

THE CAUCHY STRESS THEOREM FOR BODIES WITH FINITE PERIMETER

ALFREDO MARZOCCHI AND ALESSANDRO MUSESTI

ABSTRACT. The Cauchy Stress Theorem is proved for bodies which has finite perimeter, without extra topological assumptions, and the notions of Cauchy flux and Cauchy interaction are extended to this case. Also bodies with an empty interior can be considered.

1. INTRODUCTION

The concept of subbody of a continuous body is, in the classical idealization, not very different from the concept of the global body itself. For instance, it is very common to regard every subbody as a whole body, the only difference being the interpretation of the forces on its surface. The development of rational continuum mechanics in this direction has tried to generalize the concept of subbody: a very wide class of sets which can be used with this sense has been introduced and studied by several authors in the last two decades [BF, Z, GWZ]. A remarkable result of this approach was the proof of the Cauchy Stress Theorem under weak conditions, provided that the density tensor field is bounded.

In the same period, the case of unbounded densities was considered in [Š1] and generalized in [Š2, DMM], by means of a distributional approach (see also [MM2] for an application to the thermodynamical context). In the latter papers the body is treated under assumptions which are different from those required for subbodies: in [Š2] the topological boundary of the body B is volume-negligible, while in [DMM] only subbodies M which satisfy the condition $\text{cl } M \subseteq \text{int } B$ are considered. This situation arises because the distributional interpretation of a balance law requires that one deals with an open body.

When no topological assumptions on the whole body are made, one may wonder if it is still possible to obtain the Cauchy Stress Theorem, *i.e.* the existence of the stress density and its linear dependence on the normal.

In this note we answer this question in the positive. Of course, we do stipulate some measure-theoretical conditions, in particular, that the body has finite perimeter: B cannot be, for instance, a fractal set in the broad sense. Our result applies also when the body has an empty interior (see Section 4 for an example), a case encountered in the theories of micro-structures and of mixtures. Notice that in these situations the classical tetrahedron argument cannot even be set up.

Finally, we introduce in this framework the concept of Cauchy interaction and prove, in the spirit of [MM1], a representation theorem also for Cauchy interactions on such bodies.

The research of the authors was partially supported by M.U.R.S.T. COFIN2000 project "Modelli matematici per la scienza dei materiali" and Gruppo Nazionale per la Fisica Matematica.

2. NOTATION

For $n \geq 1$, \mathcal{L}^n will denote n -dimensional outer Lebesgue measure, and \mathcal{H}^k k -dimensional outer Hausdorff measure on \mathbb{R}^n . Given a Borel subset $E \subseteq \mathbb{R}^n$, we denote by $\mathfrak{B}(E)$ the collection of all Borel subsets of E . Moreover, $E \Delta F$ will denote the set $(E \setminus F) \cup (F \setminus E)$.

Consider a set $M \subseteq \mathbb{R}^n$. The topological boundary, closure and interior of M will be denoted by $\text{bd } M$, $\text{cl } M$ and $\text{int } M$, respectively. Denoting by $B_r(x)$ the open ball with radius r centered at x , we introduce *the measure-theoretic interior* of M

$$M_* = \left\{ x \in \mathbb{R}^n : \lim_{r \rightarrow 0^+} \frac{\mathcal{L}^n(B_r(x) \setminus M)}{\mathcal{L}^n(B_r(x))} = 0 \right\},$$

the measure-theoretic boundary of M

$$\partial_* M = \mathbb{R}^n \setminus (M_* \cup (\mathbb{R}^n \setminus M)_*)$$

and *the measure-theoretic closure* of M

$$M^* = M \cup \partial_* M.$$

The following proposition is a standard matter of measure theory.

Proposition 2.1. *Let $M \subseteq \mathbb{R}^n$. Then the following properties hold:*

- (a) $M_*, M^*, \partial_* M \in \mathfrak{B}(\mathbb{R}^n)$;
- (b) $\text{int } M \subseteq M_* \subseteq M^* \subseteq \text{cl } M$;
- (c) M is \mathcal{L}^n -measurable if and only if $\mathcal{L}^n(M \Delta M_*) = 0$.

Definition 2.2. We say that $M \subseteq \mathbb{R}^n$ is *normalized*, if $M_* = M$.

Let \mathcal{V} be the linear space associated to \mathbb{R}^n ; we now introduce the concept of *outer normal* to the measure-theoretic boundary of a set. Let $M \subseteq \mathbb{R}^n$ and $x \in \partial_* M$. We denote by $\mathbf{n}^M(x) \in \mathcal{V}$ a unit vector such that, as $r \rightarrow 0^+$,

$$\begin{aligned} \mathcal{L}^n(\{\xi \in B_r(x) \cap M : (\xi - x) \cdot \mathbf{n}^M(x) > 0\}) / r^n &\longrightarrow 0, \\ \mathcal{L}^n(\{\xi \in B_r(x) \setminus M : (\xi - x) \cdot \mathbf{n}^M(x) < 0\}) / r^n &\longrightarrow 0. \end{aligned}$$

No more than one such vector can exist. If the limits do not both obtain, we set $\mathbf{n}^M(x) = \mathbf{0}$. The bounded map $\mathbf{n}^M : \partial_* M \rightarrow \mathcal{V}$ is called *the unit outer normal* to M . It turns out that \mathbf{n}^M is a Borel map, that is, $(\mathbf{n}^M)^{-1}(A) \in \mathfrak{B}(\partial_* M)$ for any open subset $A \subseteq \mathcal{V}$.

Definition 2.3. Let $M \subseteq \mathbb{R}^n$. We say that M is a *set with finite perimeter*, if $\mathcal{H}^{n-1}(\partial_* M) < +\infty$.

Now we turn to more specific definitions.

Definition 2.4. We call *body* a set $B \subseteq \mathbb{R}^n$ which is bounded, normalized, with finite perimeter. We denote by $\mathcal{M}(B)$ the family of normalized subsets of B with finite perimeter. Moreover, we set

$$\begin{aligned} \mathcal{N}(B) &= \{C \subseteq \mathbb{R}^n : C \in \mathcal{M}(B) \text{ or } (\mathbb{R}^n \setminus C)_* \in \mathcal{M}(B)\}, \\ \mathfrak{D}(B) &= \{(A, C) \in \mathcal{M}(B) \times \mathcal{N}(B) : A \cap C = \emptyset\}. \end{aligned}$$

Our choice of sets with finite perimeter is motivated by the fact that the unit outer normal exists \mathcal{H}^{n-1} -a.e. on the measure-theoretic boundary and the Divergence Theorem holds in a weak sense (see [F, Theorem 4.5.6]).

In order to define a flux as a set function, we need the concept of *material surface*.

Definition 2.5. A *material surface* in the body B is a pair $S = (\widehat{S}, \mathbf{n}_S)$, where \widehat{S} is a Borel subset of B and $\mathbf{n}_S : \widehat{S} \rightarrow \mathcal{V}$ is a Borel map such that there exists $M \in \mathcal{M}(B)$ with $\widehat{S} \subseteq \partial_* M$ and $\mathbf{n}_S = \mathbf{n}^M|_{\widehat{S}}$. In this case, we say that S is *subordinate* to M . We denote by $\mathcal{S}(B)$ the collection of material surfaces in the body B .

We call \mathbf{n}_S the *normal* to the surface S . Two material surfaces S and T are said to be *compatible*, if they both are subordinate to the same M .

Let now Ω be a Borel subset of \mathbb{R}^n . We denote by $\mathfrak{M}(\Omega)$ the collection of Borel measures $\mu : \mathfrak{B}(\Omega) \rightarrow [0, +\infty]$ with $\mu(\Omega) < +\infty$ and by $\mathcal{L}_+^1(B)$ the set of Borel functions $h : \Omega \rightarrow [0, +\infty]$ with $\int_\Omega h d\mathcal{L}^n < +\infty$. The following definition extends the notion of “almost all” already introduced in [Š2] and [DMM].

Definition 2.6. Given $h \in \mathcal{L}_+^1(B)$ and $\nu \in \mathfrak{M}(B)$, we set

$$\begin{aligned} \mathcal{M}(B)_{h\nu} &= \left\{ A \in \mathcal{M}(B) : \int_{B \cap \partial_* A} h d\mathcal{H}^{n-1} < +\infty, \nu(B \cap \partial_* A) = 0 \right\}, \\ \mathcal{N}(B)_{h\nu} &= \{ C \in \mathcal{N}(B) : C \in \mathcal{M}(B)_{h\nu} \text{ or } (\mathbb{R}^n \setminus C)_* \in \mathcal{M}(B)_{h\nu} \}, \\ \mathfrak{D}(B)_{h\nu} &= \mathfrak{D}(B) \cap (\mathcal{M}(B)_{h\nu} \times \mathcal{N}(B)_{h\nu}), \\ \mathcal{S}(B)_{h\nu} &= \{ S \in \mathcal{S}(B) : S \text{ is subordinate to some } A \in \mathcal{M}(B)_{h\nu} \}. \end{aligned}$$

We will say that a property π *holds on almost all of* $\mathcal{M}(B)$, if there are $h \in \mathcal{L}_+^1(B)$ and $\nu \in \mathfrak{M}(B)$ such that π holds on $\mathcal{M}(B)_{h\nu}$, and in a similar fashion for $\mathcal{N}(B)$, $\mathfrak{D}(B)$ and $\mathcal{S}(B)$.

Definition 2.7. Let X be a vector space and $\mathcal{P} \subseteq \mathcal{M}(B)$. We say that a function $F : \mathcal{P} \rightarrow X$ is *additive*, if for every $M, N \in \mathcal{P}$ with $(M \cup N)_* \in \mathcal{P}$ and $M \cap N = \emptyset$ one has

$$F((M \cup N)_*) = F(M) + F(N).$$

Let now $\mathcal{P} \subseteq \mathfrak{D}(B)$. A function $F : \mathcal{P} \rightarrow X$ is *biadditive*, if the functions $F(\cdot, C)$ and $F(A, \cdot)$ are additive.

Finally we come to the main definitions. For simplicity, we consider scalar-valued fluxes and interactions; the extension to the vectorial case is trivial.

Definition 2.8. Let \mathcal{P} be a set containing almost all of $\mathcal{S}(B)$ and consider a function $Q : \mathcal{P} \rightarrow \mathbb{R}$ such that:

- (a) if S, T are compatible and disjoint and $S \cup T \in \mathcal{P}$, then

$$Q(S \cup T) = Q(S) + Q(T);$$

- (b) there exists $h \in \mathcal{L}_+^1(B)$ with

$$|Q(S)| \leq \int_S h d\mathcal{H}^{n-1}$$

for almost every $S \in \mathcal{S}(B)$;

(c) there exists $\nu \in \mathfrak{M}(B)$ with

$$|Q(B \cap \partial_* A)| \leq \nu(A)$$

for almost every $A \in \mathcal{M}(B)$.

Then Q is said to be a *balanced Cauchy flux* on B .

Definition 2.9. Consider a set $\mathcal{P} \subseteq \mathfrak{D}(B)$ containing almost all of $\mathfrak{D}(B)$ and a function $I : \mathcal{P} \rightarrow \mathbb{R}$ such that:

- (a) I is biadditive;
 (b) there exist $h \in \mathcal{L}_+^1(B)$, $\eta \in \mathfrak{M}(B \times B)$ and $\eta_e \in \mathfrak{M}(B)$ with

$$|I(A, C)| \leq \begin{cases} \int_{B \cap \partial_* A \cap \partial_* C} h d\mathcal{H}^{n-1} + \eta(A \times C) & \text{if } C \subseteq B, \\ \int_{B \cap \partial_* A \cap \partial_* C} h d\mathcal{H}^{n-1} + \eta(A \times (C \cap B)) + \eta_e(A) & \text{otherwise,} \end{cases}$$

on almost all of $\mathfrak{D}(B)$;

(c) there exists $\nu \in \mathfrak{M}(B)$ with

$$\partial_* A \subseteq \partial_* C \Rightarrow |I(A, C)| \leq \nu(A)$$

on almost all of $\mathfrak{D}(B)$.

Then I is said to be a *balanced Cauchy interaction* on B .

In [DMM] and [MM1], a subbody was defined to be a set $M \in \mathcal{M}(B)$ such that $\text{cl } M \subseteq \text{int } B$. Clearly, in this way the topological boundary of a subbody cannot meet $\partial_* B$. In the present work, we set

$$\mathcal{M}^\circ(B) = \{M \in \mathcal{M}(B) : \text{cl } M \subseteq \text{int } B\};$$

analogous definitions yield $\mathcal{N}^\circ(B)$, $\mathfrak{D}^\circ(B)$ and $\mathcal{S}^\circ(B)$. In the above papers, the concept of *almost all* was restricted to the above classes by introducing $\mathcal{M}^\circ(B)_{h\nu}$, $\mathfrak{D}^\circ(B)_{h\nu}$, $\mathcal{S}^\circ(B)_{h\nu}$; corresponding notions of Cauchy flux and Cauchy interaction were given. We refer here to those notions as *inner Cauchy flux* and *inner Cauchy interaction*, respectively. Two important results about inner Cauchy fluxes and interactions are the integral representation theorems [DMM, Theorem 7.1] and [MM1, Theorem 7.4].

Another useful notion is the class of almost all n -intervals in the interior of B , a notion made precise by the following definition.

Definition 2.10. A *grid* G is an ordered triple

$$G = (x_0, (e_1, \dots, e_n), \widehat{G}),$$

where $x_0 \in \mathbb{R}^n$, (e_1, \dots, e_n) is a positively oriented orthonormal basis in \mathbb{R}^n and \widehat{G} is a Borel subset of \mathbb{R} . If G_1, G_2 are two grids, we write $G_1 \subseteq G_2$ if $\widehat{G}_1 \subseteq \widehat{G}_2$ and they share the point x_0 and the list (e_1, \dots, e_n) . A grid G is said to be *full*, if $\mathcal{L}^1(\mathbb{R} \setminus \widehat{G}) = 0$.

Let G be a grid; a set $I \subseteq \mathbb{R}^n$ is said to be a *G -interval*, if

$$I = \left\{ x \in \mathbb{R}^n : a^{(j)} < (x - x_0) \cdot e_j < b^{(j)} \quad \forall j = 1, \dots, n \right\}$$

for some $a^{(1)}, b^{(1)}, \dots, a^{(n)}, b^{(n)} \in \widehat{G}$. We set

$$\mathcal{J}^\circ(B)_G = \{I : I \text{ is a } G\text{-interval with } \text{cl } I \subseteq \text{int } B\}.$$

We also denote by $\mathcal{S}^\circ(B)_G$ the family of all the oriented surfaces $S = (\widehat{S}, \mathbf{n}_S)$ such that $\text{cl } \widehat{S} \subseteq \text{int } B$ and, for some $1 \leq j \leq n$,

$$\widehat{S} = \left\{ x \in \mathbb{R}^n : (x - x_0) \cdot e_j = s, a^{(i)} < (x - x_0) \cdot e_i < b^{(i)} \quad \forall i \neq j \right\}, \quad \mathbf{n}_S = e_j,$$

where $s, a^{(1)}, b^{(1)}, \dots, a^{(j-1)}, b^{(j-1)}, a^{(j+1)}, b^{(j+1)}, \dots, a^{(n)}, b^{(n)} \in \widehat{G}$. This means that the elements of $\mathcal{S}^\circ(B)_G$ are open sides of G -intervals, equipped with the induced normal.

The next proposition, together with [DMM, Theorem 7.1], says that there exists an integral representation of an inner Cauchy flux on $\mathcal{S}^\circ(B)_G$ for some full grid G .

Proposition 2.11. *Let (e_1, \dots, e_n) be a positively oriented orthonormal basis in \mathbb{R}^n and $x_0 \in \mathbb{R}^n$. Then for every $h \in \mathcal{L}_+^1(\text{int } B)$ and $\nu \in \mathfrak{M}(\text{int } B)$ there exists a full grid $G = (x_0, (e_1, \dots, e_n), \widehat{G})$ such that $\mathcal{S}^\circ(B)_G \subseteq \mathcal{S}^\circ(B)_{h\nu}$.*

Proof. See [DMM, Proposition 4.5]. □

3. THE STRESS THEOREM

Let us first prove a useful “localization” property of the density of an inner Cauchy flux: if it happens that the flux concentrates around a subbody M , *i.e.* only the parts which meet M can have a non-zero contribution, then the density vanishes almost everywhere outside M .

Lemma 3.1. *Let Q° be an inner Cauchy flux on B such that there is an $M \in \mathcal{M}^\circ(B)$ with*

$$(3.1) \quad Q^\circ(S) = Q^\circ(S \cap M)$$

on almost all of $\mathcal{S}^\circ(B)$. Let $\mathbf{q} \in L^1(B; \mathcal{V})$ be the density associated with Q° , in accordance with [DMM, Theorem 7.1]. Then $\mathbf{q}(x) = \mathbf{0}$ for a.e. $x \in B \setminus M$.

Proof. Let $G = (x_0, (e_1, \dots, e_n), \widehat{G})$ be a full grid such that the integral representation of Q° holds on $\mathcal{S}^\circ(B)_G$. For any

$$I = \left\{ x \in \mathbb{R}^n : a^{(i)} < (x - x_0) \cdot e_i < b^{(i)} \quad \forall i = 1, \dots, n \right\} \in \mathcal{J}_G^\circ$$

and any $j = 1, \dots, n$, by (3.1) and Fubini’s Theorem one has

$$\begin{aligned} \int_{I \setminus M} \mathbf{q}^{(j)} d\mathcal{L}^n &= \int_{a^{(j)}}^{b^{(j)}} \left[\int_{\sigma_{j,s}(I) \setminus M} \mathbf{q}(x) \cdot e_j d\mathcal{H}^{n-1}(x) \right] d\mathcal{L}^1(s) \\ &= \int_{a^{(j)}}^{b^{(j)}} Q^\circ(\sigma_{j,s}(I) \setminus M) d\mathcal{H}^{n-1}(x) = 0, \end{aligned}$$

where

$$\sigma_{j,s}(I) = \{x \in I : (x - x_0) \cdot e_j = s\}.$$

Take now $x \notin M$ such that x is a Lebesgue point for the functions \mathbf{q} and $\chi_M \mathbf{q}$, where χ_M denotes the characteristic function of M . Consider a sequence of cubes $(J_k) \subseteq \mathcal{J}_G^\circ$ with $x \in J_k$ and $\text{diam } J_k \rightarrow 0$ as $k \rightarrow +\infty$. It follows that

$$\mathbf{q}(x) = \lim_{k \rightarrow +\infty} \frac{\int_{J_k} \mathbf{q} d\mathcal{L}^n}{\mathcal{L}^n(J_k)} = \lim_{k \rightarrow +\infty} \frac{\int_{J_k \cap M} \mathbf{q} d\mathcal{L}^n}{\mathcal{L}^n(J_k)} = \chi_M(x) \mathbf{q}(x) = \mathbf{0}$$

and the proof is complete. □

Now we state our main results.

Theorem 3.2. *Let Q be a balanced Cauchy flux on B . Then there exists an essentially unique function $\mathbf{q} \in L^1(\mathbb{R}^n; \mathcal{V})$ with divergence measure such that $\mathbf{q} = \mathbf{0}$ a.e. in $\mathbb{R}^n \setminus B$, the total variation of $\operatorname{div} \mathbf{q}$ is bounded on \mathbb{R}^n , and*

$$Q(S) = \int_S \mathbf{q} \cdot \mathbf{n}_S d\mathcal{H}^{n-1}$$

on almost all of $\mathcal{S}(B)$.

Proof. Let h and ν be such that the domain of Q contains $\mathcal{S}(B)_{h\nu}$ and Definition 2.8 holds on $\mathcal{S}(B)_{h\nu}$. Given $R > 0$ such that $\operatorname{cl} B \subseteq B_R(0)$, we set $B_R = B_R(0)$ and consider the families $\mathcal{M}^\circ(B_R)$, $\mathcal{N}^\circ(B_R)$, $\mathcal{D}^\circ(B_R)$ and $\mathcal{S}^\circ(B_R)$. Then consider the function $\widehat{h} \in \mathcal{L}_+^1(B_R)$ which extends h to zero outside B and the measure $\widehat{\nu} \in \mathfrak{M}(B_R)$ defined by $\widehat{\nu}(E) = \nu(E \cap B)$. It can be verified that the function $Q^\circ : \mathcal{S}^\circ(B_R)_{\widehat{h}\widehat{\nu}} \rightarrow \mathbb{R}$ defined by

$$Q^\circ(S) = Q(S \cap B)$$

is an inner Cauchy flux on B_R . Then one can apply [DMM, Theorem 7.1], finding a vector field $\widehat{\mathbf{q}} \in L^1(B_R; \mathcal{V})$ with divergence measure such that

$$Q^\circ(S) = \int_S \widehat{\mathbf{q}} \cdot \mathbf{n}_S d\mathcal{H}^{n-1}$$

on almost all of $\mathcal{S}^\circ(B_R)$. In particular, one has

$$Q(S) = \int_S \widehat{\mathbf{q}} \cdot \mathbf{n}_S d\mathcal{H}^{n-1}$$

for almost every $S \in \mathcal{S}(B)$. Moreover, taking into account Lemma 3.1, one has that $\widehat{\mathbf{q}} = \mathbf{0}$ for a.e. $x \in B_R \setminus B$. If \mathbf{q} is the extension of $\widehat{\mathbf{q}}$ to \mathbb{R}^n with value $\mathbf{0}$ outside B_R , then $\mathbf{q} \in L^1(\mathbb{R}^n; \mathcal{V})$, \mathbf{q} has divergence measure, the total variation of $\operatorname{div} \mathbf{q}$ is bounded on \mathbb{R}^n and $\mathbf{q} = \mathbf{0}$ a.e. in $\mathbb{R}^n \setminus B$. Finally, such a \mathbf{q} is unique \mathcal{L}^n -a.e. by [DMM, Corollary 5.7]. \square

Theorem 3.3. *Let I be a balanced Cauchy interaction. Then there exist $\mu \in \mathfrak{M}(B \times B)$, $\mu_e \in \mathfrak{M}(B)$, two Borel functions $b : B \times B \rightarrow \mathbb{R}$, $b_e : B \rightarrow \mathbb{R}$ and a field $\mathbf{q} \in L^1(\mathbb{R}^n; \mathcal{V})$ with divergence measure, such that $\mathbf{q} = \mathbf{0}$ a.e. in $\mathbb{R}^n \setminus B$, the total variation of $\operatorname{div} \mathbf{q}$ is bounded on \mathbb{R}^n and the formula*

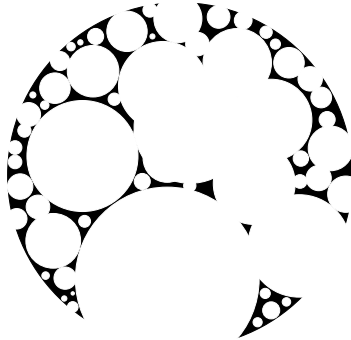
$$I(A, C) = \begin{cases} \int_{A \times C} b d\mu + \int_{\partial_* A \cap \partial_* C} \mathbf{q} \cdot \mathbf{n}^A d\mathcal{H}^{n-1} & \text{if } C \subseteq B, \\ \int_{A \times (C \cap B)} b d\mu + \int_A b_e d\mu_e + \int_{\partial_* A \cap \partial_* C} \mathbf{q} \cdot \mathbf{n}^A d\mathcal{H}^{n-1} & \text{otherwise,} \end{cases}$$

holds almost everywhere in $\mathfrak{D}(B)$.

Proof. Following the ideas in the proof of the previous theorem, we define a function $I^\circ : \mathfrak{D}^\circ(B_R)_{\widehat{h}\widehat{\nu}} \rightarrow \mathbb{R}$ as

$$I^\circ(A, C) = \begin{cases} I(A \cap B, C \cap B) & \text{if } C \subseteq B_R, \\ I(A \cap B, C \cap B) + I(A \cap B, (\mathbb{R}^n \setminus B)_*) & \text{otherwise.} \end{cases}$$

It can be verified that I° is an inner Cauchy interaction on B_R . Then it is enough to apply [MM1, Theorem 7.4], which gives the integral representation for I° . \square

FIGURE 1. The construction of B .

4. AN EXAMPLE

In this section we construct a set $B \subseteq \mathbb{R}^n$ which is bounded, normalized, with finite perimeter, and such that $\text{int } B = \emptyset$ and $\mathcal{L}^n(B) \geq \alpha > 0$.

Take $x \in \mathbb{R}^n$ and $r > 0$ and consider the set $C = \{y \in \mathbb{R}^n : |x - y| \leq r\}$. Let $\{x_k : k \in \mathbb{N}\} \subseteq C$ an enumeration of all points in C with rational components. Put $\omega_n = \mathcal{L}^n(B_1(0))$ and consider, for $0 < \alpha < \omega_n r^n$, the decreasing sequence

$$r_k = \sqrt[n]{\frac{\omega_n r^n - \alpha}{2\omega_n}} 2^{-k/n}$$

Now set

$$D = C \setminus \bigcup_{k \in \mathbb{N}} B_{r_k}(x_k), \quad B = D_*.$$

Then:

- (a) B is obviously bounded and normalized.
- (b) $\text{int } B = \emptyset$: the set D does not contain any rational point, hence $\text{int } D = \emptyset$; moreover, D is a closed subset of \mathbb{R}^n , thus $B \subseteq D$ by virtue of (b) of Proposition 2.1.
- (c) $\mathcal{L}^n(B) \geq \alpha$, since, by (c) of Proposition 2.1, $\mathcal{L}^n(B) = \mathcal{L}^n(D)$ and

$$\mathcal{L}^n(D) \geq \mathcal{L}^n(C) - \mathcal{L}^n\left(\bigcup_{k \in \mathbb{N}} B_{r_k}(x_k)\right) \geq \omega_n r^n - \omega_n \sum_{k \in \mathbb{N}} r_k^n = \alpha.$$

- (d) B has finite perimeter; indeed $\partial_* B = \partial_* D$ and

$$\begin{aligned} \mathcal{H}^{n-1}(\partial_* D) &\leq \mathcal{H}^{n-1}(\text{bd } C) + \sum_{k \in \mathbb{N}} \mathcal{H}^{n-1}(\text{bd } B_{r_k}(x_k)) \\ &= \left(r^{n-1} + \left(\frac{\omega_n r^n - \alpha}{2\omega_n} \right)^{(n-1)/n} \sum_{k \in \mathbb{N}} 2^{-k(n-1)/n} \right) \mathcal{H}^{n-1}(\text{bd } B_1(0)). \end{aligned}$$

Acknowledgments. The authors thank M. Degiovanni for very helpful discussions and W. O. Williams for having improved the language.

REFERENCES

- [BF] C. BANFI, M. FABRIZIO, Sul concetto di sottocorpo nella meccanica dei continui, *Rend. Acc. Naz. Lincei* **LXVI** (1979), 136–142.
- [DMM] M. DEGIOVANNI, A. MARZOCCHI & A. MUSESTI, Cauchy fluxes associated with tensor fields having divergence measure, *Arch. Rational Mech. Anal.* **147** (1999) 197–223.
- [F] H. FEDERER, *Geometric Measure Theory*, Springer-Verlag, Berlin, 1969.
- [GWZ] M. E. GURTIN, W. O. WILLIAMS & W. P. ZIEMER, Geometric measure theory and the axioms of continuum thermodynamics, *Arch. Rational Mech. Anal.* **92** (1986) 1–22.
- [MM1] A. MARZOCCHI & A. MUSESTI, Decomposition and integral representation of Cauchy interactions associated with measures, *Cont. Mech. Thermodyn.* **13** (2001) 149–169.
- [MM2] A. MARZOCCHI, A. MUSESTI, On the measure-theoretic foundations of the Second Law of Thermodynamics, *Math. Models Methods Appl. Sci.* in press.
- [Š1] M. ŠILHAVÝ, The existence of the flux vector and the divergence theorem for general Cauchy fluxes, *Arch. Ration. Mech. Anal.* **90** (1985), 195–211.
- [Š2] M. ŠILHAVÝ, Cauchy’s stress theorem and tensor fields with divergences in L^p , *Arch. Rational Mech. Anal.* **116** (1991), 223–255.
- [Z] W. P. ZIEMER, Cauchy flux and sets of finite perimeter, *Arch. Ration. Mech. Anal.* **84** (1983), 189–201.

ALFREDO MARZOCCHI, DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI BRESCIA,
VIA VALOTTI 9, I-25133 BRESCIA, ITALY

E-mail address: amarzocc@ing.unibs.it

ALESSANDRO MUSESTI, DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI BRESCIA,
VIA VALOTTI 9, I-25133 BRESCIA, ITALY

E-mail address: a.musesti@dmf.unicatt.it