{u},\mathsf{u}) &\le |b(\mathsf{u},\mathsf{h},\mathsf{u})| + |b(\mathsf{h},\mathsf{h},\mathsf{u})| + |\langle \varphi, \mathsf{u} \rangle| \\ &\le \frac{\nu}{2} \|\mathsf{u}\|_X^2 + \left(c_1 \|\mathsf{h}\|_X^2 + \|\varphi\|_{X'} \right) \|\mathsf{u}\|_X \;; \end{aligned}$$

from which we have the *a priori* estimate

$$\|\mathsf{u}\|_X \le \frac{2}{\nu} c_1 \|\mathsf{h}\|_X^2 + \frac{2}{\nu} \|\varphi\|_{X'} =: R < +\infty \,. \tag{18}$$

Take now $\lambda > 1$ and assume that $\Phi(\mathsf{u}) = \lambda\mathsf{u}$; it means that

$$\langle\, L(\lambda\mathsf{u}), \mathsf{u} \,\rangle + \langle\, K_{\mathsf{h}}(\mathsf{u}), \mathsf{u} \,\rangle = \langle\, \varphi, \mathsf{u} \,\rangle$$

and, following the argument leading to (18), we obtain

$$\|\mathsf{u}\|_X \le (2\lambda - 1)^{-1} R < R \,.$$

Hence the set S introduced in the Fixed Point theorem is bounded and there exists a fixed point $\mathsf{u} \in X$ for Φ. We can then conclude that $\mathsf{u} + \mathsf{h}$ is a stationary solution for Problem 1. □

We can also prove the following uniqueness result when the data h and φ are "small" compared to the coercivity constant ν.

Theorem 4. *Assume that*

$$4c_1^2 \|\mathsf{h}\|_X^2 + 4c_1 \|\varphi\|_{X'} < \nu^2$$

where φ, h and the constant c_1 are taken as in the proof of Theorem 3.

Then there exists a unique solution $\mathsf{u} \in X$ of equation (16).

Proof. Let u_1 and u_2 be two solutions of (16) and set $\mathsf{w} := \mathsf{u}_1 - \mathsf{u}_2$. Clearly $\mathsf{w} \in X$ and we can take the difference of

$$\langle\, L(\mathsf{u}_1), \mathsf{w} \,\rangle + \langle\, K_{\mathsf{h}}(\mathsf{u}_1), \mathsf{w} \,\rangle = \langle\, \varphi, \mathsf{w} \,\rangle \,,$$

$$\langle\, L(\mathsf{u}_2), \mathsf{w} \,\rangle + \langle\, K_{\mathsf{h}}(\mathsf{u}_2), \mathsf{w} \,\rangle = \langle\, \varphi, \mathsf{w} \,\rangle \,,$$

obtaining

$$\langle\, L(\mathsf{u}_1 - \mathsf{u}_2), \mathsf{w} \,\rangle + \langle\, K_{\mathsf{h}}(\mathsf{u}_1) - K_{\mathsf{h}}(\mathsf{u}_2), \mathsf{w} \,\rangle = 0 \,,$$

that is

$$a(\mathsf{w}, \mathsf{w}) + b(\mathsf{u}_1, \mathsf{u}_1, \mathsf{w}) - b(\mathsf{u}_2, \mathsf{u}_2, \mathsf{w}) + b(\mathsf{w}, \mathsf{h}, \mathsf{w}) + b(\mathsf{h}, \mathsf{w}, \mathsf{w}) = 0 \,.$$

It is easy to check that

$$b(\mathsf{u}_1, \mathsf{u}_1, \mathsf{w}) - b(\mathsf{u}_2, \mathsf{u}_2, \mathsf{w}) = b(\mathsf{w}, \mathsf{w}, \mathsf{w}) + b(\mathsf{w}, \mathsf{u}_2, \mathsf{w}) + b(\mathsf{u}_2, \mathsf{w}, \mathsf{w})$$

and then, by Lemma 2,

$$a(\mathsf{w}, \mathsf{w}) + b(\mathsf{w}, \mathsf{u}_2, \mathsf{w}) + b(\mathsf{w}, \mathsf{h}, \mathsf{w}) = 0 \,.$$

Applying again Lemma 3, we have

$$\nu \|\mathsf{w}\|_X^2 \le a(\mathsf{w}, \mathsf{w}) \le |b(\mathsf{w}, \mathsf{u}_2, \mathsf{w})| + \frac{\nu}{2} \|\mathsf{w}\|_X^2 \le \left(\frac{\nu}{2} + c_1 \|\mathsf{u}_2\|_X \right) \|\mathsf{w}\|_X^2$$

and, since $\|\mathsf{u}_2\|_X \le R$ by the bound (18), we get

$$\nu \|\mathsf{w}\|_X^2 \le \left(\frac{\nu}{2} + \frac{2c_1^2}{\nu} \|\mathsf{h}\|_X^2 + \frac{2c_1}{\nu} \|\varphi\|_{X'} \right) \|\mathsf{w}\|_X^2 \,,$$

and hence

$$\left(\frac{2c_1^2}{\nu}\,\|\mathsf{h}\|_X^2 + \frac{2c_1}{\nu}\,\|\varphi\|_{X'} - \frac{\nu}{2}\right)\|\mathsf{w}\|_X^2 \geq 0\ .$$

It is now clear that, when

$$4c_1^2\,\|\mathsf{h}\|_X^2 + 4c_1\,\|\varphi\|_{X'} < \nu^2\ ,$$

we necessarily have $\|\mathsf{w}\|_X^2 = 0$ and then $\mathsf{u}_1 = \mathsf{u}_2$. □

4.4. The evolutionary problem

We now come to analyze the evolutionary problem. Take $\mathsf{u} \in L^2([0,T];X)$, whose norm is

$$\|\mathsf{u}\|_{L^2([0,T];X)}^2 := \int_0^T \|\mathsf{u}(s)\|_X^2\,ds\ ,$$

the divergence-free datum $\mathsf{g} \in L^2([0,T];H^2(B))$ and $\varphi \in L^2([0,T];X')$.

With the notations of the previous section we can state our main result.

Theorem 5. *There exists* $\mathsf{u} \in L^2([0,T];X)$ *such that*

$$\mathsf{u}(0) = \mathsf{u}_0\ , \tag{19}$$

$$\int_B \frac{\partial \mathsf{u}}{\partial t}\cdot \mathsf{v} + a(\mathsf{u},\mathsf{v}) + b(\mathsf{u}+\mathsf{g},\mathsf{u}+\mathsf{g},\mathsf{v}) = \langle\,\varphi,\mathsf{v}\,\rangle\ , \tag{20}$$

for every $\mathsf{v} \in L^2([0,T];X)$.

Remark 4. Notice that the time derivative of u has to be interpreted as a linear form in $L^2([0,T];X')$, whose representation enters equation (20), and thus we will take $\mathsf{u} \in H^1([0,T];X')$. A key rôle in the evolutionary problem is played by the initial datum u_0 which belongs to H. At first the initial condition (19) should be understood in X', but we will see that it actually takes place in H, as u belongs to $C([0,T];H)$.

Proof. In order to proceed we need some estimates; first we set $\mathsf{v} = \mathsf{u}$ in (20) and apply Lemma 2 obtaining

$$\int_B \frac{\partial \mathsf{u}}{\partial t}\cdot \mathsf{u} + a(\mathsf{u},\mathsf{u}) + b(\mathsf{u},\mathsf{g},\mathsf{u}) + b(\mathsf{g},\mathsf{g},\mathsf{u}) = \langle\,\varphi,\mathsf{u}\,\rangle\ .$$

Integrating in time and defining $\langle\,\tilde\varphi,\mathsf{u}\,\rangle = \langle\,\varphi,\mathsf{u}\,\rangle - b(\mathsf{g},\mathsf{g},\mathsf{u})$, we get

$$\frac{1}{2}\int_0^t \frac{d}{ds}\,\|\mathsf{u}(s)\|_{L^2}^2\,ds + \int_0^t a(\mathsf{u},\mathsf{u})\,ds = -\int_0^t b(\mathsf{u},\mathsf{g},\mathsf{u})\,ds + \int_0^t \langle\,\tilde\varphi,\mathsf{u}\,\rangle\,ds\ ,$$

hence

$$\frac{1}{2}\,\|\mathsf{u}(t)\|_{L^2}^2 + \nu\int_0^t \|\mathsf{u}\|_X^2\,ds \leq \frac{1}{2}\,\|\mathsf{u}(0)\|_{L^2}^2 + \int_0^t |b(\mathsf{u},\mathsf{g},\mathsf{u})|ds + \int_0^t \|\tilde\varphi\|_{X'}\,\|\mathsf{u}\|_X\,ds.$$

By Lemma 3 and Young's inequality,

$$\frac{1}{2}\,\|\mathsf{u}(t)\|_{L^2}^2 + \nu\int_0^t \|\mathsf{u}\|_X^2\,ds \leq \frac{1}{2}\,\|\mathsf{u}_0\|_{L^2}^2 + \frac{\nu}{2}\int_0^t \|\mathsf{u}\|_X^2\,ds + c_2\int_0^t \|\tilde\varphi\|_{X'}^2\,ds\ ,$$

that gives the first estimate for a.e. $t \in [0,T]$:

$$\|\mathsf{u}(t)\|_{L^2}^2 + \nu \int_0^t \|\mathsf{u}\|_X^2 \, ds \leq \|\mathsf{u}_0\|_{L^2}^2 + 2c_2 \int_0^t \|\tilde{\varphi}\|_{X'}^2 \, ds =: M \, .$$

This *a priori* bound tells us that any solution u of our problem belongs to a bounded subset of $L^2([0,T];X) \cap L^\infty([0,T];H)$. We now introduce the following theorem whose proof is given in [5, Chap. 3, Sec. 4.4].

Theorem (J. L. Lions). *Let X and H be two Hilbert spaces, with X dense and continuously embedded in H; identify H with its dual in such a way that $X \subset H \subset X'$ and fix $T > 0$. Consider a bilinear form $a_t(\mathsf{u},\mathsf{v}) : X \times X \to \mathbb{R}$ such that:*

i) *the function* $t \mapsto a_t(\mathsf{u},\mathsf{v})$ *is measurable for every* $\mathsf{u},\mathsf{v} \in X$*;*
ii) $|a_t(\mathsf{u},\mathsf{v})| \leq C_1 \|\mathsf{u}\|_X \|\mathsf{v}\|_X$ *for a.e.* $t \in [0,T]$*, for every* $\mathsf{u},\mathsf{v} \in X$*;*
iii) $a_t(\mathsf{v},\mathsf{v}) \geq \alpha \|\mathsf{v}\|_X^2 - C_2 \|\mathsf{v}\|_H^2$ *for a.e.* $t \in [0,T]$*, for every* $\mathsf{v} \in X$*;*

where $\alpha > 0$, C_1 and C_2 are constant.

Then for every $\mathsf{f} \in L^2([0,T];X')$ and for every $\mathsf{u}_0 \in H$ there exists only one u such that

$$\mathsf{u} \in L^2([0,T];X) \cap C([0,T];H) \cap H^1([0,T];X') =: \mathfrak{X}$$

$$\mathsf{u}(0) = \mathsf{u}_0$$

$$\langle \frac{d\mathsf{u}}{dt}(t), \mathsf{v} \rangle + a_t(\mathsf{u}(t),\mathsf{v}) = \langle \mathsf{f}(t), \mathsf{v} \rangle \quad \textit{for a.e. } t \in [0,T], \textit{ for every } \mathsf{v} \in X \, .$$

It is easy to see that the spaces X, H as earlier defined and the bilinear form $a_t(\mathsf{u},\mathsf{v}) := a(\mathsf{u}(t),\mathsf{v}(t))$ fulfill the hypotheses of the previous theorem; then the function

$$L : \begin{cases} \mathfrak{X} & \to \quad H \times L^2([0,T];X') \\ \mathsf{u} & \mapsto \quad \left(\mathsf{u}(0) \, , \, \dfrac{d\mathsf{u}}{dt}(t) + a(\mathsf{u}(t), \cdot\,) \right) \end{cases}$$

is a homeomorphism.

We can now write equations (19)–(20) in $H \times L^2([0,T];X')$ as

$$L(\mathsf{u}) + (\, 0 \, , \, K_{\mathsf{g}}(\mathsf{u}) \,) = (\, \mathsf{u}_0 \, , \, \varphi \,) \, ,$$

from which

$$\mathsf{u} = L^{-1}(\, \mathsf{u}_0 \, , \, \varphi - K_{\mathsf{g}}(\mathsf{u}) \,) =: \Phi(\mathsf{u}) \, .$$

By Theorem 2 and composition arguments, Φ turns out to be a compact operator, and we can then apply the Fixed Point theorem in the Banach space $\mathfrak{X}$. Arguing as in the previous section we take $\lambda > 1$ and assume $\Phi(\mathsf{u}) = \lambda \mathsf{u}$; in particular $\mathsf{u}_0 = \lambda \mathsf{u}(0)$ and

$$\frac{\lambda}{2} \int_0^t \frac{d}{ds} \|\mathsf{u}(s)\|_{L^2}^2 \, ds + \lambda \int_0^t a(\mathsf{u},\mathsf{u}) \, ds = - \int_0^t b(\mathsf{u},\mathsf{g},\mathsf{u}) \, ds + \int_0^t \langle \tilde{\varphi}, \mathsf{u} \rangle \, ds \, .$$

We then obtain

$$\frac{\lambda}{2} \|\mathsf{u}(t)\|_{L^2}^2 + \lambda\nu \int_0^t \|\mathsf{u}\|_X^2 \, ds \leq \frac{\lambda}{2} \|\mathsf{u}(0)\|_{L^2}^2 + \int_0^t |b(\mathsf{u},\mathsf{g},\mathsf{u})| ds + \int_0^t \|\tilde{\varphi}\|_{X'} \|\mathsf{u}\|_X \, ds,$$

and, by Lemma 3 and Young's inequality,

$$\frac{\lambda}{2}\left\|\mathsf{u}(t)\right\|_{L^2}^2 + \lambda\nu\int_0^t \left\|\mathsf{u}\right\|_X^2 ds \le \frac{\lambda}{2}\left\|\lambda^{-1}\mathsf{u}_0\right\|_{L^2}^2 + \frac{\nu}{2}\int_0^t \left\|\mathsf{u}\right\|_X^2 ds + c_2\int_0^t \left\|\tilde{\varphi}\right\|_{X'}^2 ds,$$

that gives

$$\lambda\left\|\mathsf{u}(t)\right\|_{L^2}^2 + (2\lambda-1)\nu\int_0^t \left\|\mathsf{u}\right\|_X^2\, ds \le \lambda^{-1}\left\|\mathsf{u}_0\right\|_{L^2}^2 + 2c_2\int_0^t \left\|\tilde{\varphi}\right\|_{X'}^2\, ds < M\,.$$

Since $2\lambda - 1 > \lambda$ we obtain

$$\left\|\mathsf{u}(t)\right\|_{L^2}^2 + \nu\int_0^t \left\|\mathsf{u}\right\|_X^2\, ds < \lambda^{-1}M$$

and the set S introduced in the Fixed Point theorem is bounded in $L^2([0,T];X)\cap L^\infty([0,T];H)$. In order to complete the proof it remains to show that S is bounded also in $H^1([0,T];X')$. If there exists $\lambda > 1$ such that $\Phi(\mathsf{u}) = \lambda\mathsf{u}$, then we have

$$\frac{\partial\mathsf{u}}{\partial t} = -a(\mathsf{u},\cdot\,) - \frac{1}{\lambda}K_{\mathsf{g}}(\mathsf{u}) + \frac{1}{\lambda}\varphi$$

in $L^2([0,T];X')$ and

$$\left\|\frac{\partial\mathsf{u}}{\partial t}\right\|_{X'} \le \left\|a(\mathsf{u},\cdot\,)\right\|_{X'} + \frac{1}{\lambda}\left\|K_{\mathsf{g}}(\mathsf{u})\right\|_{X'} + \frac{1}{\lambda}\left\|\varphi\right\|_{X'}\ .$$

By the continuity of a and the embeddings mentioned in the proof of Theorem 2 we can write

$$\left\|\frac{\partial\mathsf{u}}{\partial t}\right\|_{X'} \le c_3\left\|\mathsf{u}\right\|_X + \frac{c_4}{\lambda}\left\|K_{\mathsf{g}}(\mathsf{u})\right\|_{L^{\frac{3}{2}}} + \frac{1}{\lambda}\left\|\varphi\right\|_{X'}\ .$$

We know that u belongs to a bounded subset of $L^2([0,T];X)$ and $\varphi\in L^2([0,T];X')$; again by the proof of Theorem 2 we deduce that

$$\int_0^T \left\|\frac{\partial\mathsf{u}}{\partial t}\right\|_{X'}^2 \le c_5 + \frac{1}{\lambda}\int_0^T \left\|K_{\mathsf{g}}(\mathsf{u})\right\|_{L^{\frac{3}{2}}} < N$$

for a fixed $N > 0$, since K_{g} maps bounded subsets of $L^2([0,T];X)\cap L^\infty([0,T];H)$ in bounded subsets of $L^2([0,T];L^{\frac{3}{2}}(B))$.

The last bound shows that $\frac{\partial\mathsf{u}}{\partial t}$ belongs to a bounded subset of $L^2([0,T];X')$ and implies that S is bounded in $\mathfrak{X}$. Hence there exists a fixed point $\mathsf{u}\in\mathfrak{X}$ for Φ and this is a solution for the Cauchy problem (19)–(20). □

Thanks to the H^2-regularity of the solution we have found, which in particular guarantees the L^∞-regularity in three dimensions, we can prove an important uniqueness result.

Theorem 6. *There exists a unique solution* $\mathsf{u}\in L^2([0,T];X)$ *of the Cauchy problem* (19)–(20).

Proof. Let u_1 and u_2 be solutions of equation (20) with the same initial datum and set $\mathsf{w} := \mathsf{u}_1 - \mathsf{u}_2$. We easily obtain, by equations (19)–(20) and Lemma 2,

$$\begin{aligned} &\mathsf{w}(0) = 0 , \\ &\int_B \frac{\partial \mathsf{w}}{\partial t} \cdot \mathsf{w} + a(\mathsf{w},\mathsf{w}) + b(\mathsf{w},\mathsf{u}_2,\mathsf{w}) + b(\mathsf{w},\mathsf{h},\mathsf{w}) = 0 , \end{aligned}$$

from which, integrating in time and applying Lemma 3 and the coercivity of a, we can deduce

$$\frac{1}{2}\,\|\mathsf{w}(t)\|^2_{L^2} + \nu \int_0^t \|\mathsf{w}\|^2_X\, ds \le \frac{\nu}{2} \int_0^t \|\mathsf{w}\|^2_X\, ds + \int_0^t |(\mathsf{w}\cdot\nabla)\mathsf{u}_2 \cdot \mathsf{w}|\, ds ,$$

and then, using also Young's inequality,

$$\begin{aligned} \|\mathsf{w}(t)\|^2_{L^2} + \nu \int_0^t \|\mathsf{w}\|^2_X\, ds &\le 2 \int_0^t \|\mathsf{w}\|_{L^\infty}\, \|\nabla \mathsf{u}_2\|_{L^2}\, \|\mathsf{w}\|_{L^2}\, ds \\ &\le c_6 \int_0^t \|\mathsf{w}\|_X\, \|\mathsf{u}_2\|_X\, \|\mathsf{w}\|_{L^2}\, ds \le \nu \int_0^t \|\mathsf{w}\|^2_X\, ds + c_7 \int_0^t \|\mathsf{u}_2\|^2_X\, \|\mathsf{w}\|^2_{L^2}\, ds; \end{aligned}$$

finally we have

$$\|\mathsf{w}(t)\|^2_{L^2} \le c_7 \int_0^t \|\mathsf{u}_2(s)\|^2_X\, \|\mathsf{w}(s)\|^2_{L^2}\, ds$$

and by Gronwall's lemma we conclude that $\|\mathsf{w}(t)\|^2_{L^2} = 0$ for every $t \in [0,T]$, that is $\mathsf{u}_1 = \mathsf{u}_2$. □

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