MEASURE-VALUED LOADS FOR A HYPERELASTIC MODEL OF SOFT TISSUES

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ABSTRACT. We study a simplified version of a class of constitutive relations used to describe large deformations of soft tissues, where the elastic energy density involves an exponential term. The class was originally introduced by Y.C. Fung as a model of many biological soft tissues in a series of papers during the Seventies. We prove existence and uniqueness of the equilibrium solution for a general measure-valued external load, under quite general boundary conditions, and study the validity of the associated Euler–Lagrange equation in the sense of distributions.

1. Introduction

Biological soft tissues are generally inelastic: these tissues, subjected to cyclic loading and unloading, usually show hysteresis phenomena as well as viscoelastic behavior. However, under a preconditioning process, it is possible to assume a drastic simplification to reduce the viscoelastic nonlinear constitutive equation of a biological tissue to a nonlinear hyperelastic one, obtaining the existence of a potential. In fact, considering a preconditioned tissue subjected to cyclic loading and unloading at constant strain rates, the stress–strain relationship becomes a curve, so that we can treat this material as elastic with a pseudo-elastic potential W.

In the seminal paper [6], Y.C. Fung proposed a one-dimensional exponential relation between stress and strain, based on experimental uniaxial tensile data from rabbit mesentery. Afterward, the same author proposed a similar model for biaxial tensile data [7]. Moreover, it was verified that in cyclic loading and unloading at constant strain rates the stress relationship is essentially independent of strain rates, and that the mechanical properties are orthotropic.

Tong and Fung [10] obtained information for the formulation of the hyperelastic potential for a loading process of the rabbit skin. In a state of plane stress, they proposed the following form for the hyperelastic potential:

$$W = \frac{1}{2} f(\alpha, \mathsf{E}(\boldsymbol{u})) + \frac{c}{2} \exp[F(a, \gamma, \mathsf{E}(\boldsymbol{u}))] \tag{1}$$

where $\mathsf{E}(\boldsymbol{u}) = (\mathsf{F}^\mathsf{T}\mathsf{F} - \mathsf{I})/2 = (\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\mathsf{T} + \nabla\boldsymbol{u}^\mathsf{T}\nabla\boldsymbol{u})/2$ is the Green strain tensor (see [9]), $\boldsymbol{u}$ is the displacement, and

$$f(\alpha, \mathsf{E}(\boldsymbol{u})) = \alpha_1 E_{xx}^2 + \alpha_2 E_{yy}^2 + \alpha_3 E_{xy}^2 + 2\alpha_4 E_{xx} E_{yy} \tag{2}$$

$$\begin{aligned} F(a, \gamma, \mathsf{E}(\boldsymbol{u})) &= a_1 E_{xx}^2 + a_2 E_{yy}^2 + a_3 E_{xy}^2 + 2a_4 E_{xx} E_{yy} \\ &\quad + \gamma_1 E_x^3 + \gamma_2 E_{yy}^3 + \gamma_4 E_{xx}^2 E_{yy} + \gamma_5 E_{xx} E_{yy}^2, \end{aligned} \tag{3}$$

2010 *Mathematics Subject Classification.* Primary 74B20; Secondary 74L15.

Key words and phrases. Nonlinear elasticity; Calculus of variations; Variational inequalities.

α_i, a_j, γ_j being constant parameters. The most important term in the equation above is the exponential one: it models the behavior of the material for large deformations, which are typical of a soft tissue. The first term accounts for the response of the material at a lower stress level.

A simplified version of the model, where all the third-order terms are dropped ($\gamma_i = 0$), writes

$$W(\boldsymbol{u}) = \frac{1}{2}\mathbb{L}\mathsf{E}(\boldsymbol{u}) \cdot \mathsf{E}(\boldsymbol{u}) + \frac{c}{2}\exp(\mathbb{M}\mathsf{E}(\boldsymbol{u}) \cdot \mathsf{E}(\boldsymbol{u})), \tag{4}$$

where $\mathbb{L}$ and $\mathbb{M}$ are two constant fourth-order tensors and $c > 0$. Moreover, as pointed out in [5], the two tensors $\mathbb{L}$ and $\mathbb{M}$ have to be positive definite in order to have the convexity of W w.r.t. $\mathsf{E}(\boldsymbol{u})$. In components the energy density writes

$$W(\boldsymbol{u}) = \frac{1}{2}\mathbb{L}_{ijhk}\mathsf{E}(\boldsymbol{u})_{ij}\mathsf{E}(\boldsymbol{u})_{hk} + \frac{c}{2}\exp(\mathbb{M}_{ijhk}\mathsf{E}(\boldsymbol{u})_{ij}\mathsf{E}(\boldsymbol{u})_{hk}). \tag{5}$$

As it is well-known, the problem of finding a static equilibrium configuration under some external loads and boundary conditions can be tackled mathematically by finding a critical point (in fact, a minimizer) of the total energy (see e.g. [2, Chapter 7]). In our case, the mathematical problem has two main nontrivial issues:

- As a feature, the regularity of the model allows for measure-valued external forces. For instance, the case of an external concentrated force, namely a Dirac delta load applied on a single point, can be studied, even if applied on the boundary.
- As a drawback, the Euler–Lagrange equation for the problem may have no sense, since the exponential term may lead to inconsistencies in the functional class where we seek the solution. Hence, a detailed mathematical study has to be performed and some *variational inequalities* have to be introduced.

As far as the second issue is concerned, it is well known (see e.g. [4, Theorem 3.37]) that under some *growth conditions* the minimizer of the energy functional associated with a density W indeed does exist and satisfies the associated Euler–Lagrange equation. For instance, it is sufficient to assume that $|W(\boldsymbol{u})| \leq |\boldsymbol{u}|^p$ for some $p > 1$. However, in the model we are studying, the presence of an exponential term prevents the energy to fulfill such a growth condition and the validity of the Euler–Lagrange equation is questionable. We will be able to prove the equivalence between the (weak) Euler–Lagrange equation and the existence of the minimizer for a special geometric setting in Theorem 4.2.

We will study the problem by means of the Direct Method in the Calculus of Variations (see again [4] for a mathematical presentation and [2, Chapter 7] for a typical application to the case of nonlinear elasticity). To this aim, we need a crucial assumption, which is a sort of partial linearization of the model, namely

$$\text{replace } \mathsf{E}(\boldsymbol{u}) \text{ with } \mathsf{e}(\boldsymbol{u}) := \mathrm{Sym}(\nabla\boldsymbol{u}), \tag{6}$$

where $\mathrm{Sym}(\nabla\boldsymbol{u})$ denotes the symmetric component of $\nabla\boldsymbol{u}$. In such a way, the measure of the strain is linear in the displacement and the problem becomes less difficult. The elastic energy then writes

$$W(\boldsymbol{u}) = \frac{1}{2}\mathbb{L}\mathsf{e}(\boldsymbol{u}) \cdot \mathsf{e}(\boldsymbol{u}) + \frac{c}{2}\exp(\mathbb{M}\mathsf{e}(\boldsymbol{u}) \cdot \mathsf{e}(\boldsymbol{u})). \tag{7}$$

The plan of the paper is the following: in Section 2 we state the mathematical problem and prove the existence and uniqueness of the minimizer. In Section 3 we give necessary conditions for the minimizer to be a solution of the corresponding Euler–Lagrange equation. Finally, in Section 4 we study a particular case of a domain lying between the graphs of two Lipschitz functions, finding necessary and sufficient conditions for the minimizer to solve the Euler–Lagrange equation.

2. Existence and uniqueness of the solution

Given an $n \times n$-matrix F, we denote with $|\mathsf{F}|$ its Frobenius norm, that is

$$|\mathsf{F}| := \Big(\sum_{i,j=1}^{n} F_{ij}^2 \Big)^{1/2} . \tag{8}$$

2.1. **Displacements and boundary conditions.**

Let Ω be a bounded connected open subset of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary. Since the hyperelastic energy (7) contains an exponential term, it is natural to consider displacements $\boldsymbol{u}$ in the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^n)$ with $n < p < \infty$, endowed with the usual norm

$$\|\boldsymbol{u}\|_{1,p} = \left(\|\boldsymbol{u}\|_p^p + \|\nabla \boldsymbol{u}\|_p^p\right)^{1/p} . \tag{9}$$

Since $p > n$, the so-called Sobolev imbedding Theorem [1, Theorem 5.4] implies that the elements of $W^{1,p}(\Omega;\mathbb{R}^n)$ admit a *continuous* representative on $\overline{\Omega}$, that is

$$W^{1,p}(\Omega;\mathbb{R}^n) \subset C^0(\overline{\Omega};\mathbb{R}^n) \tag{10}$$

(the representative is indeed Hölder-continuous). Moreover, the classic *Korn inequality* (see for instance [8, Sec. 5.6]) reads

$$\|\boldsymbol{u}\|_{1,p}^p \leq c_1 \left(\|\boldsymbol{u}\|_p^p + \|\mathsf{e}(\boldsymbol{u})\|_p^p\right) \tag{11}$$

where $\mathsf{e}(\boldsymbol{u}) = (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)/2$.

We assume homogeneous Dirichlet boundary conditions only on a part of the boundary: let $\Gamma \subset \partial\Omega$ be a closed subset of the boundary of Ω such that

$$\{\boldsymbol{0}\} = \left\{\boldsymbol{u} \in W^{1,p}(\Omega;\mathbb{R}^n) \colon \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma \ \text{ and } \ \mathsf{e}(\boldsymbol{u}) = \mathsf{0} \text{ on } \Omega\right\}, \tag{12}$$

namely, the identity is the only rigid displacement keeping all the points of Γ fixed. For instance, Γ can consist of three non collinear points (remember that $\boldsymbol{u}$ is continuous, so that it is defined on points). The domain of the energy functional is the set

$$W_\Gamma^{1,p} := \left\{\boldsymbol{u} \in W^{1,p}(\Omega;\mathbb{R}^n) : \ \boldsymbol{u}(x) = \boldsymbol{0} \text{ for any } x \in \Gamma\right\} \tag{13}$$

endowed with the norm $\|\cdot\|_{1,p}$. Notice that $W_\Gamma^{1,p}$ is a closed linear subspace of $W^{1,p}(\Omega;\mathbb{R}^n)$.

Now we prove a version of the Korn inequality which is adapted to our general boundary conditions. Recall that a sequence u_h *weakly converges* to u in L^p, $u_h \rightharpoonup u$, if

$$\int [u_h(x) - u(x)] v(x)\, dx \to 0 \tag{14}$$

for every $v \in L^{p'}$, where $p' = p/(p-1)$ (see [3, Definition 1.15]). Analogously, in the Sobolev space $W^{1,p}$ the weak convergence $u_h \rightharpoonup u$ means that $u_h \rightharpoonup u$ and $\nabla u_h \rightharpoonup \nabla u$ in L^p.

Proposition 2.1 (Korn inequality)**.** *There exists $c > 0$ such that*

$$\forall \boldsymbol{u} \in W^{1,p}_{\Gamma} : \ \|\boldsymbol{u}\|_{1,p} \le c \|\mathsf{e}(\boldsymbol{u})\|_p \,. \tag{15}$$

Proof. By contradiction, let $(\boldsymbol{u}_h)$ be a sequence in $W^{1,p}_{\Gamma}$ with $\|\boldsymbol{u}_h\|_{1,p} = 1$ and $\|\mathsf{e}(\boldsymbol{u}_h)\|_p < \frac{1}{h}$. Since $(\boldsymbol{u}_h)$ is bounded in $W^{1,p}$, then there exists $\boldsymbol{u} \in W^{1,p}_{\Gamma}$ such that

$$\boldsymbol{u}_h \rightharpoonup \boldsymbol{u} \quad \text{weakly in } W^{1,p} \tag{16}$$

up to a subsequence; in particular, $\mathsf{e}(\boldsymbol{u}) = \mathsf{0}$ and $\boldsymbol{u}_h \to \boldsymbol{u}$ in L^p up to another subsequence. By (12) one gets $\boldsymbol{u} = \boldsymbol{0}$, hence $\boldsymbol{u}_h \to \boldsymbol{0}$ in L^p. Then the classic Korn inequality (11) yields

$$1 = \|\boldsymbol{u}_h\|^p_{1,p} \le c_1 \left(\|\boldsymbol{u}_h\|^p_p + \|\mathsf{e}(\boldsymbol{u}_h)\|^p_p \right) \to 0 \tag{17}$$

as $h \to \infty$, a contradiction. □

2.2. **External loads.** We denote with $\overline{\Omega}$ the closure of the set Ω. Let $\boldsymbol{\mu}$ be a vector measure on $\overline{\Omega}$, that is, a vector-valued function defined on the Borel subsets of $\overline{\Omega}$ which is countably additive on pairwise disjoint sequences of subsets. We recall that the *total variation* of $\boldsymbol{\mu}$ is a positive measure defined by

$$|\boldsymbol{\mu}|(A) = \sup \left\{ \sum_{h=0}^{\infty} \|\boldsymbol{\mu}(A_h)\| : \bigcup_{h=0}^{\infty} A_h = A, \ A_h \text{ Borel and pairwise disjoint} \right\} \tag{18}$$

where $\|\cdot\|$ denotes the norm in the vector space. The external load $\boldsymbol{F}$ is assumed to be a vector measure with bounded total variation on $\overline{\Omega}$. Notice that the external load can be very general: for instance, one can consider both *internal* and *boundary* Dirac delta concentrated loads, as well as loads concentrated on curves, discontinuous loads, and so on. As an example, consider a point $x_0 \in \overline{\Omega}$ and define

$$\boldsymbol{F}(A) := \begin{cases} \boldsymbol{e}_1 & \text{if } x_0 \in A \\ \boldsymbol{0} & \text{if } x_0 \notin A; \end{cases} \tag{19}$$

hence $\boldsymbol{F}$ represents a Dirac delta measure supported at the point x_0 with value the vector $\boldsymbol{e}_1$, that is, $\boldsymbol{F} = \delta_{x_0} \boldsymbol{e}_1$.

Since the elements of $W^{1,p}(\Omega;\mathbb{R}^n)$ admit a continuous representative on $\overline{\Omega}$, then the integral

$$\int_{\overline{\Omega}} \boldsymbol{u} \cdot d\boldsymbol{F} \tag{20}$$

is well-defined. In the example above where $\boldsymbol{F} = \delta_{x_0} \boldsymbol{e}_1$, one has

$$\int_{\overline{\Omega}} \boldsymbol{u} \cdot d\boldsymbol{F} = \boldsymbol{u}(x_0) \cdot \boldsymbol{e}_1 . \tag{21}$$

2.3. **The energy functional.** Let us denote by $\mathrm{Sym}(\mathbb{R}^n)$ the set of symmetric tensors on $\mathbb{R}^n$. We consider an elastic energy density $W : \mathrm{Sym}(\mathbb{R}^n) \to [0, +\infty)$ such that:

- W is C^1;
- W is strictly convex;
- W satisfies the growth condition

$$\liminf_{|\mathsf{E}| \to \infty} \frac{W(\mathsf{E})}{|\mathsf{E}|^q} > 0 \quad \text{for some } q > n/2 \,. \tag{22}$$

Notice that the constitutive equation (7) is smooth and satisfies the growth condition for any $q > 1$. Assuming that $\mathbb{L}$ and $\mathbb{M}$ are positive definite, it is also strictly convex.

Choosing $p = 2q > n$, we will study the energy functional

$$J : W^{1,p}_{\Gamma} \to (-\infty, +\infty]$$
$$J(\boldsymbol{u}) := \int_{\Omega} W(\mathsf{e}(\boldsymbol{u}))\, dx - \int_{\overline{\Omega}} \boldsymbol{u} \cdot d\boldsymbol{F}\,, \tag{23}$$

which is clearly convex.

Now, by the usual Direct Method in the Calculus of Variations, we want to prove that the functional J attains its minimum. The two key ingredients are the *weakly lower semicontinuity* and the *coerciveness* of the functional J (see e.g. [3, Chapter 3]), as we detail in the next two lemmata.

We recall that Sobolev imbedding Theorem [1, Theorem 5.4] implies the so-called *Morrey's inequality*, that is

$$\forall \boldsymbol{u} \in W^{1,p}(\Omega; \mathbb{R}^n): \quad \|\boldsymbol{u}\|_\infty \leq K \|\boldsymbol{u}\|_{1,p} \tag{24}$$

where $\|\boldsymbol{u}\|_\infty := \sup_{\overline{\Omega}} |\boldsymbol{u}|$ and $K > 0$ does not depend on $\boldsymbol{u}$.

Lemma 2.2. *The functional J is weakly lower semicontinuous, that is*

$$J(\boldsymbol{u}) \leq \liminf_{h\to\infty} J(\boldsymbol{u}_h) \quad \textit{for every sequence } \boldsymbol{u}_h \rightharpoonup \boldsymbol{u} \textit{ in } W^{1,p}_{\Gamma}. \tag{25}$$

Proof. Since W is convex, it is enough to prove that J is strongly lower semicontinuous. Take a sequence $\boldsymbol{u}_h \to \boldsymbol{u}$ in $W^{1,p}_{\Gamma}$; then $\nabla \boldsymbol{u}_h \to \nabla \boldsymbol{u}$ in L^p and, up to a subsequence, $\nabla \boldsymbol{u}_h(x) \to \nabla \boldsymbol{u}(x)$ for a.e. $x \in \Omega$. Moreover $\mathsf{e}(\boldsymbol{u_h}) \to \mathsf{e}(\boldsymbol{u})$ in $L^1(\Omega)$ since Ω is bounded and $p > 2$.

The convexity of W then yields

$$W(\mathsf{e}(\boldsymbol{u}_h)) \geq W(0) + W'(0) \cdot \mathsf{e}(\boldsymbol{u}_h) \quad \text{for every } h \in \mathbb{N}. \tag{26}$$

Since the right-hand side converges in $L^1(\Omega)$, we can apply Fatou's Lemma to $W(\mathsf{e}(\boldsymbol{u}_h))$, obtaining

$$\int_{\Omega} W(\mathsf{e}(\boldsymbol{u}))\, dx \leq \liminf_{h\to\infty} \int_{\Omega} W(\mathsf{e}(\boldsymbol{u}_h))\, dx. \tag{27}$$

Hence the first term of J is lower semicontinuous.

Moreover, the linear term of J is continuous by Morrey's inequality (24):

$$\left| \int_{\overline{\Omega}} \boldsymbol{u} \cdot d\boldsymbol{F} \right| \leq \int_{\overline{\Omega}} |\boldsymbol{u}|\, d|\boldsymbol{F}| \leq \|\boldsymbol{u}\|_\infty |\boldsymbol{F}|(\overline{\Omega}) \leq K \|\boldsymbol{u}\|_{1,p} |\boldsymbol{F}|(\overline{\Omega}). \tag{28}$$

Then the whole J is lower semicontinuous. □

Lemma 2.3. *The sublevels $\{J \leq c\}$ of the functional J are bounded in $W^{1,p}_{\Gamma}$.*

Proof. The growth condition (22) of the density W yields the existence of $a > 0$ and $b \in \mathbb{R}$ such that

$$\forall \mathsf{E} \in \mathrm{Sym}(\mathbb{R}^n): \quad W(\mathsf{E}) \geq a|\mathsf{E}|^p - b. \tag{29}$$

Then for every $\boldsymbol{u} \in W^{1,p}_{\Gamma}$ one has, by Morrey's inequality,

$$J(\boldsymbol{u}) = \int_{\Omega} W(\mathsf{e}(\boldsymbol{u}))\, dx - \int_{\overline{\Omega}} \boldsymbol{u} \cdot d\boldsymbol{F} \geq a\|\mathsf{e}(\boldsymbol{u})\|_p^p - b|\Omega| - K|\boldsymbol{F}|(\overline{\Omega})\|\boldsymbol{u}\|_{1,p} \tag{30}$$

where K does not depend on $\boldsymbol{u}$. By Proposition 2.1 there exists $c_2 > 0$ such that

$$\forall \boldsymbol{u} \in W^{1,p}_\Gamma : \quad J(\boldsymbol{u}) \geq c_2 \|\boldsymbol{u}\|^p_{1,p} - b|\Omega| - K|\boldsymbol{F}|(\overline{\Omega})\|\boldsymbol{u}\|_{1,p} . \tag{31}$$

Now consider a sublevel $\{J \leq c\}$, that is $\{\boldsymbol{u} \in W^{1,p}_\Gamma : \ J(\boldsymbol{u}) \leq c\}$: by the previous inequality one has

$$\boldsymbol{u} \in \{J \leq c\} \ \Rightarrow \ c_2\|\boldsymbol{u}\|^p_{1,p} - b|\Omega| - K|\boldsymbol{F}|(\overline{\Omega})\|\boldsymbol{u}\|_{1,p} \leq c. \tag{32}$$

Since $p > 1$ and $c_2 > 0$, the sublevel has to be bounded. □

Now the proof of the following theorem is straightforward.

Theorem 2.4. *The functional J attains its minimum on $W^{1,p}_\Gamma$. Moreover, the minimizer $\bar{\boldsymbol{u}}$ is unique and $W(\mathsf{e}(\bar{\boldsymbol{u}})) \in L^1(\Omega)$.*

Proof. Let $c > J(\mathbf{0})$. By the previous lemmas, $\{J \leq c\}$ is nonempty, convex, closed and bounded in $W^{1,p}_\Gamma$, hence it is weakly compact in $W^{1,p}_\Gamma$. In particular, J has a minimizer $\bar{\boldsymbol{u}}$ in $\{J \leq c\}$, which is obviously a minimizer in $W^{1,p}_\Gamma$ with $J(\bar{\boldsymbol{u}}) \leq J(\mathbf{0})$, hence $W(\mathsf{e}(\bar{\boldsymbol{u}})) \in L^1$.

Now let $\boldsymbol{v}$ be another minimizer. Then

$$\inf J \leq J\Big(\frac{\bar{\boldsymbol{u}} + \boldsymbol{v}}{2}\Big) \leq \frac{1}{2}J(\bar{\boldsymbol{u}}) + \frac{1}{2}J(\boldsymbol{v}) = \inf J\,, \tag{33}$$

hence the equality holds and

$$\int_\Omega \left[\frac{1}{2}W(\mathsf{e}(\bar{\boldsymbol{u}})) + \frac{1}{2}W(\mathsf{e}(\boldsymbol{v})) - W\left(\frac{\mathsf{e}(\bar{\boldsymbol{u}}) + \mathsf{e}(\boldsymbol{v})}{2}\right)\right] dx = 0\,. \tag{34}$$

Since W is strictly convex, one has $\mathsf{e}(\bar{\boldsymbol{u}})(x) = \mathsf{e}(\boldsymbol{v})(x)$ a.e. in Ω, hence $\bar{\boldsymbol{u}} = \boldsymbol{v}$ by assumption (12). □

3. The Euler–Lagrange equation

We now investigate some properties of the minimizer of J, focusing in particular on the Euler–Lagrange equation. First of all, we characterize the unique minimizer by means of a variational inequality.

Notice that, since the material is hyperelastic, $W'(\mathsf{e}(\boldsymbol{u}))$ corresponds to the elastic stress for the displacement $\boldsymbol{u}$.

Theorem 3.1. *Let $\bar{\boldsymbol{u}} \in W^{1,p}_\Gamma$ be such that $W(\mathsf{e}(\bar{\boldsymbol{u}})) \in L^1(\Omega)$. Then $\bar{\boldsymbol{u}}$ is the minimizer of J if and only if*

$$W'(\mathsf{e}(\bar{\boldsymbol{u}})) \cdot (\mathsf{e}(\boldsymbol{u}) - \mathsf{e}(\bar{\boldsymbol{u}})) \in L^1(\Omega)\,, \tag{35}$$

$$\int_\Omega W'(\mathsf{e}(\bar{\boldsymbol{u}})) \cdot (\mathsf{e}(\boldsymbol{u}) - \mathsf{e}(\bar{\boldsymbol{u}}))\, dx \geq \int_{\overline{\Omega}} (\boldsymbol{u} - \bar{\boldsymbol{u}}) \cdot d\boldsymbol{F} \tag{36}$$

for every $\boldsymbol{u} \in W^{1,p}_\Gamma$ such that $W(\mathsf{e}(\boldsymbol{u})) \in L^1(\Omega)$.

Proof. Let $\bar{\boldsymbol{u}}$ be the minimizer of J and let $\boldsymbol{u} \in W^{1,p}_\Gamma$ be such that $W(\mathsf{e}(\boldsymbol{u})) \in L^1(\Omega)$. For any $h \geq 1$ the convexity of W yields

$$h\left[W\Big(\mathsf{e}(\bar{\boldsymbol{u}}) + \frac{1}{h}(\mathsf{e}(\boldsymbol{u}) - \mathsf{e}(\bar{\boldsymbol{u}}))\Big) - W(\mathsf{e}(\bar{\boldsymbol{u}}))\right] \leq W(\mathsf{e}(\boldsymbol{u})) - W(\mathsf{e}(\bar{\boldsymbol{u}}))\,, \tag{37}$$

and letting $h \to \infty$ one gets

$$W'(\mathsf{e}(\bar{\boldsymbol{u}})) \cdot (\mathsf{e}(\boldsymbol{u}) - \mathsf{e}(\bar{\boldsymbol{u}})) \leq W(\mathsf{e}(\boldsymbol{u})) - W(\mathsf{e}(\bar{\boldsymbol{u}}))\,. \tag{38}$$

In particular $[W'(\mathsf{e}(\bar{\boldsymbol{u}})) \cdot (\mathsf{e}(\boldsymbol{u}) - \mathsf{e}(\bar{\boldsymbol{u}}))]^+ \in L^1(\Omega)$, where the notation $[f]^+$ denotes the positive part of a real-valued function f. Moreover, Fatou's Lemma yields

$$\int_\Omega W'(\mathsf{e}(\bar{\boldsymbol{u}}))\cdot(\mathsf{e}(\boldsymbol{u})-\mathsf{e}(\bar{\boldsymbol{u}}))\,dx \geq \limsup_h h\left[W\Big(\mathsf{e}(\bar{\boldsymbol{u}}) + \frac{1}{h}(\mathsf{e}(\boldsymbol{u}) - \mathsf{e}(\bar{\boldsymbol{u}}))\Big) - W(\mathsf{e}(\bar{\boldsymbol{u}}))\right]. \tag{39}$$

Since Lebesgue's Theorem entails

$$\lim_h \frac{1}{h}\int_{\overline{\Omega}} (\boldsymbol{u} - \bar{\boldsymbol{u}})\cdot d\boldsymbol{F} = 0\,, \tag{40}$$

by the convexity of J it follows that

$$\begin{aligned}\int_\Omega W'(\mathsf{e}(\bar{\boldsymbol{u}}))\cdot(\mathsf{e}(\boldsymbol{u})-\mathsf{e}(\bar{\boldsymbol{u}}))\,dx - \int_{\overline{\Omega}} (\boldsymbol{u} - \bar{\boldsymbol{u}})\cdot d\boldsymbol{F}& \\ \geq \limsup_h h\Big[J\Big(\bar{\boldsymbol{u}} + \frac{1}{h}(\boldsymbol{u} - \bar{\boldsymbol{u}})\Big) - J(\bar{\boldsymbol{u}})\Big] &\geq 0\,,\end{aligned} \tag{41}$$

whence (35) and (36).

On the other hand, let $\bar{\boldsymbol{u}} \in W^{1,p}_\Gamma$ be such that (35) and (36) hold. Let $\boldsymbol{u} \in W^{1,p}_\Gamma$; if $W(\mathsf{e}(\boldsymbol{u})) \notin L^1(\Omega)$, then $J(\boldsymbol{u}) = +\infty$, hence $J(\boldsymbol{u}) \geq J(\bar{\boldsymbol{u}})$. If $W(\mathsf{e}(\boldsymbol{u})) \in L^1(\Omega)$, then we can rewrite (38) as

$$W(\mathsf{e}(\boldsymbol{u})) \geq W(\mathsf{e}(\bar{\boldsymbol{u}})) + W'(\mathsf{e}(\bar{\boldsymbol{u}}))\cdot(\mathsf{e}(\boldsymbol{u}) - \mathsf{e}(\bar{\boldsymbol{u}}))\,. \tag{42}$$

Integrating on Ω and taking into account (36) we get

$$\begin{aligned}\int_\Omega W(\mathsf{e}(\boldsymbol{u}))\,dx &\geq \int_\Omega W(\mathsf{e}(\bar{\boldsymbol{u}}))\,dx + \int_\Omega W'(\mathsf{e}(\bar{\boldsymbol{u}}))\cdot(\mathsf{e}(\boldsymbol{u}) - \mathsf{e}(\bar{\boldsymbol{u}}))\,dx \\ &\geq \int_\Omega W(\mathsf{e}(\bar{\boldsymbol{u}}))\,dx + \int_{\overline{\Omega}} (\boldsymbol{u} - \bar{\boldsymbol{u}})\cdot d\boldsymbol{F}\,,\end{aligned} \tag{43}$$

which implies that $J(\boldsymbol{u}) \geq J(\bar{\boldsymbol{u}})$, hence $\bar{\boldsymbol{u}}$ is the minimizer. □

Remark 3.2. If $\bar{\boldsymbol{u}}$ is the minimizer of J, then it is easy to see that

$$W'(\mathsf{e}(\bar{\boldsymbol{u}}))\cdot\mathsf{e}(\bar{\boldsymbol{u}}) \in L^1(\Omega) \quad \text{and} \quad \int_\Omega W'(\mathsf{e}(\bar{\boldsymbol{u}}))\cdot\mathsf{e}(\bar{\boldsymbol{u}})\,dx \leq \int_{\overline{\Omega}} \bar{\boldsymbol{u}}\cdot d\boldsymbol{F}\,. \tag{44}$$

Indeed, $\mathbf{0} \in W^{1,p}_\Gamma$ and $W(\mathbf{0}) \in L^1(\Omega)$, hence one can apply the previous theorem.

Finally we prove that the minimizer $\bar{\boldsymbol{u}}$ satisfies the Euler–Lagrange equation in a distributional sense. We would like to prove also the converse, that is, the Euler–Lagrange equation is a characterization of the minimizer; at the moment, we are able to prove it only for particular domains (see Section 4).

Let us denote by $C^\infty_\Gamma(\overline{\Omega};\mathbb{R}^n)$ the family of vector fields $\boldsymbol{w} \in C^\infty(\overline{\Omega};\mathbb{R}^n)$ such that $\Gamma \cap \operatorname{supt}\boldsymbol{w} = \varnothing$.

Theorem 3.3. *Let $\bar{\boldsymbol{u}} \in W^{1,p}_\Gamma$ be the minimizer of J and let $\boldsymbol{w} \in C^\infty_\Gamma(\overline{\Omega};\mathbb{R}^n)$. Then $W'(\mathsf{e}(\bar{\boldsymbol{u}}))\cdot\mathsf{e}(\boldsymbol{w}) \in L^1(\Omega)$ and*

$$\int_\Omega W'(\mathsf{e}(\bar{\boldsymbol{u}}))\cdot\mathsf{e}(\boldsymbol{w})\,dx = \int_{\overline{\Omega}} \boldsymbol{w}\cdot d\boldsymbol{F}\,. \tag{45}$$

In particular, $-\operatorname{div}(W'(\mathsf{e}(\bar{\boldsymbol{u}}))) = \boldsymbol{F}$ in the sense of distributions on Ω.

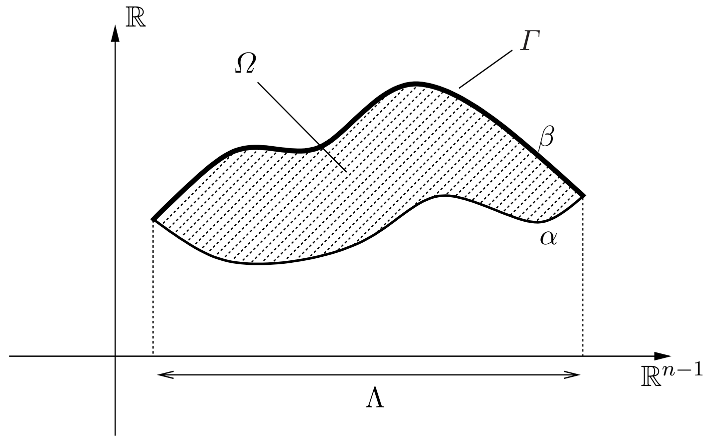

FIGURE 1. A domain satisfying assumption (50).

Proof. Let $\boldsymbol{w} \in C^\infty_\Gamma(\overline{\Omega};\mathbb{R}^n)$ and $t > 0$. Since $t\boldsymbol{w} \in C^\infty_\Gamma(\overline{\Omega};\mathbb{R}^n)$, then $t\mathsf{e}(\boldsymbol{w}) \in L^\infty(\Omega)$ and $W(t\mathsf{e}(\boldsymbol{w})) \in L^1(\Omega)$. Hence we can apply Theorem 3.1:

$$\int_\Omega W'(\mathsf{e}(\bar{\boldsymbol{u}})) \cdot (t\mathsf{e}(\boldsymbol{w}) - \mathsf{e}(\bar{\boldsymbol{u}}))\, dx \geq \int_{\overline{\Omega}} (t\boldsymbol{w} - \bar{\boldsymbol{u}}) \cdot d\boldsymbol{F}\,. \tag{46}$$

In particular, dividing by t one has

$$\int_\Omega W'(\mathsf{e}(\bar{\boldsymbol{u}})) \cdot \mathsf{e}(\boldsymbol{w})\, dx - \frac{1}{t}\int_\Omega W'(\mathsf{e}(\bar{\boldsymbol{u}})) \cdot \mathsf{e}(\bar{\boldsymbol{u}})\, dx \geq \int_{\overline{\Omega}} \boldsymbol{w} \cdot d\boldsymbol{F} - \frac{1}{t}\int_{\overline{\Omega}} \bar{\boldsymbol{u}} \cdot d\boldsymbol{F} \tag{47}$$

an letting $t \to +\infty$ (taking into account Remark 3.2)

$$\forall \boldsymbol{w} \in C^\infty_\Gamma(\overline{\Omega};\mathbb{R}^n): \quad \int_\Omega W'(\mathsf{e}(\bar{\boldsymbol{u}})) \cdot \mathsf{e}(\boldsymbol{w})\, dx \geq \int_{\overline{\Omega}} \boldsymbol{w} \cdot d\boldsymbol{F}\,. \tag{48}$$

Now, changing $\boldsymbol{w}$ with $-\boldsymbol{w}$ gives the opposite inequality, hence

$$\int_\Omega W'(\mathsf{e}(\bar{\boldsymbol{u}})) \cdot \mathsf{e}(\boldsymbol{w})\, dx = \int_{\overline{\Omega}} \boldsymbol{w} \cdot d\boldsymbol{F}\,. \tag{49}$$

□

4. NECESSARY AND SUFFICIENT CONDITIONS FOR A PARTICULAR LIPSCHITZ DOMAIN

We would like to prove the converse of Theorem 3.3. We are able to do this only in the case when Ω is a domain lying between the graphs of two functions, as specified below, with Γ one of the two graphs.

Precisely, we assume (see Fig. 1) that there exist a bounded open set $\Lambda \subset \mathbb{R}^{n-1}$ and two Lipschitz functions $\alpha, \beta : \overline{\Lambda} \to \mathbb{R}$ such that $\alpha(y) < \beta(y)$ for every $y \in \Lambda$, $\alpha(y) = \beta(y)$ for every $y \in \partial\Lambda$ and

$$\begin{aligned} \Omega &= \{(y, x_n) \in \Lambda \times \mathbb{R} : \ \alpha(y) < x_n < \beta(y)\}, \\ \Gamma &= \{(y, x_n) \in \overline{\Lambda} \times \mathbb{R} : \ x_n = \beta(y)\}. \end{aligned} \tag{50}$$

Lemma 4.1. *Let Y denote the set of vector fields $\boldsymbol{w} \in W^{1,p}_\Gamma$ with $\boldsymbol{w} = 0$ in a neighborhood of Γ and such that there exist a neighborhood $\widetilde{\Omega}$ of Ω and an extension $\widetilde{\boldsymbol{w}} \in W^{1,p}(\widetilde{\Omega};\mathbb{R}^n)$ of $\boldsymbol{w}$ with*

$$\int_{\widetilde{\Omega}} W(\mathsf{e}(\widetilde{\boldsymbol{w}}))\,dx < +\infty\,. \tag{51}$$

Assume that Ω and Γ fulfil (50). *Then*

$$\inf_{W^{1,p}_\Gamma} J = \inf_Y J\,. \tag{52}$$

Proof. Clearly we have to prove the inequality

$$\inf_Y J \le \inf_{W^{1,p}_\Gamma} J\,. \tag{53}$$

Let $\boldsymbol{w} \in W^{1,p}_\Gamma$ and consider the extension

$$\widetilde{\boldsymbol{w}}(x) = \begin{cases} \boldsymbol{w}(x) & \text{for } x \in \Omega \\ \boldsymbol{0} & \text{for } x \in H^+ \setminus \Omega \end{cases} \tag{54}$$

where $H^+ = \{(y, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \; x_n > 0\}$ is the upper half-space. Then $\widetilde{\boldsymbol{w}} \in W^{1,p}(\Omega \cup H^+;\mathbb{R}^n)$. For $h \ge 1$ consider the set $\widetilde{\Omega}_h = (\Omega \cup H^+) - \frac{1}{h}\boldsymbol{e}_n$, which is a neighborhood of Ω, and define the function $\widetilde{\boldsymbol{w}}_h : \widetilde{\Omega}_h \to \mathbb{R}^n$ as

$$\widetilde{\boldsymbol{w}}_h(x) = \widetilde{\boldsymbol{w}}\left(y, x_n + \frac{1}{h}\right) \quad \text{where } x = (y, x_n). \tag{55}$$

Then $\widetilde{\boldsymbol{w}}_h \in Y$ and $\widetilde{\boldsymbol{w}}_h \to \boldsymbol{w}$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ as $h \to +\infty$. Moreover, the positivity of W yields

$$\begin{aligned} \int_\Omega W(\mathsf{e}(\widetilde{\boldsymbol{w}}_h))\,dx &= \int_{\Omega\cap\left(\Omega-\frac{1}{h}\boldsymbol{e}_n\right)} W(\mathsf{e}(\widetilde{\boldsymbol{w}}_h))\,dx \\ &= \int_{\left(\Omega+\frac{1}{h}\boldsymbol{e}_n\right)\cap\Omega} W(\mathsf{e}(\boldsymbol{w}))\,dx \le \int_\Omega W(\mathsf{e}(\boldsymbol{w}))\,dx\,. \end{aligned} \tag{56}$$

Keeping into account the continuity of the linear term of J, one concludes the proof. ☐

Finally, we can state and prove the main theorem, where we are able to characterize the minimizer by means of a variational inequality and the Euler–Lagrange equation in the sense of distributions.

Theorem 4.2. *Let Ω satisfy* (50). *Then $\bar{\boldsymbol{u}} \in W^{1,p}_\Gamma$ is the minimizer of J if and only if the following properties hold:*

(i) $W(\mathsf{e}(\bar{\boldsymbol{u}})) \in L^1(\Omega)$;
(ii) $W'(\mathsf{e}(\bar{\boldsymbol{u}})) \in L^1(K;\mathbb{R}^n)$ *for every compact set* $K \subseteq \overline{\Omega}$ *with* $K \cap \Gamma = \varnothing$;
(iii) $W'(\mathsf{e}(\bar{\boldsymbol{u}})) \cdot \mathsf{e}(\bar{\boldsymbol{u}}) \in L^1(\Omega)$ *and*

$$\int_\Omega W'(\mathsf{e}(\bar{\boldsymbol{u}})) \cdot \mathsf{e}(\bar{\boldsymbol{u}})\,dx \le \int_{\overline{\Omega}} \bar{\boldsymbol{u}} \cdot d\boldsymbol{F}; \tag{57}$$

(iv) *the Euler–Lagrange equation*

$$-\operatorname{div}(W'(\mathsf{e}(\bar{\boldsymbol{u}}))) = \boldsymbol{F} \tag{58}$$

holds in the sense of distributions on Ω, that is,

$$\forall \boldsymbol{w} \in C^\infty_\Gamma(\overline{\Omega};\mathbb{R}^n): \quad \int_\Omega W'(\mathsf{e}(\bar{\boldsymbol{u}}))\cdot \mathsf{e}(\boldsymbol{w})\,dx = \int_{\overline{\Omega}} \boldsymbol{w}\cdot d\boldsymbol{F}\,. \tag{59}$$

Proof. Let $\bar{\boldsymbol{u}} \in W^{1,p}_\Gamma$ be the minimizer of J. Then (i) follows from Theorem 2.4, (iii) from Remark 3.2 and (iv) from Theorem 3.3. We prove (ii): by combining (35) with Remark 3.2 one has

$$W'(\mathsf{e}(\bar{\boldsymbol{u}}))\cdot \mathsf{e}(\boldsymbol{u}) \in L^1(\Omega) \tag{60}$$

for every $\boldsymbol{u} \in W^{1,p}_\Gamma$ such that $W(\mathsf{e}(\boldsymbol{u})) \in L^1(\Omega)$. Given a compact set $K \subseteq \overline{\Omega}$ with $K\cap\Gamma = \varnothing$, there exists a Lipschitz map $\theta:\mathbb{R}^n \to [0,1]$ such that $\theta = 1$ on K and θ vanishes on Γ. For every $\mathsf{M}\in \mathrm{Sym}(\mathbb{R}^n)$, we consider the function

$$\boldsymbol{u}(x) := \theta(x)\mathsf{M}x \tag{61}$$

Since $\boldsymbol{u} \in W^{1,p}_\Gamma$ and $W(\mathsf{e}(\boldsymbol{u})) \in L^1(\Omega)$, by (60) it follows that $W'(\mathsf{e}(\bar{\boldsymbol{u}}))\cdot\mathsf{e}(\boldsymbol{u}) \in L^1(K)$. It is easy to see that $\mathsf{e}(\boldsymbol{u}) = \mathsf{M}$ on K, hence

$$\forall \mathsf{M}\in\mathrm{Sym}(\mathbb{R}^n):\ W'(\mathsf{e}(\bar{\boldsymbol{u}}))\cdot\mathsf{M} \in L^1(K) \tag{62}$$

which gives (ii).

Conversely, suppose that (i)–(iv) hold for some $\bar{\boldsymbol{u}} \in W^{1,p}_\Gamma$. By Lemma 4.1 it is enough to prove that $J(\bar{\boldsymbol{u}}) \le J(\boldsymbol{w})$ for every $\boldsymbol{w}\in Y$. Given a mollifier ρ, define

$$\boldsymbol{w}_h(x) := \rho_h * \boldsymbol{w} = \int_{\Omega_h} \rho_h(x-y)\boldsymbol{w}(y)\,dy \tag{63}$$

where $\Omega_h = \{x\in\mathbb{R}^n : \mathrm{dist}(x,\Omega) < 1/h\}$ is the set of points with distance less than $1/h$ from Ω and $\rho_h(x) := \frac{1}{h}\rho(hx)$ is such that $\mathrm{supt}(\rho_h * \boldsymbol{w}) \subset \widetilde{\Omega}$ for h sufficiently large. Then $\boldsymbol{w}_h \in C^\infty_\Gamma(\overline{\Omega};\mathbb{R}^n)$ and

$$\mathsf{e}(\boldsymbol{w}_h)(x) = \int_{\Omega_h} \rho_h(x-y)\mathsf{e}(\boldsymbol{w})(y)\,dy. \tag{64}$$

By the convexity of W and Jensen's inequality one gets

$$W\big(\mathsf{e}(\boldsymbol{w}_h)(x)\big) \le \int_{\Omega_h} \rho_h(x-y) W\big(\mathsf{e}(\boldsymbol{w})(y)\big)\,dy \tag{65}$$

and, integrating on Ω and applying Fubini's Theorem,

$$\begin{aligned}\int_\Omega W\big(\mathsf{e}(\boldsymbol{w}_h)(x)\big)\,dx &\le \int_{\Omega_h} W\big(\mathsf{e}(\boldsymbol{w})(y)\big)\int_\Omega \rho_h(x-y)\,dx\,dy\\ &\le \int_{\Omega_h} W\big(\mathsf{e}(\boldsymbol{w})(y)\big)\,dy,\end{aligned} \tag{66}$$

where we used the fact that $W \ge 0$. By Fatou's Lemma we get

$$\limsup_h \int_\Omega W\big(\mathsf{e}(\boldsymbol{w}_h)(x)\big)\,dx \le \int_\Omega W\big(\mathsf{e}(\boldsymbol{w})(x)\big)\,dx, \tag{67}$$

hence

$$\inf_Y J = \inf_{C^\infty_\Gamma(\overline{\Omega};\mathbb{R}^n)} J. \tag{68}$$

Applying Lemma 4.1 it follows that

$$\inf_{W^{1,p}_\Gamma} J = \inf_{C^\infty_\Gamma(\overline{\Omega};\mathbb{R}^n)} J. \tag{69}$$

Since by combining (iii) and (iv) we get the variational inequality

$$\int_{\Omega} W'(\mathsf{e}(\bar{\boldsymbol{u}})) \cdot (\mathsf{e}(\boldsymbol{w}) - \mathsf{e}(\bar{\boldsymbol{u}}))\, dx \geq \int_{\overline{\Omega}} (\boldsymbol{w} - \bar{\boldsymbol{u}}) \cdot d\boldsymbol{F} \tag{70}$$

for every $\boldsymbol{w} \in C^{\infty}_{\Gamma}(\overline{\Omega}; \mathbb{R}^n)$, by using the same argument as in the proof of Theorem 3.1 one can prove that $\bar{\boldsymbol{u}}$ is the minimizer of J on $W^{1,p}_{\Gamma}$. □

5. Conclusion

Due to the peculiar behavior of the stress-strain relation at large deformations, many soft tissues are modeled by using an exponential term in the constitutive equation, as pointed out by Fung [7]. In this paper we studied the mathematical problem of finding equilibrium solutions in the case of hyperelastic materials whose energy density satisfies the growth condition (22), so that the energy has to rapidly increase for large deformations.

After a severe simplification of the model, that is, replacing the Green strain tensor E with $\mathsf{e}(\boldsymbol{u})$, the symmetric part of the gradient of displacement, we studied the connection between the minimizer of the elastic energy and the solution of the associated Euler–Lagrange equation. We were able to prove, under very general boundary conditions and external data, that the minimizer has to satisfy the distributional version of the Euler–Lagrange equation. With an additional assumption on the shape of the elastic body and the boundary conditions (Section 4), we proved that the solution of the Euler–Lagrange equation, together with some boundedness assumptions, is the unique minimizer of the energy.

Hence, the two typical approaches to the elastic equilibrium problem, that is, solving the steady elastic differential equation or finding a minimum for the elastic energy, are equivalent, at least under suitable assumptions.

Our results could be improved in many ways. From the mathematical viewpoint, we would like to avoid the assumption (50), which is merely technical, in the proof of Theorem 4.2. From a modeler's viewpoint, a major achievement would be to prove our results for an elastic energy involving the Green strain tensor E and not the symmetrized gradient $\mathsf{e}(\boldsymbol{u})$, that is, to bypass (6). Finally, the incompressibility constraint, which is quite customary in Biomechanics, could be taken into account, as well as some viscoelastic and other inelastic effects.

Acknowledgement

The authors wish to thank Elisabetta Nesci, who studied a preliminary version of the problem, and Marco Degiovanni for many insightful suggestions. Many thanks also to the anonymous reviewer for giving very useful advice.

This work has been partially supported by National Group of Mathematical Physics (GNFM-INdAM).

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