Meccanica **38:** 369–389, 2003.

Balanced Powers in Continuum Mechanics

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(Received: 26 March 2002; accepted in revised form 19 September 2002)

Abstract. An approach to weak balance laws in Continuum Mechanics is presented, involving densities with only divergence measure, which relies on the balance of power. An equivalence theorem between Cauchy powers and Cauchy fluxes is proved. As an application of this method, the construction of the stress tensor when the body is an orientable differential manifold is achieved under very general assumptions.

Key words: Balance laws, Virtual power, Continuum mechanics.

1. Introduction

It is widely known that the balance of momentum in Continuum Mechanics leads to the notion of Cauchy stress tensor field and to the formulation of a differential or an integral law. In this framework, the stress is regarded as a primitive concept and the celebrated Cauchy stress theorem holds, at least with continuous stress functions as in the original proof of Cauchy.

Because of its formal elegance, this approach has been followed in the axiomatization of Continuum Mechanics made by Noll [15] and Truesdell [20] and, probably due to this influence, it has been subjected to several refinements, like in the papers of Gurtin and Martins [12] who introduced the idea of Cauchy flux, of Šilhavý [18, 19] who considered possibly unbounded flux densities, and of Degiovanni *et al.* [2] who generalized them to stress densities with divergence measure. By introducing the notion of 'almost every part', Šilhavý also proved the equivalence between fluxes of class L^p and densities with divergence in L^p. Parallel to this development, the concept of subbody from the choice of Kellogg's regions (see [7, Sections 8 and 9]) has been more and more generalized up to the use of sets with finite perimeter by Ziemer [21] and the idea of normalized subsets of Šilhavý [19]. In [2, 13, 14] it has also been shown [2] that the class of normalized sets with finite perimeter is a sort of minimal class, in the sense that it suffices to state the balance law on very simple subbodies in order to have it uniquely extended on almost every set with finite perimeter.

From the existence of the stress tensor field, it is then classical and customary to derive the principle of virtual powers based on the concept of mechanical power. Also this concept has been weakened by Antman and Osborn [1].

On the other hand, during the 1970s Germain [8–10] showed how the balance of momentum can be given assuming as a postulate the Principle of Virtual Power, thus recovering the classical approach of d'Alembert. An interesting feature of this approach is that the mechanical power has more resemblance to a distribution than a Cauchy flux, thus allowing better a functional-analytic treatment, as already pointed out in Germain's work. Finally, since powers are applied to a velocity field, the principle of virtual power implies the balance of stresses when constant velocity fields are chosen.

This approach seems also to be more fruitful when dealing with stresses which are concentrated in low-dimensional parts of the body, as shown also by Dell'Isola and Seppecher [3] and Di Carlo and Tatone [4]. In this case, a weakening of the theory based on stresses in the spirit of Noll and Virga [16] raises some difficulties. Moreover, if the body is a differential manifold which is not an open subset of $\mathbb{R}^n$, and the interaction is of a vectorial kind, the approach via the virtual powers seems to be mandatory.

In this paper we investigate Germain's approach in two directions: first, we give it the same degree of generality and 'weakness' as it appears in [2]; second, we find conditions which connect the two formulations. More precisely, we introduce the notion of *Cauchy power* as a function of a subbody and of a velocity field, and prove an equivalence theorem between Cauchy fluxes and Cauchy powers. In this way we are able to recover the existence of the stress tensor field from the balance of mechanical power and to state these results in the framework of n-intervals. Moreover, we give necessary and sufficient conditions in order to have the symmetry of the tensor field or, equivalently, the balance of angular momentum.

In the last section, we apply this theory to the case where the body is an orientable differential manifold. The application is interesting in itself because mechanical bodies can be curved surfaces, and also mathematically not trivial because the concept of constant vector field cannot be given on a manifold. We are thus able to state and prove in this case the main results obtained in $\mathbb{R}^n$ and recognize that the stress tensor field is a $(0, n)$ tensor field. All these results do not rely on a Riemannian structure. The counterpart based on stresses has been treated in [17], but only for scalar fluxes and with the existence of a density given as an assumption.

2. Cauchy Fluxes and Interactions

In this section we define *the subbodies*, a class of sets by which we state the balance law in an integral form. We ask these sets to be *normalized*, which – roughly speaking – correspond to take regularly open sets in a measure-theoretic sense; moreover, we consider sets with finite perimeter, so that we can apply the Gauss–Green theorem. Finally, the sets will be taken with closure in the interior of the body, because we want their measure-theoretic boundary not to meet the boundary of the body.

For $n \geqslant 1$, $\mathcal{L}^n$ will denote the n-dimensional Lebesgue outer measure and $\mathcal{H}^k$ the k-dimensional Hausdorff outer measure on $\mathbb{R}^n$. Given a Borel subset $E \subseteq \mathbb{R}^n$, we denote with $\mathfrak{B}(E)$ the collection of all Borel subsets of E. Moreover, $E \Delta F$ will denote the set $(E \backslash F) \cup (F \backslash E)$.

Consider a set $M \subseteq \mathbb{R}^n$. The topological closure and interior of M will be denoted as usual by $\operatorname{cl} M$ and $\operatorname{int} M$, respectively. Denoting with $B_r(x)$ the open ball with radius r centered in x, we introduce *the measure-theoretic interior* of M

$$M_* = \left\{ x \in \mathbb{R}^n \colon \lim_{r \to 0^+} \frac{\mathcal{L}^n(B_r(x) \backslash M)}{\mathcal{L}^n(B_r(x))} = 0 \right\},$$

the measure-theoretic boundary of M

$$\partial_* M = \mathbb{R}^n \backslash (M_* \cup (\mathbb{R}^n \backslash M)_*)$$

and *the measure-theoretic closure* of M

$$M^* = M \cup \partial_* M.$$

They are all Borel subsets of $\mathbb{R}^n$.

DEFINITION 1. We say that $M \subseteq \mathbb{R}^n$ is *normalized*, if $M_* = M$.

DEFINITION 2. If for a subset M of $\mathbb{R}^n$ one has $\mathcal{H}^{n-1}(\partial_* M) < +\infty$, then we say that M is *a set with finite perimeter*, also said *a Caccioppoli set*.

A set with finite perimeter is in some sense regular (see [11] and Proposition 1 below); in particular, it is $\mathcal{L}^n$-measurable.

Now we introduce the concept of *outer normal* to the measure-theoretic boundary of a set. Let $M \subseteq \mathbb{R}^n$ and $x \in \partial_* M$. We denote by $\mathbf{n}^M(x) \in \mathbb{R}^n$ a unitary vector such that

$$\mathcal{L}^n(\{\xi \in B_r(x) \cap M : (\xi - x) \cdot \mathbf{n}^M(x) > 0\})/r^n \to 0,$$
$$\mathcal{L}^n(\{\xi \in B_r(x) \backslash M : (\xi - x) \cdot \mathbf{n}^M(x) < 0\})/r^n \to 0$$

as $r \to 0^+$. No more than one such vector can exist. Setting $\mathbf{n}^M(x) = 0$ in the other case, we can consider the map $\mathbf{n}^M : \partial_* M \to \mathbb{R}^n$, which is called *the unit outer normal* to M. It turns out that $\mathbf{n}^M$ is Borel and bounded.

The following propositions state the main features of sets with finite perimeter which we are interested in.

PROPOSITION 1. *If $M \subseteq \mathbb{R}^n$ is a set with finite perimeter, then $|\mathbf{n}^M(x)| = 1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial_* M$ and the Gauss–Green theorem*

$$\int_M \mathbf{v} \cdot \nabla f \, \mathrm{d}\mathcal{L}^n = \int_{\partial_* M} f \mathbf{v} \cdot \mathbf{n}^M \, \mathrm{d}\mathcal{H}^{n-1} - \int_M f \operatorname{div} \mathbf{v} \, \mathrm{d}\mathcal{L}^n$$

holds whenever $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz continuous with compact support.

PROPOSITION 2. *Let M, N be two normalized subsets of $\mathbb{R}^n$ with finite perimeter and let*

$$E = \{x \in \partial_* M \cap \partial_* N : \mathbf{n}^M(x) \neq 0, \mathbf{n}^N(x) \neq 0, \mathbf{n}^M(x) \neq -\mathbf{n}^N(x)\},$$
$$F = \{x \in \partial_* M \cap \partial_* N : \mathbf{n}^M(x) \neq 0, \mathbf{n}^N(x) \neq 0, \mathbf{n}^M(x) \neq \mathbf{n}^N(x)\}.$$

Then there exist $R_k \subseteq \partial_* M \cap \partial_* N$, $k = 1, 2, 3$, with $\mathcal{H}^{n-1}(R_k) = 0$ and

$$\partial_*(M \cup N) = (\partial_* M \backslash N^*) \cup (\partial_* N \backslash M^*) \cup E \cup R_1, \tag{1}$$

$$\partial_*(M \cap N) = (M \cap \partial_* N) \cup (N \cap \partial_* M) \cup E \cup R_2, \tag{2}$$

$$\partial_*(M \backslash N) = (\partial_* M \backslash N^*) \cup (M \cap \partial_* N) \cup F \cup R_3, \tag{3}$$

where the unions are disjoint.

PROPOSITION 3. *Let M_1, M_2, M_3 be three mutually disjoint subsets of $\mathbb{R}^n$ with finite perimeter. Then*

$$\mathcal{H}^{n-1}(\partial_* M_1 \cap \partial_* M_2 \cap \partial_* M_3) = 0.$$

For a proof of the first, we refer to [6, Theorem 4.5.6] or [22, Theorem 5.8.2], while the second can be found in [13, Proposition 2.2] and the last is an easy consequence of the properties of the unit outer normal.

Let now Ω be an open subset of $\mathbb{R}^n$. We denote by $\mathfrak{M}(\Omega)$ the set of Borel measures $\mu : \mathfrak{B}(\Omega) \to [0, +\infty]$ finite on compact subsets of Ω and by $\mathcal{L}^1_{\mathrm{loc},+}(\Omega; \mu)$ the set of Borel functions $h : \Omega \to [0, +\infty]$ with $\int_K h \, \mathrm{d}\mu < +\infty$ for every compact subset $K \subseteq \Omega$. When $\mu = \mathscr{L}^n$, we will write simply $\mathcal{L}^1_{\mathrm{loc},+}(\Omega)$. For a finite-dimensional normed space X, we denote by $\mathcal{L}^1_{\mathrm{loc}}(\Omega; X)$ the set of Borel maps $\mathbf{v} : \Omega \to X$ with $\int_K \|\mathbf{v}\| \, \mathrm{d}\mathscr{L}^n < +\infty$ for any compact subset K of Ω.

Throughout the remainder of this work, B will denote a bounded normalized subset with $\mathbb{R}^n$ with finite perimeter, which we call *a body*.

DEFINITION 3. We denote with $\mathcal{M}^\circ$ the collection of all normalized subsets M of B of finite perimeter such that $\mathrm{cl}\, M \subseteq \mathrm{int}\, B$. We call such sets *the subbodies*.

If $h \in \mathcal{L}^1_{\mathrm{loc},+}(\mathrm{int}\, B)$ and $\nu \in \mathfrak{M}(\mathrm{int}\, B)$, we set

$$\mathcal{M}^\circ_{h\nu} = \left\{ M \in \mathcal{M}^\circ : \int_{\partial_* M} h \, \mathrm{d}\mathscr{H}^{n-1} < +\infty, \nu(\partial_* M) = 0 \right\}.$$

Remark. If $M, N \in \mathcal{M}^\circ_{h\nu}$, by Proposition 2 it follows that $(M \cup N)_*$, $M \cap N$, $(M \backslash N)_* \in \mathcal{M}^\circ_{h\nu}$.

The notion we are going to define has been introduced in [19] and revised in [2].

DEFINITION 4. We say that $\mathcal{P} \subseteq \mathcal{M}^\circ$ *contains almost all of* $\mathcal{M}^\circ$, if $\mathcal{M}^\circ_{h\nu} \subseteq \mathcal{P}$ for some $h \in \mathcal{L}^1_{\mathrm{loc},+}(\mathrm{int}\, B)$ and $\nu \in \mathfrak{M}(\mathrm{int}\, B)$.

A property π *holds on almost all of* $\mathcal{M}^\circ$, if the set

$$\{M \in \mathcal{M}^\circ : \pi(M) \text{ is defined and } \pi(M) \text{ holds}\}$$

contains almost all of $\mathcal{M}^\circ$.

The following proposition states an interesting feature of the notion above: given a countable set of properties such that each of them holds on almost all of $\mathcal{M}^\circ$, they simultaneously hold on almost all of $\mathcal{M}^\circ$.

PROPOSITION 4. *If* (h_m), (ν_m) *are sequences in* $\mathcal{L}^1_{\mathrm{loc},+}(\mathrm{int}\, B)$ *and* $\mathfrak{M}(\mathrm{int}\, B)$, *respectively, then there exist* $h \in \mathcal{L}^1_{\mathrm{loc},+}(\mathrm{int}\, B)$ *and* $\nu \in \mathfrak{M}(\mathrm{int}\, B)$ *such that*

$$\mathcal{M}^\circ_{h\nu} \subseteq \bigcap_{m \in \mathbb{N}} \mathcal{M}^\circ_{h_m \nu_m}.$$

Proof. Let (K_m) be an increasing sequence of compact subsets of $\mathrm{int}\, B$ such that $\mathrm{int}\, B = \bigcup_{m \in \mathbb{N}} \mathrm{int}\, K_m$. Setting

$$\forall x \in \mathrm{int}\, B : \quad h(x) = \sum_{m \in \mathbb{N}} \frac{h_m(x)}{2^{m+1}\left(1 + \int_{K_m} h_m \, \mathrm{d}\mathscr{L}^n\right)},$$

$$\forall E \in \mathfrak{B}(\mathrm{int}\, B) : \quad \nu(E) = \sum_{m \in \mathbb{N}} \frac{\nu_m(E)}{2^{m+1}(1 + \nu_m(K_m))},$$

one can prove that h and ν have the required properties. □

To define a flux as a set function, we need the concept of *material surface*.

DEFINITION 5. We call *material surface in the body B* a pair $S = (\hat{S}, \mathbf{n}_S)$, where $\hat{S}$ is a Borel subset of $\mathbb{R}^n$ and $\mathbf{n}_S : \hat{S} \to \mathbb{R}^n$ is a Borel map such that there exists $M \in \mathcal{M}^\circ$ with $\hat{S} \subseteq \partial_* M$ and $\mathbf{n}_S = \mathbf{n}^M|_{\hat{S}}$. In this case, we say that S is *subordinated* to M. We denote by $\mathcal{S}^\circ$ the collection of the material surfaces in the body B.

We call $\mathbf{n}_S$ *the normal* to the surface S. If also $(\hat{S}, -\mathbf{n}_S)$ is a material surface, we denote it with $-S$. Let S, T be two material surfaces. We shall write $S \subseteq T$ if $\hat{S} \subseteq \hat{T}$ and $\mathbf{n}_T$ extends $\mathbf{n}_S$. S and T are said to be *disjoint* if $\hat{S} \cap \hat{T} = \emptyset$. They are said to be *compatible* if both are subordinated to the same M. In this case, we denote by $S \cup T$ the material surface $(\hat{S} \cup \hat{T}, \mathbf{n}_{S\cup T})$, where

$$\mathbf{n}_{S\cup T}(x) = \begin{cases} \mathbf{n}_S(x) & \text{if } x \in \hat{S}, \\ \mathbf{n}_T(x) & \text{if } x \in \hat{T}. \end{cases}$$

In the following, we shall sometimes identify $\hat{S}$ with S, provided that the reference to $\mathbf{n}_S$ is clear.

DEFINITION 6. For every $h \in \mathcal{L}^1_{\mathrm{loc},+}(\mathrm{int}\, B)$ and $\nu \in \mathfrak{M}(\mathrm{int}\, B)$ we set

$$\mathcal{S}^\circ_{h\nu} = \{S \in \mathcal{S}^\circ\colon S \text{ is subordinated to some } M \in \mathcal{M}^\circ_{h\nu}\}.$$

Given a set $\mathcal{R} \subseteq \mathcal{S}^\circ$, we say that $\mathcal{R}$ *contains almost all of* $\mathcal{S}^\circ$, if $\mathcal{S}^\circ_{h\nu} \subseteq \mathcal{R}$ for some $h \in \mathcal{L}^1_{\mathrm{loc},+}(\mathrm{int}\, B)$ and $\nu \in \mathfrak{M}(\mathrm{int}\, B)$; we say that a property π *holds on almost all of* $\mathcal{S}^\circ$, if the set

$$\{S \in \mathcal{S}^\circ\colon \pi(S) \text{ is defined and } \pi(S) \text{ holds}\}$$

contains almost all of $\mathcal{S}^\circ$.

It is readily seen that if $S \in \mathcal{S}^\circ_{h\nu}$ and T is a material surface with $T \subseteq S$, then $T \in \mathcal{S}^\circ_{h\nu}$. Moreover, when $S_1, S_2 \in \mathcal{S}^\circ_{h\nu}$ are compatible, then $S_1 \cup S_2 \in S^\circ_{h\nu}$. Finally, by Proposition 4 we find that the intersection of a countable family of sets containing almost all of $\mathcal{S}^\circ$ contains itself almost all of $\mathcal{S}^\circ$.

We define now a particularly simple class of subbodies and material surfaces.

DEFINITION 7. A full grid G *is an ordered triple*

$$G = (x_0, (e_1, \ldots, e_n), \hat{G}),$$

where $x_0 \in \mathbb{R}^n$, $(e_1, \ldots, e_n)$ is a positively oriented orthonormal basis in $\mathbb{R}^n$ and $\hat{G}$ is a Borel subset of $\mathbb{R}$ with $\mathcal{L}^1(\mathbb{R}\backslash\hat{G}) = 0$. If G_1, G_2 are two full grids, we write $G_1 \subseteq G_2$ if the first two components coincide and $\hat{G}_1 \subseteq \hat{G}_2$.

DEFINITION 8. Let $G = (x_0, (e_1, \ldots, e_n), \hat{G})$ be a full grid; a subset I of $\mathbb{R}^n$ is called *a G-interval*, if

$$I = \{x \in \mathbb{R}^n\colon a_j < (x - x_0) \cdot e_j < b_j \ \forall j = 1, \ldots, n\}$$

for some $a_1, b_1, \ldots, a_n, b_n \in \hat{G}$. We set

$$\mathcal{I}^\circ_G = \{I\colon I \text{ is a } G\text{-interval with } \mathrm{cl}\, I \subseteq \mathrm{int}\, B\}.$$

For $1 \leqslant j \leqslant n$, we denote by $\mathcal{S}^{\circ}_{G,j}$ the family of all the material surfaces $(\hat{S}, \mathbf{n}_S)$ such that $\mathbf{n}_S = e_j$ and $\mathrm{cl}\,\hat{S} \subseteq \mathrm{int}\,B$, where

$$\hat{S} = \{x \in \mathbb{R}^n\colon (x - x_0) \cdot e_j = s, a_i < (x - x_0) \cdot e_i < b_i \ \forall i \neq j\}$$

for some $a_1, b_1, \ldots, s, \ldots, a_n, b_n \in \hat{G}$. We set also

$$\mathcal{S}^{\circ}_G = \bigcup_{j=1}^{n} \mathcal{S}^{\circ}_{G,j}.$$

Finally we recall a property of the above class of subbodies and material surfaces.

PROPOSITION 5. *Let $x_0 \in \mathbb{R}^n$ and $(e_1, \ldots, e_n)$ be a positively oriented orthonormal basis in $\mathbb{R}^n$. Then for every $h \in \mathcal{L}^1_{\mathrm{loc},+}(\mathrm{int}\,B)$ and $\nu \in \mathfrak{M}(\mathrm{int}\,B)$ there exists a full grid $G = (x_0, (e_1, \ldots, e_n), \hat{G})$ such that $\mathcal{I}^{\circ}_G \subseteq \mathcal{M}^{\circ}_{h\nu}$ and $\mathcal{S}^{\circ}_G \subseteq \mathcal{S}^{\circ}_{h\nu}$.*

For a proof, we refer the reader to [2, Proposition 4.5].

3. The Cauchy Power

In the classical framework, the existence of the stress density $\mathbf{t}(x, \mathbf{n})$ is assumed and one can define the contact power of the stress of a subbody M on a vector field $\mathbf{v}$ setting

$$P(M, \mathbf{v}) = \int_{\partial M} \mathbf{t}(x, \mathbf{n}^M) \cdot \mathbf{v} \, \mathrm{d}\mathcal{H}^{n-1}.$$

In particular, it is clear that P is linear in $\mathbf{v}$ and the inequality

$$|P(M, \mathbf{v})| \leqslant \int_{\partial M} |\mathbf{v}| h \, \mathrm{d}\mathcal{H}^{n-1}$$

holds with $h = |\mathbf{t}|$. Moreover, the Cauchy stress theorem proves the existence of a tensor field T such that $\mathbf{t}(x, \mathbf{n}) = \mathsf{T}(x)\mathbf{n}$. Supposing T, $\mathbf{v}$ and ∂M smooth, one can apply the Gauss–Green theorem, obtaining

$$P(M, \mathbf{v}) = \int_M [(\mathrm{div}\,\mathsf{T}) \cdot \mathbf{v} + \mathsf{T} \cdot \nabla \mathbf{v}] \, \mathrm{d}\mathcal{L}^n,$$

from which one deduces that P is additive in the first argument and

$$|P(M, \mathbf{v})| \leqslant \|\mathbf{v}\|_\infty \eta(M) + \|\nabla \mathbf{v}\|_\infty \int_M \hat{h} \, \mathrm{d}\mathcal{L}^n,$$

exactly with $\mathrm{d}\eta = |\mathrm{div}\,\mathsf{T}| \, \mathrm{d}\mathcal{L}^n$ and $\hat{h} = |\mathsf{T}|$.

In the spirit of [2], we aim to generalize this to the case where the stress tensor field T can have as divergence a measure not necessarily absolutely continuous with respect to the Lebesgue measure. We consider the last inequalities as assumptions for some fixed η, h, $\hat{h}$ and deduce the existence of the stress tensor field, in the sense specified below. These assumptions do not imply the symmetry of the tensor field T; to deduce it, the more restrictive inequality

$$|P(M, \mathbf{v})| \leqslant \|\mathbf{v}\|_\infty \eta(M) + \|D[\mathbf{v}]\|_\infty \int_M \hat{h} \, \mathrm{d}\mathcal{L}^n,$$

where $D[\mathbf{v}] = \frac{1}{2}[\nabla \mathbf{v} + (\nabla \mathbf{v})']$, is needed. We postpone this development to Theorem 8.

In this very general framework, we suppose the velocity field having values in $\mathbb{R}^N$, while the dimension of the body is n.

DEFINITION 9. Let X be a vector space and $\mathcal{D} \subseteq \mathcal{M}^\circ$. We say that a function $F : \mathcal{D} \to X$ is additive, if for every $M, N \in \mathcal{D}$ with $(M \cup N)_* \in \mathcal{D}$ and $M \cap N = \emptyset$ one has

$$F((M \cup N)_*) = F(M) + F(N).$$

DEFINITION 10. A Cauchy power on B is a function

$$P : \mathcal{D} \times C_c^\infty(\operatorname{int} B; \mathbb{R}^N) \to \mathbb{R},$$

where $\mathcal{D}$ contains almost all of $\mathcal{M}^\circ$ and the following properties hold:

(a) $P(\cdot, \mathbf{v})$ is additive for every $\mathbf{v} \in C_c^\infty(\operatorname{int} B; \mathbb{R}^N)$;
(b) $P(M, \cdot)$ is linear for every $M \in \mathcal{D}$;
(c) there exists $h \in \mathcal{L}^1_{\mathrm{loc},+}(\operatorname{int} B)$ such that

$$|P(M, \mathbf{v})| \leqslant \int_{\partial_* M} |\mathbf{v}| h \, \mathrm{d}\mathcal{H}^{n-1}$$

for every $\mathbf{v} \in C_c^\infty(\operatorname{int} B; \mathbb{R}^N)$ and every $M \in \mathcal{D}$.

Remark. Taking into account property (c), it is easy to see that $P(M, \mathbf{v})$ depends only on the values of $\mathbf{v}$ on the measure-theoretic boundary of M, that is, if $\mathbf{v}_1(x) = \mathbf{v}_2(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial_* M$, then $P(M, \mathbf{v}_1) = P(M, \mathbf{v}_2)$.

DEFINITION 11. Let $\mathcal{R} \subseteq \mathcal{S}^\circ$ be a set containing almost all of $\mathcal{S}^\circ$ and consider $Q : \mathcal{R} \to \mathbb{R}^N$. We say that Q is *a Cauchy flux* on B, if the following properties hold:

(a) if $S, T \in \mathcal{R}$ are compatible and disjoint with $S \cup T \in \mathcal{R}$, then

$$Q(S \cup T) = Q(S) + Q(T);$$

(b) there exists $h \in \mathcal{L}^1_{\mathrm{loc},+}(\operatorname{int} B)$ such that the inequality

$$|Q(S)| \leqslant \int_S h \, \mathrm{d}\mathcal{H}^{n-1}$$

holds on almost all of S°.

Let us prove a first representation theorem about general Cauchy fluxes.

THEOREM 1. *Let Q be a Cauchy flux. Then for almost every $M \in \mathcal{M}^\circ$ there exists a unique* (*up to $\mathcal{H}^{n-1}$-negligible sets*) *Borel map $\mathbf{t}_{Q,M} : \partial_* M \to \mathbb{R}^N$ such that*

$$Q(S) = \int_S \mathbf{t}_{Q,M} \, \mathrm{d}\mathcal{H}^{n-1}$$

for every material surface S subordinated to M. Moreover, if h is a function satisfying (b) *of Definition* 11, *then $|\mathbf{t}_{Q,M}| \leqslant h$.*

Proof. Let h and ν be such that Q is defined on $\mathcal{M}^\circ_{h\nu}$, and (a) and (b) of Definition 11 hold. Let $M \in \mathcal{M}^\circ_{h\nu}$; then Q is additive on $\mathfrak{B}(\partial_* M)$ and for any material surface $S \subseteq \partial_* M$ we have

$$|Q(S)| \leqslant \int_S h \, \mathrm{d}\mathcal{H}^{n-1} \leqslant \int_{\partial_* M} h \, \mathrm{d}\mathcal{H}^{n-1} = c_M,$$

hence Q is a finite vector measure on $\partial_* M$ with $Q(S) = 0$ whenever $\mathcal{H}^{n-1}(S) = 0$. By the Radon–Nikodym theorem there exists a function $\mathbf{t}_{Q,M} : \partial_* M \to \mathbb{R}^N$ such that $|\mathbf{t}_{Q,M}| \leqslant h$ and $Q(S) = \int_S \mathbf{t}_{Q,M} \,\mathrm{d}\mathcal{H}^{n-1}$. □

Now we introduce a further assumption on Cauchy fluxes, which connects them with Cauchy powers, as we will see in Theorems 2 and 3.

DEFINITION 12. Let Q be a Cauchy flux on B. We say that Q is *equilibrated*, if the condition

$$Q(-S) = -Q(S)$$

holds on almost all of $\mathcal{S}^\circ$.

DEFINITION 13. We say that an equilibrated Cauchy flux Q and a Cauchy power P are *associated*, if for almost every $M \in \mathcal{M}^\circ$ the formula

$$P(M, \mathbf{v}) = \int_{\partial_* M} \mathbf{v} \cdot \mathbf{t}_{Q,M} \,\mathrm{d}\mathcal{H}^{n-1}$$

holds for every $\mathbf{v} \in C_c^\infty(\operatorname{int} B; \mathbb{R}^N)$.

In the following theorems, we prove the one-to-one relation between Cauchy powers and equilibrated Cauchy fluxes.

LEMMA 1. *Let Q be an equilibrated Cauchy flux. Then for almost every $M, N \in \mathcal{M}^\circ$ with $M \cap N = \emptyset$ one has*

$$\begin{aligned} \int_{\partial_* M \cap \partial_* N} \mathbf{v} \cdot \mathbf{t}_{Q,M} \,\mathrm{d}\mathcal{H}^{n-1} &= -\int_{\partial_* M \cap \partial_* N} \mathbf{v} \cdot \mathbf{t}_{Q,N} \,\mathrm{d}\mathcal{H}^{n-1}, \\ \int_{\partial_* M \setminus \partial_* N} \mathbf{v} \cdot \mathbf{t}_{Q,M} \,\mathrm{d}\mathcal{H}^{n-1} &= \int_{\partial_* M \setminus \partial_* N} \mathbf{v} \cdot \mathbf{t}_{Q,(M \cup N)_*} \,\mathrm{d}\mathcal{H}^{n-1} \end{aligned}$$

for every $\mathbf{v} \in C_c^\infty(\operatorname{int} B; \mathbb{R}^N)$.

Proof. We drop the subscript Q from $\mathbf{t}$. Let h and ν be such that Q is defined and Theorem 1 holds on $\mathcal{M}^\circ_{h\nu}$. Let $\mathbf{v} \in C_c^\infty(\operatorname{int} B; \mathbb{R}^N)$; then for every $i = 1, \ldots, N$ there exist two sequences $(E_{i,h})$, $(F_{i,h})$ of Borel subsets of $\operatorname{int} B$ such that

$$e_i \cdot \mathbf{v}(x) = \sum_{h=1}^{\infty} \frac{1}{h} \chi_{E_{i,h}} - \sum_{h=1}^{\infty} \frac{1}{h} \chi_{F_{i,h}}$$

(see [5, Section 1.1]), where $(e_1, \ldots, e_n)$ is an orthonormal basis in $\mathbb{R}^n$. Given M, N in $\mathcal{M}^\circ_{h\nu}$ with $M \cap N = \emptyset$, by Proposition 3 one has

$$\mathcal{H}^{n-1}(\partial_* M \cap \partial_* N \cap \partial_*(M \cup N)) = 0.$$

Moreover, by (1) it follows that $\mathbf{n}^M(x) = -\mathbf{n}^N(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial_* M \cap \partial_* N$. Denoting by $\mathbf{v}_i$ and $\mathbf{t}_{i,M}$ the components along $\mathbf{e}_i$ of $\mathbf{v}$ and $\mathbf{t}_M$, respectively, we have

$$\begin{aligned}
&\int_{\partial_* M\cap\partial_* N} \mathbf{v}_i \mathbf{t}_{i,M}\, \mathrm{d}\mathcal{H}^{n-1} \\
&\quad = \sum_{h=1}^{\infty} \frac{1}{h}\left[\int_{\partial_* M\cap\partial_* N\cap E_{i,h}} \mathbf{t}_{i,M}\, \mathrm{d}\mathcal{H}^{n-1} - \int_{\partial_* M\cap\partial_* N\cap F_{i,h}} \mathbf{t}_{i,M}\, \mathrm{d}\mathcal{H}^{n-1}\right] \\
&\quad = \sum_{h=1}^{\infty} \frac{1}{h}[Q(\partial_* M\cap\partial_* N\cap E_{i,h}, \mathbf{n}^M) - Q(\partial_* M\cap\partial_* N\cap F_{i,h}, \mathbf{n}^M)]\cdot e_i \\
&\quad = -\sum_{h=1}^{\infty} \frac{1}{h}[Q(\partial_* M\cap\partial_* N\cap E_{i,h}, \mathbf{n}^N) - Q(\partial_* M\cap\partial_* N\cap F_{i,h}, \mathbf{n}^N)]\cdot e_i \\
&\quad = -\int_{\partial_* M\cap\partial_* N} \mathbf{v}_i \mathbf{t}_{i,N}\, \mathrm{d}\mathcal{H}^{n-1}
\end{aligned}$$

and the first formula is proved.

Now take $\hat{S} \subseteq \partial_* M\backslash\partial_* N$; we have that the material surface $(\hat{S}, \mathbf{n}^M)$ is subordinated to M, hence it is in the domain of Q. Moreover, taking into account (2) and that M, N are disjoint, it follows that $\hat{S}$ is subordinated also to $(M\cup N)_*$ up to a set of zero $\mathcal{H}^{n-1}$-measure, thus

$$\int_{\hat{S}} \mathbf{t}_M\, \mathrm{d}\mathcal{H}^{n-1} = \int_{\hat{S}} \mathbf{t}_{(M\cup N)_*}\, \mathrm{d}\mathcal{H}^{n-1}.$$

Then we can prove the other formula using the same technique as above. □

THEOREM 2. *For every equilibrated Cauchy flux Q, there exists a Cauchy power P associated with Q. Moreover, if $\check{P}$ is another Cauchy power associated with Q, then for almost every $M \in \mathcal{M}^\circ$ one has $\check{P}(M, \mathbf{v}) = P(M, \mathbf{v})$ for every $\mathbf{v} \in C_c^\infty(\operatorname{int} B; \mathbb{R}^N)$.*

Proof. Let $h \in \mathcal{L}^1_{\mathrm{loc},+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ be such that Definition 12 holds and $\mathbf{t}_{Q,M}$ exists for every $M \in \mathcal{M}^\circ_{h\nu}$. We show that the function

$$P(M, \mathbf{v}) = \int_{\partial_* M} \mathbf{v}\cdot \mathbf{t}_{Q,M}\, \mathrm{d}\mathcal{H}^{n-1}$$

defined on $\mathcal{M}^\circ_{h\nu} \times C_c^\infty(\operatorname{int} B; \mathbb{R}^N)$ is a Cauchy power. Linearity on the second argument is obvious as well as the inequality

$$|P(M, \mathbf{v})| \leqslant \int_{\partial_* M} |\mathbf{v}| h\, \mathrm{d}\mathcal{H}^{n-1}.$$

To prove the additivity of P, take two disjoint subsets $M, N \in \mathcal{M}^\circ_{h\nu}$ and $\mathbf{v} \in C_c^\infty(\operatorname{int} B; \mathbb{R}^N)$. Keeping into account (1), one has $\partial_*(M\cup N) = \partial_* M \Delta \partial_* N$ up to a set of zero $\mathcal{H}^{n-1}$-measure, Hence, by Lemma 1

$$\begin{aligned} P((M \cup N)_*, \mathbf{v}) &= \int_{\partial_*(M\cup N)} \mathbf{v} \cdot \mathbf{t}_{Q,(M\cup N)_*} \, \mathrm{d}\mathcal{H}^{n-1} \\ &= \int_{\partial_* M \setminus \partial_* N} \mathbf{v} \cdot \mathbf{t}_{Q,M} \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\partial_* N \setminus \partial_* M} \mathbf{v} \cdot \mathbf{t}_{Q,N} \, \mathrm{d}\mathcal{H}^{n-1} \\ &= \int_{\partial_* M} \mathbf{v} \cdot \mathbf{t}_{Q,M} \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\partial_* N} \mathbf{v} \cdot \mathbf{t}_{Q,N} \, \mathrm{d}\mathcal{H}^{n-1} \\ &= P(M, \mathbf{v}) + P(N, \mathbf{v}) \end{aligned}$$

and P is a Cauchy power. It is clear that P is associated with Q.

Finally, if $\check{P}$ is another Cauchy power associated with Q, then for almost every $M \in \mathcal{M}^\circ$ and every $\mathbf{v} \in C_c^\infty(\operatorname{int} B; \mathbb{R}^N)$ we have

$$\check{P}(M, \mathbf{v}) = \int_{\partial_* M} \mathbf{v} \cdot \mathbf{t}_{Q,M} \, \mathrm{d}\mathcal{H}^{n-1} = P(M, \mathbf{v}),$$

which concludes the proof. □

THEOREM 3. *For every Cauchy power P there exists an equilibrated Cauchy flux Q associated with P. Moreover, if $\check{Q}$ is another equilibrated Cauchy flux associated with P, then $\check{Q} = Q$ on almost all of $\mathcal{S}^\circ$.*

Proof. Let $h \in \mathcal{L}^1_{\mathrm{loc},+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ be such that P is defined and Definition 10 holds on $\mathcal{M}^\circ_{h\nu}$. Let us fix $M \in \mathcal{M}^\circ_{h\nu}$; the function $P(M, \cdot) : C_c^\infty(\operatorname{int} B; \mathbb{R}^n) \to \mathbb{R}$ is linear and

$$|P(M, \mathbf{v})| \leqslant \|\mathbf{v}\|_\infty \int_{\partial_* M} h \, \mathrm{d}\mathcal{H}^{n-1},$$

hence $P(M, \cdot)$ is a vector distribution of order zero. By the Riesz representation theorem, there exist a unique $\mu \in \mathfrak{M}(\operatorname{int} B)$ and a μ-essentially unique Borel function $\mathbf{c}_M : \operatorname{int} B \to \mathbb{R}^N$ such that $|\mathbf{c}_M| = 1$, μ-almost everywhere in $\operatorname{int} B$ and

$$P(M, \mathbf{v}) = \int_{\operatorname{int} B} \mathbf{c}_M \cdot \mathbf{v} \, \mathrm{d}\mu.$$

Moreover, since

$$|P(M, \mathbf{v})| \leqslant \int_{\partial_* M} |\mathbf{v}| h \, \mathrm{d}\mathcal{H}^{n-1},$$

we have that μ is absolutely continuous with respect to $\mathcal{H}^{n-1} \llcorner \partial_* M$, hence

$$P(M, \mathbf{v}) = \int_{\partial_* M} \mathbf{b}_M \cdot \mathbf{v} \, \mathrm{d}\mathcal{H}^{n-1}$$

for an $\mathcal{H}^{n-1}$-essentially unique Borel function $\mathbf{b}_M : \partial_* M \to \mathbb{R}^N$ with $|\mathbf{b}_M| \leqslant h$.

Let now S be a material surface in $\mathcal{S}^\circ_{h\nu}$ subordinated to $M, N \in \mathcal{M}^\circ_{h\nu}$; then clearly $\hat{S} \subseteq \partial_* M \cap \partial_* N$ and $\mathbf{n}^M = \mathbf{n}^N$ on $\hat{S}$. Moreover, by (2) and (3), the sets $\hat{S} \setminus \partial_*(M \cap N)$ and $\hat{S} \cap \partial_*(M \setminus N)$ are $\mathcal{H}^{n-1}$-negligible. Suppose the set $\hat{S}$ to be compact and fix an element $\mathbf{a}$ of

$\mathbb{R}^N$. Then there exists a sequence $(\mathbf{v}_h)$ in $C_c^\infty(\operatorname{int} B; \mathbb{R}^n)$ such that $|\mathbf{v}_h| \leqslant |\mathbf{a}|$ and $\mathbf{v}_h \to \chi_{\hat{S}}\mathbf{a}$ pointwise. Since P is additive, it follows

$$\int_{\partial_* M} \mathbf{v}_h \cdot \mathbf{b}_M \, d\mathcal{H}^{n-1} = \int_{\partial_*(M\setminus N)} \mathbf{v}_h \cdot \mathbf{b}_{M\setminus N} \, d\mathcal{H}^{n-1} + \int_{\partial_*(M\cap N)} \mathbf{v}_h \cdot \mathbf{b}_{M\cap N} \, d\mathcal{H}^{n-1},$$

and by the dominated convergence theorem

$$\mathbf{a} \cdot \int_{\hat{S}} \mathbf{b}_M \, d\mathcal{H}^{n-1} = \mathbf{a} \cdot \int_{\hat{S}} \mathbf{b}_{M\cap N} \, d\mathcal{H}^{n-1}.$$

Exchanging M with N, by the arbitrariness of $\mathbf{a}$ one has

$$\int_{\hat{S}} \mathbf{b}_M \, d\mathcal{H}^{n-1} = \int_{\hat{S}} \mathbf{b}_N \, d\mathcal{H}^{n-1}.$$

If $\hat{S}$ is not compact, we can find a sequence (S_h) in $\mathcal{S}^\circ_{h\nu}$ such that

$$\hat{S} = \bigcup_{h\in\mathbb{N}} \hat{S}_h \cup N \quad \text{with } \mathcal{H}^{n-1}(N) = 0,$$

then

$$\int_{\hat{S}} \mathbf{b}_M \, d\mathcal{H}^{n-1} = \lim_h \int_{\hat{S}_h} \mathbf{b}_M \, d\mathcal{H}^{n-1} = \lim_h \int_{\hat{S}_h} \mathbf{b}_N \, d\mathcal{H}^{n-1} = \int_{\hat{S}} \mathbf{b}_N \, d\mathcal{H}^{n-1}.$$

Hence we can define a function $Q : \mathcal{S}^\circ_{h\nu} \to \mathbb{R}^N$ setting

$$Q(S) = \int_{\hat{S}} \mathbf{b}_M \, d\mathcal{H}^{n-1},$$

where S is subordinated to M. It is clear that Q is a Cauchy flux; we want to prove that it is equilibrated. Let S be a material surface and take two disjoint sets $M, N \in \mathcal{M}^\circ_{h\nu}$ such that $S, -S$ are subordinated to M, N, respectively. By Proposition 3 it follows that $\hat{S} \cap \partial_*(M \cup N)$ is $\mathcal{H}^{n-1}$-negligible. Let $\mathbf{a} \in \mathbb{R}^N$; if $\hat{S}$ is compact, there exists a sequence $(\mathbf{v}_h)$ in $C_c^\infty(\operatorname{int} B; \mathbb{R}^n)$ such that $\mathbf{v}_h \to \chi_{\hat{S}}\mathbf{a}$ pointwise. Since P is additive, we have

$$\int_{\partial_*(M\cup N)} \mathbf{v}_h \cdot b_{(M\cup N)_*} \, d\mathcal{H}^{n-1} = \int_{\partial_* M} \mathbf{v}_h \cdot \mathbf{b}_M \, d\mathcal{H}^{n-1} + \int_{\partial_* N} \mathbf{v}_h \cdot \mathbf{b}_N \, d\mathcal{H}^{n-1},$$

hence $Q(-S) = -Q(S)$ by the dominated convergence theorem. If $\hat{S}$ is not compact, we can apply the same technique as above, which yields that Q is equilibrated. Moreover, we readily have $\mathbf{b}_M = \mathbf{t}_{Q,M}$, hence Q is associated with P.

Finally, if $\check{Q}$ is another equilibrated Cauchy flux associated with P, for almost every $M \in \mathcal{M}^\circ$ it follows that

$$\int_{\partial_* M} \mathbf{v} \cdot \mathbf{t}_{Q,M} \, d\mathcal{H}^{n-1} = P(M, \mathbf{v}) = \int_{\partial_* M} \mathbf{v} \cdot \mathbf{t}_{\check{Q},M} \, d\mathcal{H}^{n-1}$$

for every $\mathbf{v} \in C_c^\infty(\operatorname{int} B; \mathbb{R}^N)$, hence $\mathbf{t}_{Q,M}(x) = \mathbf{t}_{\check{Q},M}(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial_* M$. Then $Q(S) = \check{Q}(S)$ for every $S \subseteq \partial_* M$ and the proof is complete. □

We now add a crucial assumption to Cauchy powers and Cauchy fluxes in order to obtain a global integral representation for both.

DEFINITION 14. We say that a Cauchy power P is *balanced*, if there exist $\eta \in \mathfrak{M}(\operatorname{int} B)$ and $h \in \mathcal{L}^1_{\mathrm{loc},+}(\operatorname{int} B)$ such that, for almost every $M \in \mathcal{M}^\circ$,

$$|P(M, \mathbf{v})| \leqslant \|\mathbf{v}\|_\infty \eta(M) + \|\nabla \mathbf{v}\|_\infty \int_M h \, \mathrm{d}\mathcal{L}^n$$

for every $\mathbf{v} \in C_c^\infty(\operatorname{int} B; \mathbb{R}^n)$.

DEFINITION 15. We say that a Cauchy flux Q is *balanced*, if there exists $\eta \in \mathfrak{M}(\operatorname{int} B)$ such that the inequality

$$|Q(\partial_* M)| \leqslant \eta(M)$$

holds on almost all of $\mathcal{M}^\circ$.

We first recall the definition of tensor field with divergence measure.

DEFINITION 16. Let $\mathsf{T} \in \mathcal{L}^1_{\mathrm{loc}}(\operatorname{int} B; \operatorname{Lin}(\mathbb{R}^n; \mathbb{R}^N))$. We say that T is a tensor field on $\operatorname{int} B$ *with divergence measure*, if $\operatorname{div} \mathsf{T}$ is a vector distribution on $\operatorname{int} B$ of order 0. This means that for every compact set $K \subseteq \operatorname{int} B$ there exists $c_K \geqslant 0$ with

$$\left| \int_{\operatorname{int} B} \mathsf{T} \nabla f \, \mathrm{d}\mathcal{L}^n \right| \leqslant c_K \max_K |f|$$

whenever $f \in C_0^\infty(\operatorname{int} B)$ and $\operatorname{supt} f \subseteq K$.

In such a case, there exist a uniquely determined $\mu \in \mathfrak{M}(\operatorname{int} B)$ and a uniquely determined μ-almost everywhere Borel map $\mathbf{u} : \operatorname{int} B \to \mathbb{R}^N$ such that $|\mathbf{u}(x)| = 1$ for μ-a.e. $x \in \operatorname{int} B$ and

$$-\int_{\operatorname{int} B} \mathsf{T} \nabla f \, \mathrm{d}\mathcal{L}^n = \int_{\operatorname{int} B} f \mathbf{u} \, \mathrm{d}\mu$$

for any Lipschitz function $f : \operatorname{int} B \to \mathbb{R}$ with compact support. We set

$$\int_M \mathbf{v} \cdot \operatorname{div} \mathsf{T} = \int_M \mathbf{v} \cdot \mathbf{u} \, \mathrm{d}\mu$$

for any $\mathbf{v} \in C_0^\infty(\operatorname{int} B; \mathbb{R}^N)$. Moreover, we put $|\operatorname{div} \mathsf{T}| = \mu$.

The following are the two main features of balanced Cauchy fluxes proved in [2]. The first states the existence of a flux density and an integral representation; the second gives an extension theorem starting from $(n-1)$-rectangles.

THEOREM 4. *Let Q be a balanced Cauchy flux on B. Then there exists a tensor field* $\mathsf{T} \in \mathcal{L}^1_{\mathrm{loc}}(\operatorname{int} B; \operatorname{Lin}(\mathbb{R}^n; \mathbb{R}^N))$ *with divergence measure such that*

$$Q(S) = \int_S \mathsf{T}\mathbf{n}_S \,\mathrm{d}\mathcal{H}^{n-1}$$

on almost all of $\mathcal{S}^\circ$. *Moreover*, T *is uniquely determined* $\mathcal{L}^n$*-almost everywhere and* $|\operatorname{div} \mathsf{T}| \leqslant \eta$, *where* η *is as in Definition* 15.

In particular, a balanced Cauchy flux is equilibrated.

THEOREM 5. *Let* $G_0 = (x_0, (e_1, \ldots, e_n), \hat{G}_0)$ *be a full grid and* $Q_0 : \mathcal{S}^\circ_{G_0} \to \mathbb{R}^n$ *be a function satisfying the following properties*:

(i) $Q_0(S) = Q_0(S_1) + Q_0(S_2)$ *whenever* $S, S_1, S_2 \in \mathcal{S}^\circ_{G_0}$, $S_1 \cap S_2 = \emptyset$ *and* $\operatorname{cl} S = \operatorname{cl} S_1 \cup \operatorname{cl} S_2$;

(ii) *there exists* $h \in \mathcal{L}^1_{\mathrm{loc},+}(\operatorname{int} B)$ *such that*
$$|Q_0(S)| \leqslant \int_S h \,\mathrm{d}\mathcal{H}^{n-1}$$
for any $S \in \mathcal{S}^\circ_{G_0}$;

(iii) *there exists* $\eta \in \mathfrak{M}(\operatorname{int} B)$ *such that*
$$\left| \sum_{j=1}^{n} (Q_0(I_j^+) - Q_0(I_j^-)) \right| \leqslant \eta(I)$$
whenever
$$I = \{x \in \mathbb{R}^n\colon a_j < (x - x_0) \cdot e_j < b_j \ \forall j = 1, \ldots, n\} \in \mathcal{I}^\circ_{G_0},$$
$$I_j^+ = \{x \in \mathbb{R}^n\colon (x - x_0) \cdot e_j = b_j, a_i < (x - x_0) \cdot e_i < b_i \ \forall i \neq j\},$$
$$I_j^- = \{x \in \mathbb{R}^n\colon (x - x_0) \cdot e_j = a_j, a_i < (x - x_0) \cdot e_i < b_i \ \forall i \neq j\}.$$
Then there exist a balanced Cauchy flux Q on B and a full grid $G \subseteq G_0$ *such that the domain of Q contains* S°_G *and*
$$\forall S \in \mathcal{S}^\circ_G: \quad Q(S) = Q_0(S).$$
Moreover, if $\check{Q}$ *also satisfies the property for some full grid* $\check{G} \subseteq G_0$, *then* $\check{Q} = Q$ *on almost all of* $\mathcal{S}^\circ$.

Now we show that balanced Cauchy fluxes and powers are intimately related; this will immediately produce an integral representation theorem and an extension property also for balanced Cauchy powers.

PROPOSITION 6. *A Cauchy power is balanced if and only if the associated Cauchy flux is balanced.*

Proof. Suppose that a Cauchy power P is balanced and consider the Cauchy flux Q associated with P; then for every $\mathbf{a} \in \mathbb{R}^N$ we have

$$|Q(\partial_* M) \cdot \mathbf{a}| = \left| \int_{\partial_* M} \mathbf{a} \cdot \mathbf{t}_{Q,M} \,\mathrm{d}\mathcal{H}^{n-1} \right| = |P(M, \mathbf{a})| \leqslant \|\mathbf{a}\| \eta(M)$$

on almost all of $\mathcal{M}^\circ$, hence in particular Q is balanced.

On the other hand, supposing that Q is balanced, by Theorem 4 one deduces that there exists a tensor field $\mathsf{T} \in L^1_{\text{loc}}(\operatorname{int} B; \operatorname{Lin}(\mathbb{R}^n; \mathbb{R}^N))$ with divergence measure such that $|\operatorname{div} \mathsf{T}| \leqslant \eta$ and $\mathbf{t}_{Q,M} = \mathsf{T}\mathbf{n}^M$ on almost all of $\mathcal{M}^\circ$. Denoting with P the Cauchy power associated with Q and setting $h(x) = |\mathsf{T}(x)|$, we have that

$$\begin{aligned} |P(M, \mathbf{v})| &= \left| \int_{\partial_* M} \mathsf{T}\mathbf{n}^M \cdot \mathbf{v} \, \mathrm{d}\mathcal{H}^{n-1} \right| = \left| \int_M \mathbf{v} \cdot \operatorname{div} \mathsf{T} + \int_M \mathsf{T} \cdot \nabla \mathbf{v} \, \mathrm{d}\mathcal{L}^n \right| \\ &\leqslant \|\mathbf{v}\|_\infty \eta(M) + \|\nabla \mathbf{v}\|_\infty \int_M h \, \mathrm{d}\mathcal{L}^n \end{aligned}$$

on almost all of $\mathcal{M}^\circ$, hence P is balanced. □

THEOREM 6. *Let P be a balanced Cauchy power and let η be as in Definition* 14. *Then there exists $\mathsf{T} \in L^1_{\text{loc}}(\operatorname{int} B; \operatorname{Lin}(\mathbb{R}^n; \mathbb{R}^N))$ with divergence measure such that $|\operatorname{div} \mathsf{T}| \leqslant \eta$ and, for almost every $M \in \mathcal{M}^\circ$,*

$$P(M, \mathbf{v}) = \int_{\partial_* M} \mathsf{T}\mathbf{n}^M \cdot \mathbf{v} \, \mathrm{d}\mathcal{H}^{n-1}$$

for every $\mathbf{v} \in C^\infty_c(\operatorname{int} B; \mathbb{R}^N)$. Moreover, T is uniquely determined $\mathcal{L}^n$-almost everywhere.

Proof. The balanced Cauchy power P is associated with a Cauchy flux that is balanced by Proposition 6. The conclusion follows from Theorem 4. □

Finally, we give an extension formula for balanced Cauchy powers, which states that the behavior of a Cauchy power on almost all n-intervals extends it to almost all of $\mathcal{M}^\circ$.

THEOREM 7. *Let G_0 be a full grid and $P_0 : \mathcal{I}^\circ_{G_0} \times C^\infty_c(\operatorname{int} B; \mathbb{R}^N) \to \mathbb{R}$ a function which satisfies the following assumptions:*

(a) *for every finite disjoint family $\{I_k\colon k \in \Lambda\} \subseteq \mathcal{I}^\circ_{G_0}$ and $\mathbf{v} \in C^\infty_c(\operatorname{int} B; \mathbb{R}^N)$ the following holds:*
$$\left(\bigcup_{k\in\Lambda} I_k \right)_* \in \mathcal{I}^\circ_{G_0} \quad \Longrightarrow \quad P_0\left(\left(\bigcup_{k\in\Lambda} I_k \right)_*, \mathbf{v} \right) = \sum_{k\in\Lambda} P_0(I_k, \mathbf{v});$$
(b) *$P_0(I, \cdot)$ is linear for every $I \in \mathcal{I}^\circ_{G_0}$;*
(c) *there exists $h \in \mathcal{L}^1_{\text{loc},+}(\operatorname{int} B)$ such that*
$$|P_0(I, \mathbf{v})| \leqslant \int_{\partial I} |\mathbf{v}| h \, \mathrm{d}\mathcal{H}^{n-1}$$
for every $\mathbf{v} \in C^\infty_c(\operatorname{int} B; \mathbb{R}^N)$ and $I \in \mathcal{I}^\circ_{G_0}$;
(d) *there exist $\eta \in \mathfrak{M}(\operatorname{int} B)$ and $\tilde{h} \in \mathcal{L}^1_{\text{loc},+}(\operatorname{int} B)$ such that*
$$|P_0(I, \mathbf{v})| \leqslant \|\mathbf{v}\|_\infty \eta(I) + \|\nabla \mathbf{v}\|_\infty \int_I \tilde{h} \, \mathrm{d}\mathcal{L}^n$$
for every $\mathbf{v} \in C^\infty_c(\operatorname{int} B; \mathbb{R}^N)$ and $I \in \mathcal{I}^\circ_{G_0}$.

Then there exist a full grid $G \subseteq G_0$ and a balanced Cauchy power P such that the domain of P contains $\mathcal{I}^\circ_G$ and $P(I) = P_0(I)$ for every $I \in \mathcal{I}^\circ_G$. Moreover, if $\check{P}$ has the same property of P for some full grid $G \subseteq G_0$, then for almost every $M \in \mathcal{M}^\circ$ one has $\check{P}(M, \mathbf{v}) = P(M, \mathbf{v})$ for every $\mathbf{v} \in C^\infty_c(\operatorname{int} B; \mathbb{R}^N)$.

Proof. Let $S \in \mathcal{S}^\circ_{G_0}$ and $I \in \mathcal{I}^\circ_{G_0}$ be such that S is subordinated to I. Given $\mathbf{a} \in \mathbb{R}^N$, there exists a sequence $(\mathbf{v}_k)$ in $C^\infty_c(\operatorname{int} B; \mathbb{R}^N)$ such that $|\mathbf{v}_k| \leqslant \mathbf{a}$ and $\mathbf{v}_k \to \chi_S \mathbf{a}$ pointwise. Define the component of $Q_0 : \mathcal{S}^\circ_{G_0} \to \mathbb{R}^N$ with respect to $\mathbf{a}$ as

$$\mathbf{a} \cdot Q_0(S) = \lim_k P_0(I, \mathbf{v}_k).$$

Then we have:

1. Q_0 does not depend on the choice of the sequence $\mathbf{v}_k$, since
$$\left| \lim_k P_0(I, \mathbf{v}_k - \tilde{\mathbf{v}}_k) \right| \leqslant \lim_k \int_{\partial_* I} |\mathbf{v}_k - \tilde{\mathbf{v}}_k| h \, \mathrm{d}\mathcal{H}^{n-1} = 0.$$
2. If $\mathcal{H}^{n-1}(S \Delta T) = 0$, then it is easy to check that $Q_0(S) = Q_0(T)$.
3. $Q_0(S)$ does not depend on the set I. Indeed, if S is subordinated to I_1, I_2, then
$$\lim_k P_0(I_1, \mathbf{v}_k) = \lim_k P_0(I_1 \cap I_2, \mathbf{v}_k) = \lim_k P_0(I_2, \mathbf{v}_k)$$
since $\mathcal{H}^{n-1}(S \cap (I_1 \Delta I_2)) = 0$ and P_0 is additive.
4. $Q_0(-S) = -Q_0(S)$ by additivity of P_0.

Now we prove that Q_0 satisfies (i), (ii) and (iii) of Theorem 5.

(i) Let $S, S_1, S_2 \in \mathcal{S}^\circ_{G_0}$ with $S_1 \cap S_2 = \emptyset$ and $\operatorname{cl} S = \operatorname{cl} S_1 \cup \operatorname{cl} S_2$. Then there exist $I_1, I_2 \in \mathcal{I}^\circ_{G_0}$ such that $I_1 \cap I_2 = \emptyset$, $(I_1 \cup I_2)_* \in I^\circ_{G_0}$ and S_j is subordinated to I_j, S is subordinated to $(I_1 \cup I_2)_*$. Given $\mathbf{a} \in \mathbb{R}^N$, let $\mathbf{v}^{(j)}_k \to \chi_{S_j} \mathbf{a}$; then we have that $\mathbf{v}^{(1)}_k + \mathbf{v}^{(2)}_k \to \chi_{S_1 \cup S_2} \mathbf{a}$. Since $\mathcal{H}^{n-1}(S \Delta (S_1 \cup S_2)) = 0$, it follows that
$$\begin{aligned} \mathbf{a} \cdot Q_0(S) &= \lim_k P_0(I, \mathbf{v}^{(1)}_k + \mathbf{v}^{(2)}_k) \\ &= \lim_k (P_0(I, \mathbf{v}^{(1)}_k) + P_0(I, \mathbf{v}^{(2)}_k)) = \mathbf{a} \cdot Q_0(S_1) + \mathbf{a} \cdot Q_0(S_2). \end{aligned}$$

(ii) It is obvious.

(iii) Let $I \in \mathcal{I}^\circ_{G_0}$; using the notation of Theorem 5, we start showing that
$$\mathbf{a} \cdot \sum_{j=1}^N (Q_0(I^+_j) - Q_0(I^-_j)) = P_0(I, \mathbf{a})$$
for every $\mathbf{a} \in \mathbb{R}^N$. The surfaces I^+_j and $-I^-_j$ are subordinated to I, hence we can take two sequences $(\mathbf{v}^{(j)+}_k)$, $(\mathbf{v}^{(j)-}_k)$ as in the definition of Q_0 relative to $I^+_j, -I^-_j$, respectively. Since
$$\sum_{j=1}^N (\mathbf{v}^{(j)+}_k + \mathbf{v}^{(j)-}_k) \to \chi_{\partial_* I} \mathbf{a},$$
it follows that
$$\mathbf{a} \cdot \sum_{j=1}^N (Q_0(I^+_j) + Q_0(-I^-_j)) = \lim_k P_0\left(I, \sum_{j=1}^N (\mathbf{v}^{(j)+}_k + \mathbf{v}^{(j)-}_k)\right) = P_0(I, \mathbf{a}).$$
Then for every $\mathbf{a} \in \mathbb{R}^N$ with $|\mathbf{a}| \leqslant 1$ we have that
$$\left| \mathbf{a} \cdot \sum_{j=1}^N (Q_0(I^+_j)) - Q(I^-_j)) \right| = |P_0(I, \mathbf{a})| \leqslant \eta(I)$$
and (iii) is proved. Finally, we apply Theorem 5. □

Let us finally consider $n = N$. We recall that a vector field $\mathbf{w} \in C_0^\infty(\operatorname{int} B; \mathbb{R}^n)$ is *an infinitesimal rigid displacement*, if $\nabla \mathbf{w}(x)$ is skew for any $x \in \operatorname{int} B$, $D[\mathbf{w}] = 0$. The following theorem gives the necessary and sufficient assumptions in order to have the symmetry of T.

THEOREM 8. *Let $\mathsf{T} \in L^1_{\mathrm{loc}}(\operatorname{int} B; \operatorname{Lin}(\mathbb{R}^n; \mathbb{R}^n))$ be a tensor field with divergence measure. Then the following conditions are equivalent*:

(a) *there exist $\eta \in \mathfrak{M}(\operatorname{int} B)$ and $h \in \mathcal{L}^1_{\mathrm{loc},+}(\operatorname{int} B)$ such that, for almost every $M \in \mathcal{M}^\circ$,*
$$\left| \int_{\partial_* M} \mathsf{T}\mathbf{n}^M \cdot \mathbf{v} \, \mathrm{d}\mathcal{H}^{n-1} \right| \leqslant \int_M |\mathbf{v}| \, \mathrm{d}\eta + \|D[\mathbf{v}]\|_\infty \int_M \hat{h} \, \mathrm{d}\mathcal{L}^n$$
for every $\mathbf{v} \in C_c^\infty(\operatorname{int} B; \mathbb{R}^N)$;

(b) *there exists $\eta \in \mathfrak{M}(\operatorname{int} B)$ such that, for almost every $M \in \mathcal{M}^\circ$,*
$$\left| \int_{\partial_* M} \mathsf{T}\mathbf{n}^M \cdot \mathbf{w} \, \mathrm{d}\mathcal{H}^{n-1} \right| \leqslant \int_M |\mathbf{w}| \, \mathrm{d}\eta$$
for every infinitesimal rigid displacement $\mathbf{w} \in C_0^\infty(\operatorname{int} B; \mathbb{R}^n)$;

(c) $\mathsf{T}(x)$ *is symmetric for $\mathcal{L}^n$-a.e. $x \in \operatorname{int} B$.*

Proof.

(a) $\Rightarrow$ (b). It is obvious, since $D[\mathbf{w}] = 0$.

(b) $\Rightarrow$ (c). Consider the infinitesimal rigid displacement $\mathbf{w}(x) = W(x - x_0)$, where W is a skew $n \times n$-matrix. Define for $a, b \in \mathbb{R}^n$ the matrix $a \wedge b = a \otimes b - b \otimes a$. Then, since $(a \wedge b) \cdot W = -2b \cdot Wa$ for every skew matrix W, it is easy to deduce
$$\left| \frac{1}{2} W \cdot \int_{\partial_* M} (x - x_0) \wedge \mathsf{T}\mathbf{n}^M \, \mathrm{d}\mathcal{H}^{n-1} \right| \leqslant |W| \int_M |x - x_0| \, \mathrm{d}\eta.$$
Since $(x - x_0) \wedge \mathsf{T}$ is also skew, then the arbitrariness of W allows us to apply [2, Theorem 8.3].

(c) $\Rightarrow$ (a). Keeping into account that $\mathsf{T} \cdot \nabla \mathbf{v} = \mathsf{T} \cdot D[\mathbf{v}]$ since T is symmetric, by the formula
$$\int_{\partial_* M} \mathsf{T}\mathbf{n}^M \cdot \mathbf{v} \, \mathrm{d}\mathcal{H}^{n-1} = \int_M \mathbf{v} \cdot \operatorname{div} \mathsf{T} + \int_M \mathsf{T} \cdot \nabla \mathbf{v} \, \mathrm{d}\mathcal{L}^n,$$
(a) easily follows. $\square$

4. The Case *B* Manifold

Let us suppose now that B is an n-dimensional orientable differential manifold (second countable, Hausdorff, paracompact). We will denote by $\{U_i, \varphi_i\}$ an atlas for the manifold. It is known that every such manifold can be endowed with a Riemannian structure. The beginning of this section is devoted to introduce some topics which are related with the Riemannian structure but are independent of it. We recall a simple lemma of prime importance.

LEMMA 2. *If g_1, g_2 are two Riemannian metrics on B, then there exist two strictly positive continuous functions C_1, C_2 on B such that*
$$\forall x \in B,\ \forall v \in T_x B: \quad C_1(x)\langle g_2(x)v, v\rangle \leqslant \langle g_1(x)v, v\rangle \leqslant C_2(x)\langle g_2(x)v, v\rangle. \tag{4}$$

DEFINITION 17. If $M \subseteq B$ and $x \in B$, we say that x is *a point of density for M in B* if, given a chart (U, φ) around x, one has that $\varphi(x) \in (\varphi(U \cap M))_*$ in $\mathbb{R}^n$.

The following lemma, which holds in $\mathbb{R}^N$, proves that this definition is independent of the chosen chart.

LEMMA 3. *If A, B are two open subsets of $\mathbb{R}^n$, $\varphi : A \to B$ is a diffeomorphism and $K \subseteq A$ is compact, then there exist $c_1, c_2 > 0$ such that for every $M \subseteq K$:*

(a) $c_1 \operatorname{diam} M \leqslant \operatorname{diam} \varphi(M) \leqslant c_2 \operatorname{diam} M$;
(b) $c_1 \mathcal{H}^k(M) \leqslant \mathcal{H}^k(\varphi(M)) \leqslant c_2 \mathcal{H}^k(M)$ *for every* $k = 0, \ldots, n$;
(c) *if* $x \in M_*$, *then* $\varphi(x) \in (\varphi(M))_*$.

Proof. Since φ is bilipschitz, (a) is obvious, while (b) can be found, for instance, in [5, Section 2.4.1]. In particular, for $k = n$ one has

$$c_1 \mathcal{L}^n(M) \leqslant \mathcal{L}^n(\varphi(M)) \leqslant c_2 \mathcal{L}^n(M).$$

In order to prove (c), we recall that in the definition of $x \in M_*$ (see Section 2), one can replace the balls $B_r(x)$ with sets I_r such that $x \in \operatorname{cl} I_r$, $\operatorname{diam} I_r \to 0$ as $r \to 0$ and there is a constant $L > 0$ with

$$\limsup_{r \to 0} \frac{(\operatorname{diam} I_r)^n}{\mathcal{L}^n(I_r)} \leqslant L.$$

Let us choose $I_r = \varphi(B_r(x))$; then they have the required properties and if $x \in M_*$ one has

$$\begin{aligned} \lim_{r \to 0} \frac{\mathcal{L}^n(I_r \backslash \varphi(M))}{\mathcal{L}^n(I_r)} &= \lim_{r \to 0} \frac{\mathcal{L}^n(\varphi(B_r(x) \backslash M))}{\mathcal{L}^n(I_r)} \\ &\leqslant \lim_{r \to 0} \frac{c_2}{c_1 \omega_n} \frac{\mathcal{L}^n(B_r(x) \backslash M)}{r^n} = 0, \end{aligned}$$

where ω_n denotes the volume of the unit ball. Hence $\varphi(x) \in (\varphi(M))_*$. □

DEFINITION 18. Let $M \subseteq B$. We denote by M_* the set of all points of density for M in B. If $M = M_*$, we shall say that M is *normalized.*

Note that the whole manifold B is normalized, because it is open in itself.

DEFINITION 19. We define the measure-theoretic boundary of M as

$$\partial_* M = B \backslash (M_* \cup (B \backslash M)_*).$$

Fix now a Riemannian structure g on B; then we can introduce on B the $(n-1)$-Hausdorff measure $\mathcal{H}_g^{n-1}$.

LEMMA 4. *Let g_1, g_2 be two Riemannian structures on B and $0 \leqslant s \leqslant n$. Then*

(a) *for every $M \subseteq B$ with compact closure, there exist $c_{1,M}, c_{2,M}$ such that*
$c_{1,M} \mathcal{H}_{g_2}^s(M) \leqslant \mathcal{H}_{g_1}^s(M) \leqslant c_{2,M} \mathcal{H}_{g_2}^s(M)$;
(b) *for every $M \subseteq B$ we have $\mathcal{H}_{g_1}^s(M) = 0$ if and only if $\mathcal{H}_{g_2}^s(M) = 0$.*

Proof. (a) It follows immediately by Lemma 2.

(b) Let $\mathcal{H}^s_{g_2}(M) = 0$ and let $\{V_j\colon j \in J\}$ be an open cover of M such that each V_j has compact closure in some coordinate neighborhood. Since B is second countable, by Lindelöf's Theorem we can extract a countable subcover $\{V_{j_k}\colon k \in \mathbb{N}\}$. We have $\mathcal{H}^s_{g_2}(M \cap V_{j_k}) = 0$, hence $\mathcal{H}^s_{g_1}(M \cap V_{j_k}) = 0$ by (a). It follows $\mathcal{H}^s_{g_1}(M) = 0$. □

Hence, the fact that $\int_M h \, \mathrm{d}\mathcal{H}^s$ or $\mathcal{H}^s(M)$ vanish is independent of the Riemannian structure. In the same way, when M has compact closure in B the fact that $\int_M h \, \mathrm{d}\mathcal{H}^s$ or $\mathcal{H}^s(M)$ is finite is independent of the Riemannian structure.

DEFINITION 20. Let $M \subseteq B$ be a set with compact closure. We say that M *has finite perimeter*, if $\mathcal{H}^{n-1}(\partial_* M) < +\infty$.

Note that this makes sense, $\partial_* M$ having compact closure in B. Since the definitions of $\mathcal{L}^1_{\mathrm{loc},+}(B, \mathcal{H}^n)$, $\mathcal{M}^\circ$, $\mathcal{M}^\circ_{h\nu}$ and 'almost all' are given in terms of sets with compact closure, they extend naturally to the case of the manifold B.

Now we are ready to give the main definition of this section. We denote by $\mathcal{X}_c(B)$ the set of all smooth vector fields on B with compact support.

DEFINITION 21. Let $\mathcal{D} \subseteq M^\circ$ be a set containing almost all of $\mathcal{M}^\circ$ and take a function $P : \mathcal{D} \times \mathcal{X}_c(B) \to \mathbb{R}$. We say that P is *a Cauchy power* on B, if the following properties hold:

(a) $P(\cdot, \mathbf{v})$ is additive for every $\mathbf{v} \in \mathcal{X}_c(B)$;
(b) $P(M, \cdot)$ is linear for every $M \in \mathcal{D}$;
(c) there exists $h \in \mathcal{L}^1_{\mathrm{loc},+}(B; \mathcal{H}^n)$ such that
$$|P(M, \mathbf{v})| \leqslant \int_{\partial_* M} |\mathbf{v}| h \, \mathrm{d}\mathcal{H}^{n-1}$$
for every $v \in \mathcal{X}_c(B)$ and every $M \in \mathcal{D}$.

It is clear that the existence of such an h as in (c) is independent of the Riemannian structure.

DEFINITION 22. A Cauchy power P is said to be *balanced* if, given a Riemannian structure on B, there exist $h \in \mathcal{L}^1_{\mathrm{loc},+}(B; \mathcal{H}^n)$ and $\eta \in \mathfrak{M}(B)$ such that, for almost every $M \in \mathcal{M}^\circ$,

$$|P(M, \mathbf{v})| \leqslant \|\mathbf{v}\|_\infty \eta(M) + \mathrm{Lip}(\mathbf{v}) \int_M h \, \mathrm{d}\mathcal{H}^n$$

for every $\mathbf{v} \in \mathcal{X}_c(B)$, where $\mathrm{Lip}(\mathbf{v})$ denotes the Lipschitz constant of $\mathbf{v}$ in the Riemannian structure induced on TB.

Again, the balance of P does not depend on the Riemannian structure.

Now we recall some notations. Let $\mathbf{a}$ be an m-vector and Q a p-form with $p > m$; we define a $(p - m)$-form $\mathbf{a} \lrcorner Q$ by

$$\langle \mathbf{a} \lrcorner Q, \xi \rangle = \langle Q, \xi \wedge \mathbf{a} \rangle$$

for every $(p-m)$-vector ξ. In the same way, if $\mathcal{Q}$ is a p-differential form and $\mathbf{v}$ an m-vector field, we define a $(p-m)$-differential form $\mathbf{v} \lrcorner \mathcal{Q}$ by

$$(\mathbf{v} \lrcorner \mathcal{Q})(x) = \mathbf{v}(x) \lrcorner \mathcal{Q}(x)$$

for every $x \in B$ (see [6, p. 351]).

DEFINITION 23. Let $0 \leqslant k \leqslant n$. A $(0, k)$-tensor field Q is said to be of class L^1_{loc}, if its representation in a chart is of class L^1_{loc}.

Remark.

1. For $1 \leqslant k \leqslant n$, if Q is a $(0, k)$-tensor field of class L^1_{loc} and $\mathbf{v} \in \mathcal{X}_c$, then $\mathsf{Q}\mathbf{v}$ is a $(0, k-1)$-tensor field of class L^1_{loc}.
2. If ω is an $(n-1)$-differential form of class L^1_{loc}, then $\int_{\partial_* M} \omega$ is well defined for almost every $m \in \mathcal{M}^\circ$.
3. If $f \in C_0^\infty(B)$ and ω is an $(n-1)$-differential form of class L^1_{loc}, then $(df) \wedge \omega$ is an n-differential form of class L^1_{loc}.

DEFINITION 24. Let ω be an $(n-1)$-differential form of class L^1_{loc}. We say that *$d\omega$ is a measure*, if for every compact set $K \subseteq B$ there exists $c_K \geqslant 0$ such that

$$\left| \int_B (df) \wedge \omega \right| \leqslant c_K \|f\|_\infty$$

for every $f \in C_0^\infty(B)$ with supt $f \subseteq K$.

The following theorem states the representation formula for a balanced Cauchy power on a manifold.

THEOREM 9. *Let P be a balanced Cauchy power on B. Then there exists a $(0, n)$-tensor field Q on B of class L^1_{loc} such that*:

(a) $\mathsf{Q}\mathbf{v}$ *is an $(n-1)$-differential form for every $\mathbf{v} \in \mathcal{X}_c(B)$, that is, for a.e. $x \in B$ the map*
$$\{(w_1, \ldots, w_{n-1}) \mapsto \mathsf{Q}(\mathbf{v}(x), w_1, \ldots, w_{n-1})\}$$
is $(n-1)$-alternating;

(b) $d(\mathsf{Q}\mathbf{v})$ *is a measure for every $\mathbf{v} \in \mathcal{X}_c(B)$*;

(c) *for almost every $M \in \mathcal{M}^\circ$, the formula*
$$P(M, \mathbf{v}) = \int_{\partial_* M} \mathsf{Q}\mathbf{v} \tag{5}$$
holds for every $\mathbf{v} \in \mathcal{X}_c(B)$.

Proof. Let $x \in B$ and (U, φ) be a chart with $\varphi(U) = B_r(\varphi(x))$ for a suitable $r > 0$. Then U is a normalized set with finite perimeter. We define a function $R : \mathcal{D} \times C_c^\infty(\varphi(U); \mathbb{R}^n) \to \mathbb{R}^n$, where $\mathcal{D}$ contains almost all of $\mathcal{M}^\circ(\varphi(U))$, setting

$$R(A, \mathbf{v}) = P(\varphi^{-1}(A), (d\varphi)^{-1}\mathbf{v}).$$

Such a function is well defined, since $(d\varphi)^{-1}\mathbf{v} \in \mathcal{X}_c(B)$ (up to an extension by zero outside U) and $\varphi^{-1}(A) \in \mathcal{M}^\circ$. We claim that R is a balanced Cauchy power on $\varphi(U)$. Additivity on the first argument and linearity on the second are obvious, while (c) of Definition 10 follows by the estimate

$$|R(A, \mathbf{v})| \leqslant \int_{\partial_*(\varphi^{-1}(A))} |(d\varphi)^{-1}\mathbf{v}| h \, \mathrm{d}\mathcal{H}^{n-1} \leqslant \int_{\partial_* A} \tilde{h} |\mathbf{v}| \, \mathrm{d}\mathcal{H}^{n-1},$$

which holds on almost all of $\mathcal{M}^\circ(\varphi(U))$ for every $\mathbf{v} \in C_c^\infty(\varphi(U); \mathbb{R}^n)$. Moreover, considering that P is balanced, one can prove that R is balanced. By applying Theorem 6 to R with $n = N$, we get an essentially unique function T_U in $L^1_{\mathrm{loc}}(\varphi(U); \mathrm{Lin}(\mathbb{R}^n; \mathbb{R}^n))$ with divergence measure such that, on almost all of $\mathcal{M}^\circ(\varphi(U))$,

$$R(A, \mathbf{v}) = \int_{\partial_* A} \mathsf{T}_U \mathbf{n}^A \cdot \mathbf{v} \, \mathrm{d}\mathcal{H}^{n-1}$$

for every $\mathbf{v} \in C_c^\infty(\varphi(U); \mathbb{R}^n)$. It is not hard to prove that if $U \cap V \neq \emptyset$, then $\mathsf{T}_U = \mathsf{T}_V$ on $\varphi(U) \cap \varphi(V)$. Denoting with $d\mathcal{L}^n$ the volume form of $\mathbb{R}^n$, the function

$$\{(v_1, \ldots, v_n) \mapsto ((\mathsf{T}_U v_1) \lrcorner d\mathcal{L}^n)(v_2, \ldots, v_n)\}$$

is a $(0, n)$-tensor field on $\varphi(U)$. Pulling it back on B, one obtains the tensor field Q which satisfies (5). Moreover, since T_U has divergence measure in $\mathbb{R}^n$, $d(\mathsf{Q}\mathbf{v})$ is a measure for every $\mathbf{v} \in \mathcal{X}_c(B)$. □

Acknowledgements

The authors thank Marco Degiovanni for engaging in very useful discussions and Antonio Di Carlo for drawing their attention to the subject.

The research was supported partially by the MURST COFIN2000 project “Modelli Matematici per la Scienza dei Materiali” and the GNFM project “Continui di grado elevato e teoria delle distribuzioni”.

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