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Edge-force densities and second-order powers

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Abstract. By means of balanced virtual powers, an axiomatic approach is developed, in the spirit of Noll, to second-gradient continua. The measure-theoretical formulation allows a considerable simplification since the existence of an edge stress density is regarded as a special case of a surface stress which is a singular measure with respect to the area. To prove our results, we introduce a particular class of subbodies, namely the sets with curvature measure.

1. Introduction and notations

The description of the stresses occurring in a continuous medium is usually given by means of the concept of material surface. For instance, Gurtin and Martins [9] introduced the notion of Cauchy flux through a material surface and were able to prove Cauchy's Stress Theorem dropping the continuity assumption of the surface force density. This led to various generalizations [16,14,10,15,2], where the regularity of the force density was weakened more and more.

The approach by material surfaces left open the problem of the above description whenever the force concentrates on low-dimensional sets, such as edges or vertices. In this direction, Noll and Virga [13] and Forte and Vianello [6] attempted to construct a theory including regular one-dimensional subsets of $\mathbb{R}^3$ and a corresponding edge interaction. However, some of their assumptions do not have strict physical evidence, and the generalization to less regular subsets seems to be quite difficult.

In 1973, Germain [7,8] introduced an alternative approach to the whole matter by stating D'Alembert's principle for continuous media. The main notion in this case is the power expended on velocity fields, a quantity which has a sense on whole subbodies and does not deal with pieces of their boundaries. In this spirit, Dell'Isola and Seppecher [3] used the balance of power to prove some of the assumptions in [13], choosing a precise dependence of the power through the derivatives of the velocity field, and found a more general form of the Cauchy stress tensor, involving a third-order tensor field called *hyperstress*.

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However, in all the papers [13,6,3] considering second-order continua, the densities of force are presented as *primitive* notions, in contrast to the set-theoretic approach of [9,10].

On the other hand, two of us showed in [12] that a power of order one, i.e. a power which depends only on the first derivatives of the velocity, is sufficient to recover the Cauchy stress theorem with a stress tensor having divergence measure. Therefore, edge contact forces cannot be described in this theory.

In this paper we study the properties of powers of order k (see Definition 8), defined as functions of two variables: a subbody and a velocity field. We define this concept for very special subbodies like n-intervals and provide an extension theorem, as in previous papers [2,11,12], for more general classes of subbodies. This is closely related to the approach to higher-order powers suggested by Di Carlo and Tatone [4] and has the advantage of avoiding the disputed question about the suitable class of subbodies.

First, we give an integral representation of the power with respect to Radon measures. At this level, force densities appear as *derived* quantities. Then, in the case $k = 2$, by means of a weak balance we are able to represent the power as an integral where stresses and hyperstresses come out in a natural way. In doing this, we introduce a special subclass of the family of sets with finite perimeter, called *sets with curvature measure*, where such an integral representation can be written by introducing suitable singular measures. In this way, edge contributions are treated as surface contributions involving a curvature that is singular with respect to the area.

Finally, let us observe that, with respect to Cauchy fluxes, the approach through the powers allows natural extensions to manifolds (see e.g. [12]). Moreover, the presence of the smooth velocity field as a test simplifies the treatment of sets and stresses with lack of regularity.

In the sequel, $\mathcal{L}^n$ will denote the n-dimensional Lebesgue outer measure and $\mathcal{H}^k$ the k-dimensional Hausdorff outer measure on $\mathbb{R}^n$. Given a Borel subset $\Omega \subseteq \mathbb{R}^n$, we denote by $\mathfrak{B}(\Omega)$ the collection of all Borel subsets of Ω.

The topological closure, interior and boundary of $E \subseteq \mathbb{R}^n$ will be denoted as usual by $\mathrm{cl}\, E$, $\mathrm{int}\, E$ and $\mathrm{bd}\, E$ respectively. Denoting by $B_r(x)$ the open ball with radius r centred at x, we introduce the *measure-theoretic interior* of E:

$$E_* = \left\{ x \in \mathbb{R}^n : \lim_{r \to 0^+} \left(r^{-n} \mathcal{L}^n (B_r(x) \setminus E) \right) = 0 \right\}$$

and *the measure-theoretic boundary* of E:

$$\partial_* E = \mathbb{R}^n \setminus \left(E_* \cup (\mathbb{R}^n \setminus E)_* \right),$$

both of which are Borel subsets of $\mathbb{R}^n$. We say that $E \subseteq \mathbb{R}^n$ is *normalized* if $E_* = E$.

Now let Ω be a Borel subset of $\mathbb{R}^n$. We denote by $\mathfrak{M}(\Omega)$ the set of Borel measures $\mu : \mathfrak{B}(\Omega) \to [0, +\infty]$ finite on compact subsets of Ω and by $\mathcal{L}^p_{loc,+}(\Omega)$,

$p \in [1, +\infty]$ the set of Borel functions $h : \Omega \to [0, +\infty]$ such that

$$\int_K h^p \, d\mathcal{L}^n < +\infty \quad (p < +\infty), \qquad \operatorname{ess\,sup}\{h(x) : x \in K\} < +\infty \quad (p = +\infty)$$

for every compact subset $K \subseteq \Omega$.

Definition 1. *Let $\eta, \lambda \in \mathfrak{M}(\Omega)$. We write $\eta \ll \lambda$ if*

$$\forall E \in \mathfrak{B}(\Omega) : \lambda(E) = 0 \Rightarrow \eta(E) = 0 .$$

We write $\eta \perp \lambda$ if there exists $S \in \mathfrak{B}(\Omega)$ such that $\lambda(S) = 0$ and

$$\forall E \in \mathfrak{B}(\Omega) : \eta(E) = \eta(E \cap S) .$$

We denote by $\eta = \eta^a + \eta^s$ the Lebesgue decomposition of η with respect to λ, *that is the unique decomposition of η such that $\eta^a \ll \lambda$ and $\eta^s \perp \lambda$.*

Definition 2. *For any integer $k \geqslant 2$, we denote by* Sym_k *the set of symmetric k-linear forms on $\mathbb{R}^n$. Then* Sym_k *is a finite-dimensional linear space whose canonical norm is defined as*

$$\forall f \in \mathrm{Sym}_k :$$
$$|f| := \sup\left\{|f(x_1, \ldots, x_k)| : x_1, \ldots, x_k \in \mathbb{R}^n, \; |x_1| \leqslant 1, \ldots, |x_k| \leqslant 1\right\} .$$

We also set $\mathrm{Sym}_0 := \mathbb{R}$ *and* $\mathrm{Sym}_1 := (\mathbb{R}^n)^*$.

If g is a k-linear form on $\mathbb{R}^n$, its symmetric part $[g]^S \in \mathrm{Sym}_k$ is defined as

$$[g]^S (x_1, \ldots, x_k) := \frac{1}{k!} \sum_{\sigma} g\left(x_{\sigma(1)}, \ldots, x_{\sigma(k)}\right) ,$$

where the sum extends over all permutations σ of $\{1, \ldots, k\}$.

Throughout the remainder of this work, Ω will denote a bounded open normalized subset of $\mathbb{R}^n$, which we call *the body*.

Definition 3. *For every integer $k \geqslant 0$ we set*

$$\mathrm{Pol}_k = \left\{v \in C^\infty(\Omega) : v \textit{ is a polynomial of degree at most } k\right\} .$$

We also set $\mathrm{Pol}_{-1} = \{0\}$.

Definition 4. A full grid *G is an ordered triple*

$$G = \left(x_0, (e_1, \ldots, e_n), \widehat{G}\right) ,$$

where $x_0 \in \mathbb{R}^n$, $(e_1, \ldots, e_n)$ is a positively oriented orthonormal basis in $\mathbb{R}^n$ and $\widehat{G}$ is a Borel subset of $\mathbb{R}$ with $\mathcal{L}^1(\mathbb{R} \setminus \widehat{G}) = 0$.

If G_1, G_2 are two full grids, we write $G_1 \subseteq G_2$ if $\widehat{G}_1 \subseteq \widehat{G}_2$ and they share the point x_0 and the list $(e_1, \ldots, e_n)$.

We define now a particularly simple class of subbodies.

Definition 5. *Let $G = \big(x_0, (e_1, \dots, e_n), \widehat{G}\big)$ be a full grid. A subset M of $\mathbb{R}^n$ is said to be* a G-interval *if*

$$M = \big\{x \in \mathbb{R}^n : a_j < (x - x_0) \cdot e_j < b_j \ \forall j = 1, \dots, n\big\}$$

for some $a_1, b_1, \dots, a_n, b_n \in \widehat{G}$. We set

$$\mathcal{M}_G = \big\{M \subseteq \mathbb{R}^n : M \text{ is a } G\text{-interval with } \mathrm{cl}\, M \subseteq \Omega\big\} .$$

Remark 1. Let $\lambda : \mathcal{M}_G \to \mathbb{R}$ be a function such that

(a) λ is $*$-additive, i.e.

$$\lambda((M_1 \cup M_2)_*) = \lambda(M_1) + \lambda(M_2)$$

for every $M_1, M_2 \in \mathcal{M}_G$ such that $(M_1 \cup M_2)_* \in \mathcal{M}_G$ and $M_1 \cap M_2 = \varnothing$;

(b) There exists $\mu \in \mathfrak{M}(\Omega)$ such that

$$\forall M \in \mathcal{M}_G : \quad |\lambda(M)| \leqslant \mu(M) .$$

Then there exists a Borel function $a : \Omega \to [-1, 1]$ such that

$$\forall M \in \mathcal{M}_G : \quad \lambda(M) = \int_M a(x)\, d\mu(x) .$$

Moreover, a is uniquely determined μ-a.e.

In fact, we can denote by L the linear space spanned by the characteristic functions χ_J of n-intervals of the form

$$J = \big\{x \in \mathbb{R}^n : a_j \leqslant (x - x_0) \cdot e_j < b_j \ \forall j = 1, \dots, n\big\} , \qquad \mathrm{cl}\, J \subseteq \Omega ,$$

with $a_1, b_1, \dots, a_n, b_n \in \widehat{G}$, and consider the linear functional $\Lambda : L \to \mathbb{R}$ such that

$$\Lambda(\chi_J) := \lambda(\mathrm{int}\, J) .$$

Then L is a lattice of functions on Ω and Λ a Daniell integral on L. The assertion follows e.g. from [5, 2.5.9].

More generally, let Y be a finite-dimensional normed space and $\lambda : \mathcal{M}_G \times Y \to \mathbb{R}$ a function such that

(a) For every $y \in Y$, $\lambda(\,\cdot\,, y)$ is $*$-additive;
(b) For every $M \in \mathcal{M}_G$, $\lambda(M, \,\cdot\,)$ is linear;
(c) There exists $\mu \in \mathfrak{M}(\Omega)$ such that

$$\forall M \in \mathcal{M}_G, \forall y \in Y : \quad |\lambda(M, y)| \leqslant |y| \mu(M) .$$

Then there exists a bounded Borel map $A : \Omega \to Y^*$ such that

$$\forall M \in \mathcal{M}_G, \forall y \in Y : \quad \lambda(M, y) = \int_M \langle A(x), y \rangle\, d\mu(x) .$$

Moreover, A is uniquely determined μ-a.e.

Definition 6. *We denote by* $\mathcal{R}$ *the class of open n-intervals* I *such that* $\operatorname{cl} I \subseteq \Omega$.

Definition 7. *Let* $\mathcal{A} \subseteq \mathcal{R}$. *We say that* $\mathcal{A}$ contains almost all of $\mathcal{R}$ *if for every* $x_0 \in \mathbb{R}^n$ *and every positively oriented orthonormal basis* $(e_1, \dots, e_n)$ *in* $\mathbb{R}^n$ *there exists a full grid*

$$G = \left(x_0, (e_1, \dots, e_n), \widehat{G}\right)$$

such that $\mathcal{M}_G \subseteq \mathcal{A}$.

2. Powers of order k

Definition 8. *Let* $\mathcal{A}$ *be a subset of* $\mathcal{R}$ *containing almost all of* $\mathcal{R}$ *and k a nonnegative integer. We say that a function* $P : \mathcal{A} \times C^\infty(\Omega) \to \mathbb{R}$ *is* a power of order k *if the following properties hold:*

(a) For every $v \in C^\infty(\Omega)$, $P(\,\cdot\,, v)$ *is countably* $*$*-additive, i.e.*

$$P\left(\left(\bigcup_{i\in\mathbb{N}} M_i\right)_*, v\right) = \sum_{i\in\mathbb{N}} P(M_i, v)$$

for every disjoint sequence $(M_i) \in \mathcal{A}$ *such that* $\left(\bigcup_{i\in\mathbb{N}} M_i\right)_* \in \mathcal{A}$;

(b) For every $M \in \mathcal{A}$, $P(M, \,\cdot\,)$ *is linear;*

(c) There exist $\mu_0, \dots, \mu_k \in \mathfrak{M}(\Omega)$ *such that*

$$\forall M \in \mathcal{A},\ \forall v \in C^\infty(\Omega): \quad |P(M, v)| \leqslant \sum_{j=0}^{k} \int_M |v^{(j)}(x)|\, d\mu_j(x)\,,$$

where $v^{(j)}(x) \in \operatorname{Sym}_j$ *denotes the* j*th derivative of* v *at* x.

Remark 2. Let $M \in \mathcal{R}$; then it is easy to prove that for every full grid G there exists a disjoint sequence $(M_i) \subseteq \mathcal{M}_G$ such that

$$\left(\bigcup_{i\in\mathbb{N}} M_i\right)_* = M.$$

Remark 3. In view of the previous remark, one could replace (a) by the following weaker assumption:

(a′) For every $v \in C^\infty(\Omega)$ and for every full grid G,

$$P\left(\left(\bigcup_{i\in\mathbb{N}} M_i\right)_*, v\right) = \sum_{i\in\mathbb{N}} P(M_i, v)$$

whenever $(M_i) \in \mathcal{A} \cap \mathcal{M}_G$ is a disjoint sequence such that $\left(\bigcup_{i\in\mathbb{N}} M_i\right)_* \in \mathcal{A}$.

Remark 4. More generally, one could consider powers $P(M, \vec{v})$, where the velocity field $\vec{v}$ takes values in $\mathbb{R}^N$, $N \geqslant 1$. If $\{e_1, \dots, e_N\}$ denotes the canonical basis in $\mathbb{R}^N$, for every $i = 1, \dots, N$ it is possible to define $P_i(M, v) = P(M, ve_i)$, which is a power in the sense of Definition 8. On the other hand, if $\vec{v} = (v_1, \dots, v_N)$, it holds that

$$P(M, \vec{v}) = \sum_{i=1}^{N} P_i(M, v_i)\,.$$

For this reason, we only treat the case of scalar velocity fields, as the results for vector velocity fields can be easily deduced from the corresponding ones in the scalar case.

2.1. Integral representation

The main result of this section is an integral representation formula (Theorem 1) for a power of order k.

In the following lemmas, we assume that $k \geqslant 0$, G is a full grid and $P : \mathcal{M}_G \times \mathrm{Pol}_k \to \mathbb{R}$ is a function which satisfies (a), (b) and (c) of Definition 8.

Lemma 1. *Let $P(M, v) = 0$ for every $M \in \mathcal{M}_G$ and $v \in \mathrm{Pol}_{k-1}$.*

Then there exists a bounded Borel map $A_k : \Omega \to \mathrm{Sym}_k^*$ such that

$$\forall M \in \mathcal{M}_G, \forall v \in \mathrm{Pol}_k : \quad P(M, v) = \int_M \langle A_k(x), v^{(k)}(x)\rangle \, d\mu_k(x)\,, \tag{1}$$

where μ_k is given by (c) of Definition 8. Moreover, A_k is uniquely determined μ_k-a.e.

Proof. First, by countable $*$-additivity and Lebesgue Theorem one has

$$\lim_{i\to\infty} P(M_i, v) = P(M, v),$$
$$\lim_{i\to\infty} \int_{M_i} \left\langle A_k(x), v^{(k)}(x)\right\rangle d\mu_k(x) = \int_M \left\langle A_k(x), v^{(k)}(x)\right\rangle d\mu_k(x)$$

whenever $(M_i) \subseteq \mathcal{M}_G$ is an increasing sequence with

$$\bigcup_{i\in\mathbb{N}} M_i = M \in \mathcal{M}_G\,.$$

In particular, it is enough to prove (1) for a full grid $G' \subseteq G$, since every set $M \in \mathcal{M}_G$ is the union of an increasing sequence in $\mathcal{M}_{G'}$. Thus, without loss of generality we assume that for every $j = 0, \dots, k$, $i = 1, \dots, n$ and $a \in \widehat{G}$, we have

$$\mu_j\left(\{x \in \Omega : (x - x_0) \cdot e_i = a\}\right) = 0\,.$$

Now we claim that

$$\forall M \in \mathcal{M}_G, \forall v \in \mathrm{Pol}_k : \quad |P(M, v)| \leqslant \int_M |v^{(k)}|\, d\mu_k(x)\,. \tag{2}$$

Actually, we may assume that $k \geqslant 1$. Given $v \in \mathrm{Pol}_k$, by Remark 1 we have

$$\forall M \in \mathcal{M}_G : \quad P(M, v) - \int_M |v^{(k)}| \, d\mu_k(x) = \int_M a_v(x) \, d\mu(x) \, ,$$

where $a_v : \Omega \to \mathbb{R}$ is Borel and bounded and $\mu := \mu_0 + \cdots + \mu_k$. Let us prove that $a_v \leqslant 0$ μ-a.e.

Let

$$x_0 \in \operatorname{supt} \mu := \{x \in \Omega : \ \mu(\Omega \cap B_r(x)) > 0 \quad \forall r > 0\} \, ,$$

and let (M_i) be a sequence of open n-cubes in $\mathcal{M}_G$ with $x_0 \in M_i$ and $\operatorname{diam} M_i \to 0$ as $i \to \infty$. We have $v = u + w$ with $u \in \mathrm{Pol}_{k-1}$ and $w \in \mathrm{Pol}_k$ satisfying $w^{(j)}(x_0) = 0$ for $0 \leqslant j \leqslant k-1$. It follows that

$$\begin{aligned} P(M_i, v) - \int_{M_i} |v^{(k)}| \, d\mu_k(x) &= P(M_i, w) - \int_{M_i} |w^{(k)}| \, d\mu_k(x) \\ &\leqslant \sum_{j=0}^{k-1} \int_{M_i} |w^{(j)}(x)| \, d\mu_j(x) \, , \end{aligned}$$

hence

$$\limsup_i \frac{P(M_i, v) - \int_{M_i} |v^{(k)}| \, d\mu_k(x)}{\mu(M_i)} \leqslant 0 \, .$$

Therefore, we have $a_v \leqslant 0$, namely

$$\forall M \in \mathcal{M}_G, \forall v \in \mathrm{Pol}_k : \quad P(M, v) - \int_M |v^{(k)}| \, d\mu_k(x) \leqslant 0 \, .$$

In a similar way it is possible to prove that

$$\forall M \in \mathcal{M}_G, \forall v \in \mathrm{Pol}_k : \quad P(M, v) + \int_M |v^{(k)}| \, d\mu_k(x) \geqslant 0 \, ,$$

whence (2).

Let now $k \geqslant 0$. For every $F \in \mathrm{Sym}_k$, let v_F be the homogeneous polynomial of degree k with $v_F^{(k)} = F$. Since (2) implies

$$\forall M \in \mathcal{M}_G, \forall F \in \mathrm{Sym}_k : \quad |P(M, v_F)| \leqslant |F| \mu_k(M) \, ,$$

according to Remark 1 there exists a bounded Borel map $A_k : \Omega \to \mathrm{Sym}_k^*$ such that

$$\begin{aligned} &\forall M \in \mathcal{M}_G, \forall F \in \mathrm{Sym}_k : \\ &\qquad P(M, v_F) = \int_M \langle A_k(x), F \rangle \, d\mu_k(x) = \int_M \left\langle A_k(x), v_F^{(k)} \right\rangle d\mu_k(x) \, . \end{aligned}$$

Moreover, A_k is uniquely determined μ_k-a.e. Since $P(M, \cdot)$ vanishes on Pol_{k-1}, the assertion follows. □

Lemma 2. *For every $j = 0, \dots, k$ there exists a bounded Borel map $A_j : \Omega \to \mathrm{Sym}_j^*$ such that*

$$\forall M \in \mathcal{M}_G, \forall v \in \mathrm{Pol}_k : \quad P(M, v) = \sum_{j=0}^{k} \int_M \left\langle A_j(x), v^{(j)}(x) \right\rangle d\mu_j(x) .$$

Moreover, each A_j is uniquely determined μ_j-a.e.

Proof. As in the proof of the previous lemma, we assume without loss of generality that

$$\mu_j \left(\{x \in \Omega : (x - x_0) \cdot e_i = a\}\right) = 0$$

for every $j = 0, \dots, k$, $i = 1, \dots, n$ and $a \in \widehat{G}$.

We argue by induction on k. For $k = 0$ the assertion immediately follows from Lemma 1.

Let now $k \geqslant 1$ and assume that the assertion is true for $k - 1$. Since $v^{(k)} = 0$ for every $v \in \mathrm{Pol}_{k-1}$, the restriction $P\Big|_{\mathcal{M}_G \times \mathrm{Pol}_{k-1}}$ satisfies (a), (b) and (c) of Definition 8. By the inductive assumption, there exist bounded and Borel maps $A_j : \Omega \to \mathrm{Sym}_j^*$, $j = 0, \dots, k - 1$ such that

$$\forall M \in \mathcal{M}_G, \forall v \in \mathrm{Pol}_{k-1} : \quad P(M, v) = \sum_{j=0}^{k-1} \int_M \left\langle A_j(x), v^{(j)}(x) \right\rangle d\mu_j(x)$$

and each A_j is uniquely determined μ_j-a.e. Then

$$\widetilde{P}(M, v) := P(M, v) - \sum_{j=0}^{k-1} \int_M \left\langle A_j(x), v^{(j)}(x) \right\rangle d\mu_j(x)$$

satisfies the assumption of Lemma 1. Therefore, there exists a bounded Borel map $A_k : \Omega \to \mathrm{Sym}_k^*$ such that

$$\forall M \in \mathcal{M}_G, \forall v \in \mathrm{Pol}_k : \quad \widetilde{P}(M, v) = \int_M \left\langle A_k(x), v^{(k)}(x) \right\rangle d\mu_k(x)$$

and A_k is uniquely determined μ_k-a.e. □

Lemma 3. *Let $A_j : \Omega \to \mathrm{Sym}_j^*$ be as in Lemma 2.*

Then, for every $j = 0, \dots, k$, $i = 1, \dots, n$ and $a \in \widehat{G}$, we have $A_j = 0$ μ_j-a.e. on

$$\{x \in \Omega : (x - x_0) \cdot e_i = a\} .$$

In particular, for every $M \in \mathcal{M}_G$ and $j = 0, \dots, k$, we have $A_j = 0$ μ_j-a.e. on $\partial_* M$.

Proof. Let $i = 1, \dots, n$ and $a \in \widehat{G}$. If $v \in \mathrm{Pol}_k$, $M_1, M_2 \in \mathcal{M}_G$, $(M_1 \cup M_2)_* \in \mathcal{M}_G$ and $M_1 \cap M_2 = \varnothing$, we have

$$\sum_{j=0}^{k} \int_{(M_1 \cup M_2)_*} \langle A_j(x), v^{(j)}(x) \rangle \, d\mu_j(x) = \sum_{j=0}^{k} \int_{M_1} \langle A_j(x), v^{(j)}(x) \rangle \, d\mu_j(x) + \sum_{j=0}^{k} \int_{M_2} \langle A_j(x), v^{(j)}(x) \rangle \, d\mu_j(x) ,$$

hence

$$\sum_{j=0}^{k} \int_{(M_1 \cup M_2)_* \setminus (M_1 \cup M_2)} \langle A_j(x), v^{(j)}(x) \rangle \, d\mu_j(x) = 0 .$$

If we set

$$\Omega' = \{x \in \Omega : \, (x - x_0) \cdot e_i = a\}$$

and we denote by $\mathcal{M}'_G$ the family of I sets of the form

$$I = \left\{x \in \mathbb{R}^n : \quad (x - x_0) \cdot e_i = a, \quad a_m < (x - x_0) \cdot e_m < b_m \ \forall m \neq i \right\}$$

with $a_m, b_m \in \widehat{G}$ and $\mathrm{cl}\, I \subseteq \Omega'$, it follows that

$$\forall I \in \mathcal{M}'_G, \forall v \in \mathrm{Pol}_k(\Omega') : \quad \sum_{j=0}^{k} \int_{I} \langle A_j(x), v^{(j)}(x) \rangle \, d\mu_j(x) = 0 .$$

From Lemma 2, applied in $n-1$ dimensions, we deduce that $A_j = 0$ μ_j-a.e in Ω' for every $j = 1, \dots, k$. □

Finally, we come to the main theorem.

Theorem 1. *Let P be a power of order k.*

Then, for every $j = 0, \dots, k$ there exists a bounded Borel map $A_j : \Omega \to \mathrm{Sym}_j^*$ such that

$$\forall M \in \mathcal{A}, \forall v \in C^\infty(\Omega) : \quad P(M, v) = \sum_{j=0}^{k} \int_{M} \langle A_j(x), v^{(j)}(x) \rangle \, d\mu_j(x) ,$$

where $\mu_0, \dots, \mu_k$ are given by (c) of Definition 8. Moreover, each A_j is uniquely determined μ_j-a.e.

Proof. Let G be a full grid with $\mathcal{M}_G \subseteq \mathcal{A}$. First of all, we claim that for every $j = 0, \dots, k$ there exists a bounded Borel map $A_j : \Omega \to \mathrm{Sym}_j^*$ such that

$$\forall M \in \mathcal{M}_G, \forall v \in C^\infty(\Omega) : \quad P(M, v) = \sum_{j=0}^{k} \int_{M} \langle A_j(x), v^{(j)}(x) \rangle \, d\mu_j(x) \tag{3}$$

and each A_j is uniquely determined μ_j-a.e.

Consider the restriction $P\big|_{\mathcal{M}_G \times \mathrm{Pol}_k}$. By Lemma 2 there exist bounded and Borel maps $A_j : \Omega \to \mathrm{Sym}_j^*$, $j = 0, \dots, k$ such that

$$\forall M \in \mathcal{M}_G, \forall v \in \mathrm{Pol}_k : \quad P(M, v) = \sum_{j=0}^{k} \int_M \left\langle A_j(x), v^{(j)}(x) \right\rangle d\mu_j(x)$$

and each A_j is uniquely determined μ_j-a.e. Given $v \in C^\infty(\Omega)$, by Remark 1 we have

$$\forall M \in \mathcal{M}_G : \quad P(M, v) - \sum_{j=0}^{k} \int_M \left\langle A_j(x), v^{(j)}(x) \right\rangle d\mu_j(x) = \int_M a_v(x)\, d\mu(x)\,,$$

where $a_v : \Omega \to \mathbb{R}$ is Borel and bounded and $\mu := \mu_0 + \cdots + \mu_k$. To prove the assertion, it is enough to show that $a_v = 0$ μ-a.e.

Let $x_0 \in \mathrm{supt}\,\mu$ and let (M_i) be a sequence of open n-cubes in $\mathcal{M}_G$ with $x_0 \in M_i$ and $\mathrm{diam}\, M_i \to 0$ as $i \to \infty$. We can write $v = u + w$ with $u \in \mathrm{Pol}_k$ and $w \in C^\infty(\Omega)$ satisfying $w^{(j)}(x_0) = 0$ for $0 \leqslant j \leqslant k$. We have

$$\begin{aligned} P(M_i, v) &- \sum_{j=0}^{k} \int_{M_i} \left\langle A_j(x), v^{(j)}(x) \right\rangle d\mu_j(x) \\ &= P(M_i, w) - \sum_{j=0}^{k} \int_{M_i} \left\langle A_j(x), w^{(j)}(x) \right\rangle d\mu_j(x)\,, \end{aligned}$$

hence, taking into account (c) of Definition 8,

$$\lim_i \frac{1}{\mu(M_i)} \left(P(M_i, v) - \sum_{j=0}^{k} \int_{M_i} \left\langle A_j(x), v^{(j)}(x) \right\rangle d\mu_j(x) \right) = 0$$

and (3) follows.

Let now $M \in \mathcal{A}$ and $(M_i) \subseteq \mathcal{M}_G$ be a disjoint sequence such that $\left(\bigcup_{i \in \mathbb{N}} M_i \right)_* = M$ (cf. Remark 2). By (3) and the countable $*$-additivity of P, for any $v \in C^\infty(\Omega)$ it follows that

$$\begin{aligned} P(M, v) = \sum_{i \in \mathbb{N}} P(M_i, v) &= \sum_{i \in \mathbb{N}} \sum_{j=0}^{k} \int_{M_i} \left\langle A_j(x), v^{(j)}(x) \right\rangle d\mu_j(x) \\ &= \sum_{j=0}^{k} \int_{\bigcup_{i \in \mathbb{N}} M_i} \left\langle A_j(x), v^{(j)}(x) \right\rangle d\mu_j(x) \\ &= \sum_{j=0}^{k} \int_M \left\langle A_j(x), v^{(j)}(x) \right\rangle d\mu_j(x), \end{aligned}$$

where the last equality takes into account Lemma 3. $\square$

It is easy to see that Theorem 1 admits a form of converse.

Proposition 1. *Let $\mu_0, \dots, \mu_k \in \mathfrak{M}(\Omega)$ and, for $j = 0, \dots, k$, let $A_j : \Omega \to \mathrm{Sym}^*_j$ be Borel and bounded.*

Then there exists a set $\mathcal{A} \subseteq \mathcal{R}$ containing almost all of $\mathcal{R}$ such that the function $P : \mathcal{A} \times C^\infty(\Omega) \to \mathbb{R}$ defined as

$$P(M, v) = \sum_{j=0}^{k} \int_M \left\langle A_j(x), v^{(j)}(x) \right\rangle d\mu_j(x) \tag{4}$$

is a power of order k.

Proof. Let $\mu = \mu_0 + \cdots + \mu_k$ and

$$\mathcal{A} = \{M \in \mathcal{R} : \ \mu(\mathrm{bd}\, M) = 0\}\,.$$

Since μ is finite on compact subsets of Ω, it is easy to see that $\mathcal{A}$ contains almost all of $\mathcal{R}$. Then it is clear that (4) defines a power of order k. □

2.2. Extension to Borel sets

A first important consequence of the representation formula is an extension result of the power to the Borel subsets of Ω. This can be done in an easy way, the only difficulty being to obtain a $*$-additive function. We need a simple definition.

Definition 9. *Let $\eta \in \mathfrak{M}(\Omega)$. We set*

$$\mathcal{B}_\eta = \{M \subseteq \mathbb{R}^n : \ M = M_*\,, \mathrm{cl}\, M \subseteq \Omega,\ \eta(\partial_* M) = 0\}\,.$$

Theorem 2. *Let P be a power of order k. Let $A_j : \Omega \to \mathrm{Sym}^*_j$ be as in Theorem 1.*

Then there exists $\eta \in \mathfrak{M}(\Omega)$ such that the function $\tilde{P} : \mathcal{B}_\eta \times C^\infty(\Omega) \to \mathbb{R}$ defined as

$$\tilde{P}(M, v) = \sum_{j=0}^{k} \int_M \left\langle A_j(x), v^{(j)}(x) \right\rangle d\mu_j(x)$$

is an extension of P which satisfies (a), (b) and (c) of Definition 8 on $\mathcal{B}_\eta$.

Proof. Let $\eta = \mu_0 + \cdots + \mu_k$. Then $\tilde{P}$ is $*$-additive on $\mathcal{B}_\eta$. Moreover, it clearly satisfies (b) and (c) of Definition 8 on $\mathcal{B}_\eta$; thus the proof is complete. □

Henceforth we shall denote by the same symbol P such an extension.

2.3. Decomposition in body and contact part

Definition 10. *A power P of order k is said to be* weakly balanced *if there exists $\nu \in \mathfrak{M}(\Omega)$ such that*

$$\forall M \in \mathcal{A},\ \forall v \in C^\infty_c(M) : \quad |P(M, v)| \leqslant \int_M |v|\, d\nu\,.$$

In particular, P is said to be a contact power *if*

$$\forall M \in \mathcal{A},\ \forall v \in C_c^\infty(M): \quad P(M,v) = 0\,,$$

namely if it is weakly balanced with $\nu = 0$.

A power P of order 0 *is said to be* a body power.

Note that a body power is always weakly balanced, by choosing trivially $\nu = \mu_0$.

Theorem 3. *Let P be a weakly balanced power of order k, and let μ_j, A_j, $0 \leqslant j \leqslant k$ be as in Theorem 1.*

Then the following facts hold:

(a) There exists a bounded Borel function $B : \Omega \to \mathbb{R}$ such that

$$\forall v \in C_c^\infty(\Omega): \quad \sum_{j=0}^{k} \int_\Omega \left\langle A_j(x), v^{(j)}(x) \right\rangle d\mu_j(x) = \int_\Omega B(x)v(x)\,d\nu(x)\,;$$

moreover, B is uniquely determined ν-a.e.;

(b) One has

$$\forall M \in \mathcal{A},\ \forall v \in C_c^\infty(M): \quad P(M,v) = \int_M B(x)v(x)\,d\nu(x)\,.$$

Proof. If $M \in \mathcal{A}$ and $v \in C_c^\infty(M)$, we have

$$\left| \sum_{j=0}^{k} \int_M \left\langle A_j(x), v^{(j)}(x) \right\rangle d\mu_j(x) \right| = |P(M,v)| \leqslant \int_M |v|\,d\nu\,.$$

Since $\mathcal{A}$ is an open cover of Ω, assertions (a) and (b) follow easily. □

Let P be a weakly balanced power of order k, let μ_j, A_j, $0 \leqslant j \leqslant k$ be as in Theorem 1, and let ν, B be as in Theorem 3. According to Proposition 1 define, for a suitable class $\mathcal{A}$ containing almost all of $\mathcal{R}$, two powers $P_b, P_c : \mathcal{A} \times C^\infty(\Omega) \to \mathbb{R}$ by

$$P_b(M,v) := \int_M B(x)v(x)\,d\nu(x)\,,$$

$$P_c(M,v) := P(M,v) - \int_M B(x)v(x)\,d\nu(x)\,.$$

It is readily seen that P_b is a body power and P_c a contact power of order k. Of course, we have $P = P_b + P_c$.

Definition 11. *P_b is said to be* the body part *of P and P_c* the contact part *of P.*

3. First-order contact powers

Let P be a contact power of order 1 such that (c) of Definition 8 holds with $\mu_1 \ll \mathcal{L}^n$. We set $\eta = \mu_0$.

According to Theorems 1 and 3, there exist a bounded Borel function $a : \Omega \to \mathbb{R}$ and $T \in L^1_{loc}(\Omega; \mathbb{R}^n)$ such that

$$\forall M \in \mathcal{B}_\eta, \forall v \in C^\infty(\Omega) : \quad P(M, v) = \int_M a v \, d\eta + \int_M T \cdot \nabla v \, d\mathcal{L}^n \,, \tag{5}$$

$$\forall v \in C^\infty_c(\Omega) : \quad \int_\Omega a v \, d\eta + \int_\Omega T \cdot \nabla v \, d\mathcal{L}^n = 0 \,. \tag{6}$$

Moreover, a is uniquely determined η-a.e. and T is uniquely determined $\mathcal{L}^n$-a.e.

We briefly recall now the concept of *outer normal* to the measure-theoretic boundary of a set. Let $M \subseteq \mathbb{R}^n$ and $x \in \partial_* M$. We denote by $\vec{n}^M(x) \in \mathbb{R}^n$ a unit vector such that

$$\mathcal{L}^n \left(\{\xi \in B_r(x) \cap M : (\xi - x) \cdot \vec{n}^M(x) > 0\}\right) / r^n \to 0,$$
$$\mathcal{L}^n \left(\{\xi \in B_r(x) \setminus M : (\xi - x) \cdot \vec{n}^M(x) < 0\}\right) / r^n \to 0$$

as $r \to 0^+$. No more than one such vector can exist. Setting $\vec{n}^M(x) = 0$ elsewhere, we can consider the map $\vec{n}^M : \partial_* M \to \mathbb{R}^n$, which is called *the unit outer normal* to M. It turns out that $\vec{n}^M$ is Borel and bounded.

Whenever $\mathcal{H}^{n-1}(\partial_* M) < +\infty$, we say that M is *a set with finite perimeter*. In that case it is well known that $\vec{n}^M(x) \neq 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial_* M$ and the Gauss–Green theorem for Lipschitz functions holds.

We now define a suitable subclass of $\mathcal{B}_\eta$ which allows us to give a representation formula for a contact power of order 1 involving only the measure-theoretic boundary of the subbodies. We refer to [15,2] for a discussion about this class.

Definition 12. *For $h \in \mathcal{L}^1_{loc,+}(\Omega)$ we set*

$$\mathcal{M}_{h\eta} = \left\{ M \in \mathcal{B}_\eta : \mathcal{H}^{n-1}(\partial_* M) < +\infty, \int_{\partial_* M} h \, d\mathcal{H}^{n-1} < +\infty \right\}.$$

Theorem 4 (Cauchy's Stress Theorem). *There exists $h \in \mathcal{L}^1_{loc,+}(\Omega)$ such that*

$$\forall M \in \mathcal{M}_{h\eta}, \forall v \in C^\infty(\Omega) : \quad P(M, v) = \int_{\partial_* M} v \, T \cdot \vec{n}^M \, d\mathcal{H}^{n-1}.$$

Proof. By [2, Theorem 5.4] and (6), there exists $h \in \mathcal{L}^1_{loc,+}(\Omega)$ such that

$$\int_M T \cdot \nabla v \, d\mathcal{L}^n = \int_{\partial_* M} v \, T \cdot \vec{n}^M \, d\mathcal{H}^{n-1} - \int_M a v \, d\eta$$

for every locally Lipschitz vector field $v : \Omega \to \mathbb{R}$ and every $M \in \mathcal{M}_{h\eta}$. Then

$$P(M, v) = \int_M a v \, d\eta + \int_M T \cdot \nabla v \, d\mathcal{L}^n = \int_{\partial_* M} v \, T \cdot \vec{n}^M \, d\mathcal{H}^{n-1}$$

for every $v \in C^\infty(\Omega)$ and $M \in \mathcal{M}_{h\eta}$. □

4. Second-order contact powers

Throughout this section we assume that P is a contact power of order 2 such that (c) of Definition 8 holds with $\mu_1 \ll \mathcal{L}^n$ and $\mu_2 \ll \mathcal{L}^n$. We set $\eta = \mu_0$.

According to Theorems 1 and 3, there exist a bounded Borel function $a : \Omega \to \mathbb{R}$, $B \in L^1_{loc}(\Omega; \mathbb{R}^n)$ and $\mathsf{C} \in L^1_{loc}(\Omega; \mathrm{Sym}_2)$ such that

$$\forall M \in \mathcal{B}_\eta, \forall v \in C^\infty(\Omega) :$$

$$P(M, v) = \int_M av\, d\eta + \int_M B \cdot \nabla v\, d\mathcal{L}^n + \int_M \mathsf{C} \cdot \nabla\nabla v\, d\mathcal{L}^n , \tag{7}$$

$$\forall v \in C^\infty_c(\Omega) : \quad \int_\Omega av\, d\eta + \int_\Omega B \cdot \nabla v\, d\mathcal{L}^n + \int_\Omega \mathsf{C} \cdot \nabla\nabla v\, d\mathcal{L}^n = 0 . \tag{8}$$

Moreover, a is uniquely determined η-a.e. and B, C are uniquely determined $\mathcal{L}^n$-a.e.

4.1. Boundary representation

When the distribution $\mathrm{div}\, \mathsf{C}$ is in fact a function, we have a representation of $P(M, v)$ in terms of the boundary of M.

Theorem 5. *Assume that* $\mathrm{div}\, \mathsf{C} \in L^1_{loc}(\Omega, \mathbb{R}^n)$. *Then there exists* $h \in \mathcal{L}^1_{loc,+}(\Omega)$ *such that*

$$P(M, v) = \int_{\partial_* M} [v(B - \mathrm{div}\, \mathsf{C}) \cdot \vec{n}^M + \nabla v \cdot \mathsf{C}\vec{n}^M]\, d\mathcal{H}^{n-1} \tag{9}$$

for every $v \in C^\infty(\Omega)$ *and* $M \in \mathcal{M}_{h\eta}$.

Proof. By [2, Theorem 5.4] there exists $h \in \mathcal{L}^1_{loc,+}(\Omega)$ such that

$$\int_M \mathsf{C} \cdot \nabla\nabla v\, d\mathcal{L}^n = -\int_M \nabla v \cdot \mathrm{div}\, \mathsf{C}\, d\mathcal{L}^n + \int_{\partial_* M} \nabla v \cdot \mathsf{C}\vec{n}^M d\mathcal{H}^{n-1},$$

$$\int_M (B - \mathrm{div}\, \mathsf{C}) \cdot \nabla v\, d\mathcal{L}^n = -\int_M v\, \mathrm{div}(B - \mathrm{div}\, \mathsf{C}) + \int_{\partial_* M} v(B - \mathrm{div}\, \mathsf{C}) \cdot \vec{n}^M d\mathcal{H}^{n-1}$$

for every $v \in C^\infty(\Omega)$ and $M \in \mathcal{M}_{h\eta}$. Moreover, rephrasing (8), we have

$$\forall v \in C^\infty_c(\Omega) : \quad -\int_\Omega (B - \mathrm{div}\, \mathsf{C}) \cdot \nabla v\, d\mathcal{L}^n = \int_\Omega av\, d\eta .$$

Hence it follows that

$$\begin{aligned} P(M, v) &= \int_M av\, d\eta + \int_M B \cdot \nabla v\, d\mathcal{L}^n + \int_M \mathsf{C} \cdot \nabla\nabla v\, d\mathcal{L}^n \\ &= \int_M av\, d\eta + \int_M (B - \mathrm{div}\, \mathsf{C}) \cdot \nabla v\, d\mathcal{L}^n + \int_{\partial_* M} \nabla v \cdot \mathsf{C}\vec{n}^M\, d\mathcal{H}^{n-1} \\ &= \int_{\partial_* M} [v(B - \mathrm{div}\, \mathsf{C}) \cdot \vec{n}^M + \nabla v \cdot \mathsf{C}\vec{n}^M]\, d\mathcal{H}^{n-1} \end{aligned}$$

for every $v \in C^\infty(\Omega)$ and $M \in \mathcal{M}_{h\eta}$. □

The condition $\operatorname{div} \mathsf{C} \in L^1_{loc}(\Omega, \mathbb{R}^n)$ has a counterpart in terms of the power P, as we will show in Theorem 6. This is quite interesting, since assumptions made on P are in general more meaningful than those made on its representation densities.

First we need a definition and a proposition.

Definition 13. *Let $G = \big(x_0, (e_1, \dots, e_n), \widehat{G}\big)$ be a full grid and $M \in \mathcal{M}_G$ of the form*

$$M = \Big\{x \in \mathbb{R}^n : a_j < (x - x_0) \cdot e_j < b_j,\ j = 1, \dots, n\Big\}, \tag{10}$$

where $a_1, b_1, \dots, a_n, b_n \in \widehat{G}$. Whenever $1 \leqslant j \leqslant n$ and $a_j \leqslant \alpha < \beta \leqslant b_j$, we set

$$M^{(j)}_{\alpha,\beta} = \Big\{x \in \mathbb{R}^n : \alpha < (x - x_0) \cdot e_j < \beta,\ a_i < (x - x_0) \cdot e_i < b_i \quad \forall i \neq j\Big\}.$$

We simply write $M^{(j)}_\beta$ in the case $\alpha = a_j$.

Proposition 2. *Let $\mathcal{M}_G \subseteq \mathcal{A}$, $M \in \mathcal{M}_G$ be represented as in* (10)*, $v \in C^\infty_c(M)$ and $1 \leqslant j \leqslant n$. Then $M^{(j)}_\beta \in \mathcal{M}_G$ for $\mathcal{L}^1$-a.e. $\beta \in (a_j, b_j]$, and the map*

$$\big\{\beta \mapsto P\big(M^{(j)}_\beta, v\big)\big\}$$

belongs to $L^\infty(a_j, b_j)$ for every $v \in C^\infty_c(\Omega)$.

Proof. We have

$$\Big|P\big(M^{(j)}_\beta, v\big)\Big| \leqslant \int_M |av|\, d\eta + \int_M \big(|B||\nabla v| + |\mathsf{C}||\nabla\nabla v|\big) d\mathcal{L}^n,$$

hence the map is bounded. To prove measurability, we apply Fubini's theorem:

$$\begin{aligned}
\int_{a_j}^{b_j} P\big(M^{(j)}_\beta, v\big)\, d\beta &= \int_{a_j}^{b_j} \int_{M^{(j)}_\beta} av\, d\eta\, d\beta \\
&\quad + \int_{a_j}^{b_j} \int_{M^{(j)}_\beta} (B \cdot \nabla v + \mathsf{C} \cdot \nabla\nabla v) d\mathcal{L}^n\, d\beta \\
&= \int_M (b_j - (x - x_0) \cdot e_j) av\, d\eta \\
&\quad + \int_M (b_j - (x - x_0) \cdot e_j)(B \cdot \nabla v + \mathsf{C} \cdot \nabla\nabla v) d\mathcal{L}^n;
\end{aligned} \tag{11}$$

since the right-hand side is integrable, the map $\{\beta \mapsto P(M^{(j)}_\beta, v)\}$ is measurable. □

Theorem 6. *We have that $\operatorname{div} \mathsf{C} \in L^1_{loc}(\Omega, \mathbb{R}^n)$ if and only if there exist $h \in \mathcal{L}^1_{loc,+}(\Omega)$ such that*

$$\left|\int_{a_j}^{b_j} P\big(M^{(j)}_\beta, v\big)\, d\beta\right| \leqslant \int_M |v| h\, d\mathcal{L}^n \tag{12}$$

for every $\mathcal{M}_G \subseteq \mathcal{A}$, $M \in \mathcal{M}_G$, $v \in C^\infty_c(M)$ and $j = 1, \dots, n$. In this case, we have $|B - 2 \operatorname{div} \mathsf{C}| \leqslant h$ on $\mathcal{L}^n$-a.a. of Ω.

Proof. Assume that $\operatorname{div}\mathsf{C} \in L^1_{loc}(\Omega, \mathbb{R}^n)$. Let $\mathcal{M}_G \subseteq \mathcal{A}$; taking into account Theorem 5, the symmetry of C and [2, Proposition 4.5], there exists a full grid $G' \subseteq G$ such that

$$P(M, v) = \int_{\operatorname{bd} M} [v(B - 2\operatorname{div}\mathsf{C}) \cdot \vec{n}^M + \operatorname{div}(\mathsf{C}v) \cdot \vec{n}^M]\, d\mathcal{H}^{n-1} \tag{13}$$

for every $v \in C^\infty(\Omega)$ and $M \in \mathcal{M}_{G'}$.

Let now $M \in \mathcal{M}_G$ and $v \in C_c^\infty(M)$. Since $v = 0$ on $\operatorname{bd} M$, without loss of generality we can assume that $M_\beta^{(j)} \in \mathcal{M}_{G'}$ for a.e. $\beta \in (a_j, b_j)$. Moreover, taking into account that $\vec{n}^{M_\beta^{(j)}}(x) = e_j$ whenever $x \in \operatorname{bd} M_\beta^{(j)}$ and $v(x) \neq 0$, by applying (13) to $M_\beta^{(j)}$, $1 \leqslant j \leqslant n$, it follows that

$$\begin{aligned}\int_{a_j}^{b_j} P\big(M_\beta^{(j)}, v\big)\, d\beta &= e_j \cdot \int_{a_j}^{b_j} \int_{\operatorname{bd} M_\beta^{(j)}} [v(B - 2\operatorname{div}\mathsf{C}) + \operatorname{div}(\mathsf{C}v)]\, d\mathcal{H}^{n-1} \\ &= e_j \cdot \int_M [v(B - 2\operatorname{div}\mathsf{C}) + \operatorname{div}(\mathsf{C}v)]\, d\mathcal{L}^n \\ &= e_j \cdot \int_M v(B - 2\operatorname{div}\mathsf{C})\, d\mathcal{L}^n,\end{aligned}$$

hence (12) is proved by choosing $h = |B - 2\operatorname{div}\mathsf{C}|$.

On the other hand, let $\mathcal{M}_G \in \mathcal{A}$, $M \in \mathcal{M}_G$ and $v \in C_c^\infty(M)$. By (11) we have

$$\int_{a_j}^{b_j} P\big(M_\beta^{(j)}, v\big)\, d\beta = \int_\Omega qav\, d\eta + \int_\Omega q(B \cdot \nabla v + \mathsf{C} \cdot \nabla\nabla v) d\mathcal{L}^n,$$

where we set $q(x) = b_j - (x - x_0) \cdot e_j$. Keeping in mind (8) and (12), by straightforward calculation one gets

$$\forall 1 \leqslant j \leqslant n : \quad |e_j \cdot (B - 2\operatorname{div}\mathsf{C})| \leqslant h \quad \text{on } \mathcal{L}^n\text{-a.a. of } \Omega\,,$$

hence $\operatorname{div}\mathsf{C} \in L^1_{loc}(\Omega, \mathbb{R}^n)$ and $|B - 2\operatorname{div}\mathsf{C}| \leqslant h$ on $\mathcal{L}^n$-a.a. of Ω. □

Remark 5. Let $\mathcal{M}_G \subseteq \mathcal{A}$, and consider the function $Q_G^{(j)} : \mathcal{M}_G \times C^\infty(\Omega) \to \mathbb{R}$ given by

$$Q_G^{(j)}(M, v) = \int_{a_j}^{b_j} P\big(M_\beta^{(j)}, v\big)\, d\beta.$$

Then $Q_G^{(j)}$ has the properties (a), (b) and (c) of Definition 8 and (12) is tantamount to saying that $Q_G^{(j)}$ is weakly balanced. We thank Paolo Podio–Guidugli for having brought this to our attention.

4.2. Going to the edges

In this subsection we will give an integral representation for P which takes into account sets with possibly non-smooth normal (such as sets with edges).

First of all, let $\mathsf{C} \in C^\infty\big(\Omega, \mathrm{Sym}_2\big)$ and $M \in \mathcal{M}$ with $\mathrm{bd}\, M$ of class C^2. Consider a C^1-extension $\vec{n}$ of $\vec{n}^M$ such that $(\nabla\vec{n})\vec{n} = 0$ on $\mathrm{bd}\, M$ (for instance, we can take $\vec{n} = \nabla g$, where $g(x) = d(x, M) - d(x, \mathbb{R}^n \setminus M)$). Then we have

$$\int_{\mathrm{bd}\, M} \nabla v \cdot [\mathsf{C}\vec{n}^M - (\mathsf{C}\vec{n}^M \cdot \vec{n}^M)\vec{n}^M]\, d\mathcal{H}^{n-1} = -\int_{\mathrm{bd}\, M} v \,\mathrm{div}(\mathsf{C}\vec{n} - (\mathsf{C}\vec{n}\cdot\vec{n})\vec{n})\, d\mathcal{H}^{n-1}$$
$$= \int_{\mathrm{bd}\, M} v[-\mathrm{div}\,\mathsf{C}\cdot\vec{n} - \mathsf{C}\cdot\nabla\vec{n} + (\nabla^s\mathsf{C})\vec{n}\vec{n}\vec{n} + (\mathrm{div}\,\vec{n})\mathsf{C}\cdot(\vec{n}\otimes\vec{n})]\, d\mathcal{H}^{n-1}$$

for every $v \in C^\infty(\Omega)$. Keeping in mind Theorem 5 and setting $D\vec{n} = \frac{1}{2}\nabla\vec{n} + \frac{1}{2}\nabla\vec{n}^T$, since C is symmetric one has

$$P(M, v) = \int_{\mathrm{bd}\, M} v[(B - 2\,\mathrm{div}\,\mathsf{C})\cdot\vec{n} + (\nabla^s\mathsf{C})\vec{n}\vec{n}\vec{n} + \mathsf{C}\cdot((\vec{n}\otimes\vec{n})\,\mathrm{div}\,\vec{n} - D\vec{n})]\, d\mathcal{H}^{n-1}$$
$$+ \int_{\mathrm{bd}\, M} \frac{\partial v}{\partial n}(\mathsf{C}\vec{n}\cdot\vec{n})\, d\mathcal{H}^{n-1}, \quad (14)$$

where the field $(\vec{n}\otimes\vec{n})\,\mathrm{div}\,\vec{n} - D\vec{n}$ is continuous.

Remark 6. The tensor field $(\vec{n}\otimes\vec{n})\,\mathrm{div}\,\vec{n} - D\vec{n}$ does not depend on the extension $\vec{n}$ of $\vec{n}^M$, provided that $(\nabla\vec{n})\vec{n} = 0$ on $\mathrm{bd}\, M$. Moreover, in that case one has

$$\mathrm{div}(\mathsf{C}\vec{n} - (\mathsf{C}\vec{n}\cdot\vec{n})\vec{n}) = \mathrm{div}_S(\mathsf{C}\vec{n} - (\mathsf{C}\vec{n}\cdot\vec{n})\vec{n})$$

on $\mathrm{bd}\, M$, where div_S denotes the surface divergence.

Now we want to introduce a more general class of sets which allows a further integration by parts.

Definition 14. *Let M be a normalized set with finite perimeter. We say that M is* a set with curvature measure *if there exist $\lambda_M \in \mathfrak{M}(\partial_* M)$ with $\lambda_M(\partial_* M) < +\infty$ and a Borel tensor field $\mathsf{U} : \partial_* M \to \mathrm{Sym}_2$ with $|\mathsf{U}(x)| = 1$ for λ_M-a.e. $x \in \partial_* M$, such that*

$$-\int_{\partial_* M} [-(\mathrm{div}\,\mathsf{C})\cdot\vec{n}^M + ((\nabla\mathsf{C})\vec{n}^M\vec{n}^M)\cdot\vec{n}^M]\, d\mathcal{H}^{n-1} = \int_{\partial_* M} \mathsf{C}\cdot\mathsf{U}\, d\lambda_M$$

for every $\mathsf{C} \in C^\infty_c(\mathbb{R}^n; \mathrm{Sym}_2)$. It turns out that λ_M is uniquely determined and U is uniquely determined λ_M-a.e.

For $h \in \mathcal{L}^1_{loc,+}(\Omega)$ we set

$$\mathcal{C}_{h\eta} = \left\{M \in \mathcal{M}_{h\eta} : M \textit{ has curvature measure and } \int_{\partial_* M} h\, d\lambda_M < +\infty\right\}.$$

Remark 7. One can prove that the elements of $\mathcal{R}$ are sets with curvature measure. Indeed, since on each face the term $[-(\operatorname{div}\mathsf{C})\cdot\vec{n}^M+((\nabla\mathsf{C})\vec{n}^M\vec{n}^M)\cdot\vec{n}^M]$ is a surface divergence, it turns out that λ_M is the Hausdorff measure $\mathcal{H}^{n-2}$ restricted to the edges, and $\mathsf{U}=\vec{n}^M\otimes\vec{N}+\vec{N}\otimes\vec{n}^M$, where $\vec{N}$ is the normal to the edge in the hyperplane of the surface.

As a first integration by parts gave the boundary representation formula (9) for P, a further integration should reduce the order of derivative of the test field v and bring up line integrals. In doing this, however, we notice that the normal derivative of v cannot be dropped, since it corresponds to a field of doublets assigned on the boundary. Moreover, line integrals will appear as surface integrals with respect to a singular measure.

Let us begin with some preliminary facts.

Definition 15. *Let* $\mathsf{C}\in L^1_{loc}(\Omega,\mathrm{Sym}_2)$. *We define* the symmetric gradient *of* C *by setting* $(\nabla^s\mathsf{C})=[\nabla\mathsf{C}]^S$, *that is to say, for any* $u,v,w\in\mathbb{R}^n$ *we define the distribution* $(\nabla^s\mathsf{C})uvw$ *on* Ω *as*

$$\langle(\nabla^s\mathsf{C})uvw,\varphi\rangle = \frac{1}{3}\int_\Omega[(\mathsf{C}v\cdot w)(\nabla\varphi\cdot u)+(\mathsf{C}u\cdot w)(\nabla\varphi\cdot v)+(\mathsf{C}u\cdot v)(\nabla\varphi\cdot w)]\,d\mathcal{L}^n$$

for every $\varphi\in C^\infty_c(\Omega)$. *The function* $\{(u,v,w)\mapsto(\nabla^s\mathsf{C})uvw\}$ *is 3-linear and symmetric; moreover,* $((\nabla\mathsf{C})uu)\cdot u=(\nabla^s\mathsf{C})uuu$ *holds for every* $u\in\mathbb{R}^n$.

Proposition 3. *Let* $f\in\mathrm{Sym}_3$, *and set* $c(u)=f(u,u,u)$ *for every* $u\in\mathbb{R}^n$. *Then* f *is uniquely determined by* c *and*

$$f(u,v,w)=\frac{1}{24}\big(c(u+v+w)-c(u+v-w)-c(u-v+w)+c(u-v-w)\big)$$

for every $(u,v,w)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$. *Moreover,*

$$|f|\leqslant\frac{3}{2}\sup\{|c(u)|:\,u\in\mathbb{R}^n,\,|u|=1\}.$$

Proof. Since it holds that

$$\begin{aligned}&c(u+v+w)=c(u)+c(v)+c(w)+6f(u,v,w)\\&+3\big[f(u,u,v)+f(u,u,w)+f(v,v,u)+f(v,v,w)+f(w,w,u)+f(w,w,v)\big],\end{aligned}$$

it follows that

$$\begin{aligned}c(u+v+w)-c(u+v-w)&=2c(w)+12f(u,v,w)+6\big[f(u,u,w)+f(v,v,w)\big],\\c(u-v-w)-c(u-v+w)&=-2c(w)+12f(u,v,w)-6\big[f(u,u,w)+f(v,v,w)\big].\end{aligned}$$

Hence we have

$$\big[c(u+v+w)-c(u+v-w)\big]+\big[c(u-v-w)-c(u-v+w)\big]=24f(u,v,w),$$

which gives the first statement.

Now set $M = \sup\{|c(u)| : u \in \mathbb{R}^n, |u| = 1\}$; it follows from the parallelogram identity that

$$
\begin{aligned}
|f(u, v, w)| &\leqslant \frac{M}{24}(|u+v+w|^3 + |u+v-w|^3 + |u-v+w|^3 + |u-v-w|^3) \\
&\leqslant \frac{M}{6}(|u|+|v|+|w|)(|u|^2+|v|^2+|w|^2).
\end{aligned}
$$

Taking the supremum as $|u|, |v|, |w| \leqslant 1$ the proof is complete. □

Lemma 4. *Let $p \in [1, +\infty]$ and consider a tensor field $\mathsf{C} \in L^p_{loc}(\Omega, \mathrm{Sym}_2)$ such that $\operatorname{div}\mathsf{C} \in L^1_{loc}(\Omega, \mathbb{R}^n)$ and $\nabla^s\mathsf{C} \in L^1_{loc}(\Omega, \mathrm{Sym}_3)$. Then there exist a sequence $(\mathsf{C}_m) \in C^\infty_c(\Omega, \mathrm{Sym}_2)$ and $h_0 \in \mathcal{L}^p_{loc,+}(\Omega)$, $h_1 \in \mathcal{L}^1_{loc,+}(\Omega)$ with*

$$
\forall x \in \Omega : \quad |\mathsf{C}_m(x)| \leqslant h_0(x), \quad |\operatorname{div}\mathsf{C}_m(x)| \leqslant h_1(x), \quad |\nabla^s\mathsf{C}_m(x)| \leqslant h_1(x),
$$

$$
h_0(x) < +\infty \Rightarrow x \text{ is a Lebesgue point of } \mathsf{C} \text{ and } \lim_m \mathsf{C}_m(x) = \widetilde{\mathsf{C}}(x),
$$

$$
h_1(x) < +\infty \Rightarrow x \text{ is a Lebesgue point of } \operatorname{div}\mathsf{C} \text{ and } \lim_m \operatorname{div}\mathsf{C}_m(x) = \widetilde{\operatorname{div}\mathsf{C}}(x),
$$

$$
h_1(x) < +\infty \Rightarrow x \text{ is a Lebesgue point of } \nabla^s\mathsf{C} \text{ and } \lim_m \nabla^s\mathsf{C}_m(x) = \widetilde{\nabla^s}\mathsf{C}(x),
$$

where $\widetilde{f}(x)$ denotes the Lebesgue limit of f in x.

Proof. Let $\rho \in C^\infty_c(\mathbb{R}^n)$ with $\rho \geqslant 0$ and $\int \rho \, d\mathcal{L}^n = 1$. Let $\rho_m = m^n\rho(mx)$. Let (K_m) be an increasing sequence of compact subsets of Ω with $\Omega = \bigcup_{m=1}^{\infty} K_m$. Let $\theta_m \in C^\infty_c(\Omega)$ with $0 \leqslant \theta_m \leqslant 1$ and $\theta_m = 1$ on K_m. We set

$$
\mathsf{C}_m(x) = \int_\Omega \rho_m(x-y)\theta_m(y)\mathsf{C}(y)\, d\mathcal{L}^n(y).
$$

Then we have that $\mathsf{C}_m \in C^\infty_c(\Omega, \mathrm{Sym}_2)$, $\mathsf{C}_m \to \widetilde{\mathsf{C}}$ in $L^1_{loc}(\Omega, \mathrm{Sym}_2)$ and

$$
|\mathsf{C}_m(x)| \leqslant \operatorname*{ess\,sup}_\Omega |\widetilde{\mathsf{C}}(x)|.
$$

By [1, Theorem IV.9] there exist $h_0 \in \mathcal{L}^p_{loc,+}(\Omega)$ and $h_1 \in \mathcal{L}^1_{loc,+}(\Omega)$ such that, up to a subsequence,

$$
\begin{aligned}
|\mathsf{C}_m(x)| \leqslant h_0(x), &\quad h_0(x) < +\infty \to \lim_m \mathsf{C}_m(x) = \widetilde{\mathsf{C}}(x), \\
|\operatorname{div}\mathsf{C}_m(x)| \leqslant h_1(x), &\quad h_1(x) < +\infty \to \lim_m \operatorname{div}\mathsf{C}_m(x) = f(x), \\
|\nabla^s\mathsf{C}_m(x)| \leqslant h_1(x), &\quad h_1(x) < +\infty \to \lim_m \nabla^s\mathsf{C}_m(x) = g(x),
\end{aligned}
$$

for some functions f, g. Moreover, one has

$$
\begin{aligned}
\int_\Omega \operatorname{div}\mathsf{C}_m \cdot \varphi \, d\mathcal{L}^n &= -\int_\Omega \mathsf{C}_m \cdot \nabla\varphi \, d\mathcal{L}^n \\
&= -\int_\Omega \nabla\varphi(x) \cdot \Big(\int_\Omega \rho_m(x-y)\theta_m(y)\widetilde{\mathsf{C}}(y)\, d\mathcal{L}^n(y)\Big) d\mathcal{L}^n(x) \\
&= -\int_\Omega \theta_m(y)\widetilde{\mathsf{C}}(y) \cdot \Big(\int_\Omega \rho_m(x-y)\nabla\varphi(x)\, d\mathcal{L}^n(x)\Big) d\mathcal{L}^n(y)
\end{aligned}
$$

for every $\varphi \in C_c^\infty(\Omega, \mathrm{Sym}_2)$. In particular

$$\int_\Omega \operatorname{div} \mathsf{C}_m \cdot \varphi \, d\mathcal{L}^n \to -\int_\Omega \widetilde{\mathsf{C}} \cdot \nabla\varphi \, d\mathcal{L}^n = \int_\Omega \operatorname{div} \widetilde{\mathsf{C}} \cdot \varphi \, d\mathcal{L}^n,$$

i.e. $\operatorname{div} \mathsf{C}_m \rightharpoonup \operatorname{div} \widetilde{\mathsf{C}} = \widetilde{\operatorname{div} \mathsf{C}}$. Hence it follows $f(x) = \widetilde{\operatorname{div} \mathsf{C}}(x)$ for $\mathcal{L}^n$-a.e. $x \in \Omega$. In a similar way, one can prove that $g(x) = \widetilde{\nabla^s \mathsf{C}}(x)$ for $\mathcal{L}^n$-a.e. $x \in \Omega$. A change of h_0, h_1 on a $\mathcal{L}^n$-negligible suitable set ends up the proof. □

Theorem 7. *Let P be a contact power of order 2 such that (c) of Definition 8 holds with $\mu_1 \ll \mathcal{L}^n$ and $\mu_2 \ll \mathcal{L}^n$, and let $\eta = \mu_0$. Assume moreover that* $\operatorname{div} \mathsf{C} \in L^1_{loc}(\Omega, \mathbb{R}^n)$ *and* $\nabla^s \mathsf{C} \in L^1_{loc}(\Omega, \mathrm{Sym}_3)$.

Then there exists $h \in \mathcal{L}^1_{loc,+}(\Omega)$ such that

$$\begin{aligned} P(M, v) = &\int_{\partial_* M} v\Big[(B - 2\operatorname{div} \mathsf{C}) \cdot \vec{n}^M + (\nabla^s \mathsf{C})\vec{n}^M\vec{n}^M\vec{n}^M\Big]\, d\mathcal{H}^{n-1} \\ &+ \int_{\partial_* M} \frac{\partial v}{\partial n}(\mathsf{C}\vec{n}^M \cdot \vec{n}^M)\, d\mathcal{H}^{n-1} + \int_{\partial_* M} v\,\mathsf{C} \cdot \mathsf{U}\, d\lambda_M \end{aligned} \tag{15}$$

for every $M \in \mathcal{C}_{h\eta}$ and $v \in C^\infty(\Omega)$.

Proof. In view of (9), it is sufficient to prove that

$$\begin{aligned} &\int_{\partial_* M} \nabla v \cdot \Big[\mathsf{C}\vec{n}^M - (\mathsf{C}\vec{n}^M \cdot \vec{n}^M)\vec{n}^M\Big]\, d\mathcal{H}^{n-1} \\ &\quad = \int_{\partial_* M} v\Big[((\nabla \mathsf{C})\vec{n}^M\vec{n}^M) \cdot \vec{n}^M - \operatorname{div} \mathsf{C} \cdot \vec{n}^M\Big]\, d\mathcal{H}^{n-1} + \int_{\partial_* M} v\,\mathsf{C} \cdot \mathsf{U}\, d\lambda_M \end{aligned} \tag{16}$$

on $\mathcal{C}_{h\eta}$ for every $v \in C^\infty(\Omega)$.

Consider a sequence $(\mathsf{C}_m) \in C_c^\infty(\Omega, \mathrm{Sym}_2)$ and $h_0, h_1 \in \mathcal{L}^1_{loc,+}(\Omega)$ as in Lemma 4. Let $h \in \mathcal{L}^1_{loc,+}(\Omega)$ be such that (9) holds on $\mathcal{C}_{h\eta}$ and $h_0, h_1 \leqslant h$. Let $v \in C^\infty(\Omega)$, and denote by $\vec{n}$ the map $\vec{n}^M$. Taking into account Definition 14, we obtain

$$\begin{aligned} &\int_{\partial_* M} v\mathsf{C}_m \cdot \mathsf{U}\, d\lambda_M = -\int_{\partial_* M} \Big[-\operatorname{div}(v\mathsf{C}_m) \cdot \vec{n} + (\nabla(v\mathsf{C}_m)\vec{n}\vec{n}) \cdot \vec{n}\Big]\, d\mathcal{H}^{n-1} \\ &= \int_{\partial_* M} \Big[v \operatorname{div} \mathsf{C}_m \cdot \vec{n} + \nabla v \cdot \mathsf{C}_m\vec{n} - (\mathsf{C}_m\vec{n} \cdot \vec{n})(\nabla v \cdot \vec{n}) - v(\nabla^s \mathsf{C}_m)\vec{n}\vec{n}\vec{n}\Big]\, d\mathcal{H}^{n-1}. \end{aligned}$$

Since h is $(\mathcal{H}^{n-1}+\lambda_M)$-summable on $\partial_* M$ and $|\mathsf{C}_m| \leqslant h$, $|\operatorname{div} \mathsf{C}_m| \leqslant h$, $|\nabla^s \mathsf{C}_m| \leqslant h$, we get (16) by Lebesgue Theorem. □

Remark 8. If $\mathsf{C} \in L^\infty_{loc}(\Omega; \mathrm{Sym}_2)$, then the previous theorem holds on *every* set in $\mathcal{M}_{h\eta}$ with curvature measure. Indeed, in that case the function h_0 of Lemma 4 can be taken in $\mathcal{L}^\infty_{loc,+}(\Omega)$, which is sufficient to apply Lebesgue Theorem to the term $\int_{\partial_* M} v\mathsf{C}_m \cdot \mathsf{U}\, d\lambda_M$.

In the same spirit as above, we show that the condition $\nabla^s \mathsf{C} \in L^1_{loc}(\Omega, \mathrm{Sym}_3)$ has a counterpart in terms of P.

Theorem 8. *We have that* $\nabla^s \mathsf{C} \in L^1_{loc}(\Omega, \mathrm{Sym}_3)$ *if and only if there exists* $h \in \mathcal{L}^1_{loc,+}(\Omega)$ *such that*

$$\left| \int_\alpha^\beta P\big(M_s^{(j)}, v\big)\, ds \right| \leqslant \int_{M_{\alpha,\beta}^{(j)}} \left(|v| + \left| \frac{\partial v}{\partial e_j} \right| \right) h \, d\mathcal{L}^n \tag{17}$$

for every $\mathcal{M}_G \subseteq \mathcal{A}$, $M \in \mathcal{M}_G$, $v \in C_c^\infty(M)$, $j = 1, \ldots, n$ *and* $a_j < \alpha < \beta < b_j$. *In this case, we have* $|\nabla^s \mathsf{C}| \leqslant \frac{3}{2}(h + |B - 2 \operatorname{div} \mathsf{C}|)$ *on* $\mathcal{L}^n$*-a.a. of* Ω.

Proof. Whenever $u \in \mathbb{R}^n$, $|u| = 1$, we have trivially

$$\nabla v \cdot \mathsf{C} u = \frac{\partial v}{\partial u} (\mathsf{C} u \cdot u) + \nabla v \cdot (\mathsf{C} u - (\mathsf{C} u \cdot u) u)\,.$$

Let $\mathcal{M}_G \subseteq \mathcal{A}$, $M \in \mathcal{M}_G$ and $v \in C_c^\infty(M)$; moreover, let $1 \leqslant j \leqslant n$ and $a_j < \alpha < \beta < b_j$. Since $\vec{n}^{M_s^{(j)}}(x) = e_j$ whenever $x \in \operatorname{bd} M_s^{(j)}$ and $v(x) \neq 0$, by combining (9) with the previous identity we obtain

$$\begin{aligned}
&\int_\alpha^\beta P\big(M_s^{(j)}, v\big)\, ds \\
&\quad = \int_{M_{\alpha,\beta}^{(j)}} \left[v (B - \operatorname{div} \mathsf{C}) \cdot e_j + \frac{\partial v}{\partial e_j} \mathsf{C} e_j \cdot e_j + \nabla v \cdot (\mathsf{C} e_j - (\mathsf{C} e_j \cdot e_j) e_j) \right] d\mathcal{L}^n \\
&\quad = \int_{M_{\alpha,\beta}^{(j)}} \left[\frac{\partial v}{\partial e_j} \mathsf{C} e_j \cdot e_j + v \big((B - 2 \operatorname{div} \mathsf{C}) \cdot e_j + (\nabla^s \mathsf{C}) e_j e_j e_j \big) \right] d\mathcal{L}^n,
\end{aligned}$$

where we applied the divergence theorem and considered the identity

$$\operatorname{div}((\mathsf{C} e_j \cdot e_j) e_j) = (\nabla^s \mathsf{C}) e_j e_j e_j\,.$$

Hence, if $\nabla^s \mathsf{C} \in L^1_{loc}(\Omega, \mathrm{Sym}_3)$ holds, then (17) is satisfied by choosing

$$h = |\mathsf{C}| + |B - 2 \operatorname{div} \mathsf{C}| + |\nabla^s \mathsf{C}|.$$

On the other hand, in view of Proposition 3 it is enough to prove that

$$\forall u \in \mathbb{R}^n, |u| = 1 : \quad ((\nabla \mathsf{C}) u u) \cdot u \in L^1_{loc}(\Omega). \tag{18}$$

Since $(B - \operatorname{div} \mathsf{C}) \in L^1_{loc}(\Omega, \mathbb{R}^n)$, the theorem is proved if one finds $h \in \mathcal{L}^1_{loc,+}(\Omega)$ such that

$$\left| \int_\Omega [v (B - \operatorname{div} \mathsf{C}) \cdot u + \nabla v \cdot (\mathsf{C} u - (\mathsf{C} u \cdot u) u)] d\mathcal{L}^n \right| \leqslant \int_\Omega |v| h \, d\mathcal{L}^n \tag{19}$$

for every $v \in C_c^\infty(\Omega)$ and $u \in \mathbb{R}^n$ with $|u| = 1$. Fix such a u and let

$$G = \big(x_0, (e_1, \ldots, e_n), \widehat{G}\big)$$

be a full grid such that $\mathcal{M}_G \subseteq \mathcal{A}$ and $u = e_1$. Let $v \in C_c^\infty(\Omega)$ and $M \in \mathcal{M}_G$ be such that $v(x) = 0$ whenever $x \notin M$. Then there exist $a_1, b_1, \ldots, a_n, b_n \in \widehat{G}$ with

$$M = \big\{ x \in \mathbb{R}^n : a_j < (x - x_0) \cdot e_j < b_j,\ j = 1, \ldots, n \big\}.$$

We define a sequence $(v_m) \subseteq L^\infty(M)$ in this way: let

$$s_{m,k} = a_1 + k\frac{b_1 - a_1}{m}, \qquad k = 1, \dots, m\,;$$

we set

$$v_m(x_1, \dots, x_n) = \begin{cases} v(s_{m,k}, x_2, \dots, x_n) & \text{for } s_{m,k} \leqslant x_1 < s_{m,k+1}, k = 1, \dots, m \\ 0 & \text{otherwise.} \end{cases}$$

Then for $\mathcal{L}^n$-a.e. $x \in M$ we have

$$\frac{\partial v_m}{\partial e_1}(x) = 0, \quad \lim_{m\to\infty} v_m(x) = v(x), \quad \lim_{m\to\infty} \frac{\partial v_m}{\partial e_i}(x) = \frac{\partial v}{\partial e_i}(x), \quad i \neq 1.$$

By (17) it follows that

$$\left| \int_\Omega [v(B - \operatorname{div} \mathsf{C}) \cdot u + \nabla v \cdot (\mathsf{C}u - (\mathsf{C}u \cdot u)u)] d\mathcal{L}^n \right|$$
$$= \left| \lim_m \sum_{k=0}^{m-1} \int_{M^{(1)}_{s_{m,k}, s_{m,k+1}}} v_m(B - \operatorname{div} \mathsf{C}) \cdot e_1 + \nabla v_m \cdot (\mathsf{C}e_1 - (\mathsf{C}e_1 \cdot e_1)e_1) d\mathcal{L}^n \right|$$
$$= \left| \lim_m \sum_{k=0}^{m-1} \int_{s_{m,k}}^{s_{m,k+1}} P\big(M^{(1)}_s, v_m\big) ds \right| \leqslant \lim_m \sum_{k=0}^{m-1} \int_{M^{(1)}_{s_{m,k}, s_{m,k+1}}} |v_m| h \, d\mathcal{L}^n = \int_\Omega |v| h \, d\mathcal{L}^n,$$

where we considered that v_m is smooth for $s_k < x_j < s_{k+1}$.

By combining (19) with Proposition 3, we get also the estimate on $|\nabla^s \mathsf{C}|$. □

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