

Asymptotics of Solutions for fully Nonlinear Elliptic Problems at Nearly Critical Growth

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Abstract. In this paper we deal with the study of limits of solutions of a class of fully nonlinear elliptic problems at nearly critical growth. By means of P.L. Lions' concentration-compactness principle, we prove an alternative result for the existence of non-trivial solutions of the limit problem.

Keywords: *Concentration-compactness principle, critical exponent, best Sobolev constant, fully nonlinear elliptic problems*

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1. Introduction

Let Ω be a bounded domain of \mathbb{R}^n , $1 < p < n$ and $p^* = \frac{np}{n-p}$. In 1989 Guedda and Veron [10] proved that the p -Laplacian problem at critical growth

$$\left. \begin{array}{ll} -\Delta_p u = u^{p^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array} \right\} \quad (*)$$

has no non-trivial solution $u \in W_0^{1,p}(\Omega)$ if the domain Ω is star-shaped. As known, this non-existence result is due to the failure of compactness for the critical Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, which causes a loss of global Palais-Smale condition for the functional associated with problem (*). On the other hand, if for instance one considers annular domains

$$\Omega_{r_1, r_2} = \{x \in \mathbb{R}^n : 0 < r_1 < |x| < r_2\},$$

then the radial embedding

$$W_{0,rad}^{1,p}(\Omega_{r_1, r_2}) \hookrightarrow L^q(\Omega_{r_1, r_2})$$

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is compact for each $q < +\infty$ and one can find a non-trivial radial solution of problem (*) (see [11]). In particular, the existence of non-trivial solutions of problem (*) depends also on the topology of the domain. In the case $p = 2$, the problem

$$\left. \begin{aligned} -\Delta u &= u^{(n+2)/(n-2)} && \text{in } \Omega \\ u &> 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \right\} \quad (**)$$

has been deeply studied and existence results have been obtained provided that Ω satisfies suitable assumptions. In the striking paper [3], Bahri and Coron have proved that if Ω has a non-trivial topology, i.e. if Ω has a non-trivial homology in some positive dimension, then problem (**) always admits a non-trivial solution.

On the other hand, Dancer [8] constructed for each $n \geq 3$ a contractible domain Ω_n , homeomorphic to a ball, for which problem (**) has a non-trivial solution. Therefore, we see how the existence of non-trivial solutions of problem (**) is related to the shape of the domain and not just to the topology. See also [15] and references therein for more recent existence and multiplicity results.

We remark that, to the authors' knowledge, this kind of achievements are not known when $p \neq 2$. In our opinion, one of the main difficulties is the fact that, differently from the case $p = 2$, it is not proven that all positive smooth solutions of the equation $-\Delta_p u = u^{p^*-1}$ in \mathbb{R}^n are Talenti's radial functions, which attain the best Sobolev constant (see Proposition 3.1).

Now, there is a second approach in the study of problem (*), which in general does not require any geometrical or topological assumption on Ω , namely to investigate the asymptotic behaviour of solutions u_ε of problems with nearly critical growth

$$\left. \begin{aligned} -\Delta_p u &= |u|^{p^*-2-\varepsilon} u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \right\} \quad (***)$$

as $\varepsilon \rightarrow 0$. If Ω is a ball and $p = 2$, Atkinson and Peletier [2] showed in 1987 the blow-up of a sequence of radial solutions. The extension to the case $p \neq 2$ was achieved by Knaap and Peletier [12] in 1989. On a general bounded domain, instead, the study of limits of solutions of problem (***) was performed by Garcia Azorero and Peral Alonso [9] around 1992.

Let now $\varepsilon > 0$ and consider the general class of Euler-Lagrange equations with nearly critical growth

$$\left. \begin{aligned} -\operatorname{div}(\nabla_\xi \mathcal{L}(x, u, \nabla u)) + D_s \mathcal{L}(x, u, \nabla u) &= |u|^{p^*-2-\varepsilon} u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \right\} \quad (\mathcal{P}_\varepsilon)$$

associated with the functional $f_\varepsilon : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ given by

$$f_\varepsilon(u) = \int_\Omega \mathcal{L}(x, u, \nabla u) \, dx - \frac{1}{p^*-\varepsilon} \int_\Omega |u|^{p^*-\varepsilon} \, dx. \quad (1)$$

As noted in [18], in general these functionals are not even locally Lipschitz under natural growth assumptions. Nevertheless, via techniques of non-smooth critical point theory (see [18] and references therein) it can be shown that for each $\varepsilon > 0$ problem $(\mathcal{P}_\varepsilon)$ admits a non-trivial solution $u_\varepsilon \in W_0^{1,p}(\Omega)$.

Let u_ε be a solution of problem $(\mathcal{P}_\varepsilon)$. The main goal of this paper is to prove that if the weak limit of $(|\nabla u_\varepsilon|^p)_{\varepsilon>0}$ has no blow-up points in Ω , then the limit problem

$$\left. \begin{aligned} -\operatorname{div}(\nabla_\xi \mathcal{L}(x, u, \nabla u)) + D_s \mathcal{L}(x, u, \nabla u) &= |u|^{p^*-2}u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \right\} \quad (\mathcal{P}_0)$$

has a non-trivial solution (the weak limit of $(u_\varepsilon)_{\varepsilon>0}$), provided that $f_\varepsilon(u_\varepsilon) \rightarrow c$ with

$$\frac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}} < c < 2 \frac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}} \tag{2}$$

where $\nu > 0$ and $\gamma \in (0, p^* - p)$ will be introduced later on. In our framework, (2) plays the role of a generalized second critical energy range (if $\gamma = 0$ and $\nu = 1$, one finds the usual range $\frac{S^{n/p}}{n} < c < 2 \frac{S^{n/p}}{n}$ for problem (***)).

The plan of the paper is as follows:

In Section 2 we shall state our main results. Section 3 contains some preliminary lemmas, namely the lower bounds of the non-vanishing Dirac masses and of the non-trivial weak limits. In Section 4 we prove our main results. In Section 5 we see that at the mountain pass levels the sequence $(u_\varepsilon)_{\varepsilon>0}$ blows up. Finally, Section 6 contains a non-existence result.

2. The main results

Let Ω be any bounded domain of \mathbb{R}^n . In the following, the space $W_0^{1,p}(\Omega)$ will be endowed with the standard norm $\|u\|_{1,p}^p = \int_\Omega |\nabla u|^p dx$ and $\|\cdot\|_p$ will denote the usual norm of $L^p(\Omega)$.

Assume that $\mathcal{L} : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is measurable in x for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$, of class C^1 in (s, ξ) a.e. in Ω , that $\mathcal{L}(x, s, \cdot)$ is strictly convex and $\mathcal{L}(x, s, 0) = 0$. Moreover, assume the following:

(A₁) There exists $b_0 > 0$ such that

$$\mathcal{L}(x, s, \xi) \leq b_0 |s|^p + b_0 |\xi|^p \tag{3}$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

(A₂) There exists $b_1 > 0$ such that for each $\delta > 0$ there exists $a_\delta \in L^1(\Omega)$ with

$$|D_s \mathcal{L}(x, s, \xi)| \leq a_\delta(x) + \delta |s|^{p^*} + b_1 |\xi|^p \tag{4}$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$. Moreover, there exist $a_1 \in L^{p'}(\Omega)$ and $\nu > 0$ such that

$$|\nabla_\xi \mathcal{L}(x, s, \xi)| \leq a_1(x) + b_1 |s|^{\frac{p^*}{p'}} + b_1 |\xi|^{p-1}, \tag{5}$$

$$\nabla_{\xi} \mathcal{L}(x, s, \xi) \cdot \xi \geq \nu |\xi|^p \tag{6}$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

(\mathcal{A}_3) For a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$,

$$D_s \mathcal{L}(x, s, \xi) s \geq 0 \tag{7}$$

and there exists $\gamma \in (0, p^* - p)$ such that

$$(\gamma + p) \mathcal{L}(x, s, \xi) - \nabla_{\xi} \mathcal{L}(x, s, \xi) \cdot \xi - D_s \mathcal{L}(x, s, \xi) s \geq 0 \tag{8}$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

Remark 2.1. The growth conditions of (\mathcal{A}_1) and (\mathcal{A}_2) and the assumptions in (\mathcal{A}_3) are natural in the fully nonlinear setting and were considered in [18], and in a stronger form in [1, 16] (see also Remark 6.2). Notice that when \mathcal{L} is p -homogeneous with respect to ξ , then condition (8) becomes $D_s \mathcal{L}(x, s, \xi) s \leq \gamma \mathcal{L}(x, s, \xi)$ for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

As an example, taking $A \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $A' \in L^\infty(\mathbb{R})$, $A(s) \geq \nu$ and $\gamma A(s) \geq A'(s)s \geq 0$ for each $s \in \mathbb{R}$, the class of Lagrangians

$$\mathcal{L}(x, s, \xi) = \frac{1}{p} A(s) |\xi|^p$$

satisfies all the previous requirements. For instance $(\gamma^{-1} + \arctan(s^2)) |\xi|^p / p$ belongs to this class for each $\gamma \in (0, p^* - p)$.

Remark 2.2. We stress that although as noted in the introduction f_ε fails to be differentiable, one may compute the derivatives along the L^∞ -directions, namely

$$f'_\varepsilon(u)(\varphi) = \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi \, dx + \int_{\Omega} D_s \mathcal{L}(x, u, \nabla u) \varphi \, dx - \int_{\Omega} |u|^{p^* - 2 - \varepsilon} u \varphi \, dx.$$

for all $u \in W_0^{1,p}(\Omega)$ and for all $\varphi \in W_0^{1,p} \cap L^\infty(\Omega)$.

The following is a general property due to Brézis and Browder [5].

Proposition 2.3. *Let $u, v \in W_0^{1,p}(\Omega)$ be such that $D_s \mathcal{L}(x, u, \nabla u) v \geq 0$ and*

$$\langle w, \varphi \rangle = \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi \, dx + \int_{\Omega} D_s \mathcal{L}(x, u, \nabla u) \varphi \, dx \quad (\varphi \in C_c^\infty(\Omega))$$

with $w \in L^1_{loc}(\Omega) \cap W^{-1,p'}(\Omega)$. Then $D_s \mathcal{L}(x, u, \nabla u) v \in L^1(\Omega)$ and

$$\langle w, v \rangle = \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \cdot \nabla v \, dx + \int_{\Omega} D_s \mathcal{L}(x, u, \nabla u) v \, dx.$$

From now on, by solution of problem (\mathcal{P}_ε) we shall always mean weak solution, namely $f'_\varepsilon(u_\varepsilon) = 0$ in the sense of distributions. The next lemma is our starting point.

Lemma 2.4. For each $\varepsilon > 0$, $(\mathcal{P}_\varepsilon)$ admits a non-trivial solution $u_\varepsilon \in W_0^{1,p}(\Omega)$.

Proof. See [18: Theorem 1.1] ■

We point out that, in our general framework, the technical aspects in the verification of the Palais-Smale condition for f_ε are, in our opinion, interesting and not trivial.

Note that since $\mathcal{L}(x, s, 0) = 0$, in view of (6) one obtains

$$\mathcal{L}(x, s, \xi) \geq \frac{\nu}{p} |\xi|^p \tag{9}$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

Lemma 2.5. Let $(u_\varepsilon)_{\varepsilon>0} \subset W_0^{1,p}(\Omega)$ be a sequence of solutions of problem $(\mathcal{P}_\varepsilon)$ such that $\lim_{\varepsilon \rightarrow 0} f_\varepsilon(u_\varepsilon) < +\infty$. Then $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$.

Proof. For each $\varepsilon > 0$ we have $f'_\varepsilon(u_\varepsilon)(\varphi) = 0$ for each $\varphi \in C_c^\infty(\Omega)$. On the other hand, taking into account (7), by Proposition 2.3 one can also take $\varphi = u_\varepsilon$. Therefore, in view of (8) and (9) one obtains

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} f_\varepsilon(u_\varepsilon) &= \lim_{\varepsilon \rightarrow 0} \left(f_\varepsilon(u_\varepsilon) - \frac{1}{p^* - \varepsilon} f'_\varepsilon(u_\varepsilon)(u_\varepsilon) \right) \\ &= \lim_{\varepsilon \rightarrow 0} \left(\int_\Omega \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) dx \right. \\ &\quad \left. - \frac{1}{p^* - \varepsilon} \int_\Omega \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon dx \right. \\ &\quad \left. - \frac{1}{p^* - \varepsilon} \int_\Omega D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon dx \right) \\ &\geq \lim_{\varepsilon \rightarrow 0} \frac{p^* - p - \varepsilon - \gamma}{p^* - \varepsilon} \int_\Omega \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) dx \\ &\geq \frac{p^* - p - \gamma}{pp^*} \nu \lim_{\varepsilon \rightarrow 0} \int_\Omega |\nabla u_\varepsilon|^p dx. \end{aligned}$$

In particular, $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$ ■

Let us now recall the classical P.L. Lions' concentration-compactness principle

Lemma 2.6. Let $(u_\varepsilon)_{\varepsilon>0} \subset W_0^{1,p}(\Omega)$ be bounded and let u be its weak limit. Then there exist two bounded positive measures μ and σ such that

$$|\nabla u_\varepsilon|^p \rightharpoonup \mu, \quad |u_\varepsilon|^{p^*} \rightharpoonup \sigma \quad (\text{in the sense of measures}) \tag{10}$$

$$\mu \geq |\nabla u|^p + \sum_{j=1}^\infty \mu_j \delta_{x_j} \quad (\mu_j \geq 0) \tag{11}$$

$$\sigma = |u|^{p^*} + \sum_{j=1}^\infty \sigma_j \delta_{x_j} \quad (\sigma_j \geq 0) \tag{12}$$

$$\mu_j \geq S \sigma_j^{\frac{p}{p^*}} \tag{13}$$

where δ_{x_j} denotes the Dirac measure at $x_j \in \bar{\Omega}$ and S denotes the best Sobolev constant for the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.

Proof. See e.g. [13, Lemma I.1] or [14] ■

Under assumptions $(\mathcal{A}_1) - (\mathcal{A}_3)$, the following is our main result.

Theorem 2.7. *Assume that $(u_\varepsilon)_{\varepsilon>0} \subset W_0^{1,p}(\Omega)$ is a sequence of solutions of problem $(\mathcal{P}_\varepsilon)$ such that $f_\varepsilon(u_\varepsilon) \rightarrow c$ and*

$$\frac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}} < c < 2 \frac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}}.$$

Then $\mu_j = 0$ for $j \geq 2$ and the following alternative holds:

- (a) $\mu_1 = 0$ and u is a non-trivial solution of problem (\mathcal{P}_0) .
- (b) $\mu_1 \neq 0$ and $u = 0$.

This result extends [9: Theorem 9] to a class of fully nonlinear elliptic problems.

Theorem 2.8. *Let $(u_\varepsilon)_{\varepsilon>0}$ be any sequence of solutions of problem $(\mathcal{P}_\varepsilon)$ with*

$$\lim_{\varepsilon \rightarrow 0} f_\varepsilon(u_\varepsilon) = \frac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}}.$$

Then $u = 0$.

As we shall see in Section 5, this is also the behaviour when one considers critical levels of mountain-pass type.

3. The weak limit of $(u_\varepsilon)_{\varepsilon>0}$

Let us briefly summarize the main properties of the best Sobolev constant [19].

Proposition 3.1. *Let $1 < p < n$ and S be the best Sobolev constant, i.e.*

$$S = \inf \left\{ \int_{\Omega} |\nabla u|^p dx : u \in W_0^{1,p}(\Omega) \text{ with } \int_{\Omega} |u|^{p^*} dx = 1 \right\}. \tag{14}$$

Then the following facts hold:

- (a) S is independent on $\Omega \subset \mathbb{R}^n$.
- (b) The infimum (14) is never achieved on bounded domains $\Omega \subset \mathbb{R}^n$.
- (c) The infimum (14) is achieved if $\Omega = \mathbb{R}^n$ by the family of functions on \mathbb{R}^n

$$T_{\delta,x_0}(x) = \left(n\delta \left(\frac{n-p}{p-1} \right)^{p-1} \right)^{\frac{n-p}{p^2}} \left(\delta + |x - x_0|^{\frac{p}{p-1}} \right)^{-\frac{n-p}{p}} \tag{15}$$

with $\delta > 0$ and $x_0 \in \mathbb{R}^n$. Moreover, T_{δ,x_0} is a solution of $-\Delta_p u = u^{p^*-1}$ on \mathbb{R}^n .

The next result establishes uniform lower bounds for the Dirac masses.

Lemma 3.2. *If $\mu_j \neq 0$, then $\sigma_j \geq \nu^{\frac{n}{p}} S^{\frac{n}{p}}$ and $\mu_j \geq \nu^{\frac{n}{p^*}} S^{\frac{n}{p^*}}$.*

Proof. Let $x_j \in \bar{\Omega}$ the point which supports the Dirac measure of coefficient σ_j . Denoting with $B(x_j, \delta)$ the open ball of center x_j and radius $\delta > 0$, we can consider a function $\psi_\delta \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq \psi_\delta \leq 1$, $|\nabla \psi_\delta| \leq \frac{2}{\delta}$, $\psi_\delta(x) = 1$ if $x \in B(x_j, \delta)$ and $\psi_\delta(x) = 0$ if $x \notin B(x_j, 2\delta)$. By Proposition 2.3 we have

$$\begin{aligned} 0 &= f'_\varepsilon(u_\varepsilon)(\psi_\delta u_\varepsilon) \\ &= \int_\Omega u_\varepsilon \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi_\delta \, dx + \int_\Omega \psi_\delta \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx \\ &\quad + \int_\Omega \psi_\delta D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \, dx - \int_\Omega |u_\varepsilon|^{p^* - \varepsilon} \psi_\delta \, dx. \end{aligned} \tag{16}$$

Applying Hölder inequality and (5) to the first term of the decomposition and keeping into account that $(u_\varepsilon)_{\varepsilon > 0}$ is bounded in $W_0^{1,p}(\Omega)$, one finds constants $c_1, c_2 > 0$ such that

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \left| \int_\Omega u_\varepsilon \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi_\delta \, dx \right| \\ &\leq \left(\int_{B(x_j, 2\delta)} |a_1|^{\frac{p}{p-1}} \, dx \right)^{\frac{p-1}{p}} \left(\int_{B(x_j, 2\delta)} |u|^{p^*} \, dx \right)^{\frac{1}{p^*}} \left(\int_{B(x_j, 2\delta)} |\nabla \psi_\delta|^n \, dx \right)^{\frac{1}{n}} \\ &\quad + b_1 \left(\int_{B(x_j, 2\delta)} |u|^{p^*} \, dx \right)^{\frac{n-1}{n}} \left(\int_{B(x_j, 2\delta)} |\nabla \psi_\delta|^n \, dx \right)^{\frac{1}{n}} \\ &\quad + \tilde{b}_1 \left(\int_{B(x_j, 2\delta)} |u|^{p^*} \, dx \right)^{\frac{1}{p^*}} \left(\int_{B(x_j, 2\delta)} |\nabla \psi_\delta|^n \, dx \right)^{\frac{1}{n}} \\ &\leq c_1 \left(\int_{B(x_j, 2\delta)} |u|^{p^*} \, dx \right)^{\frac{1}{p^*}} + c_2 \left(\int_{B(x_j, 2\delta)} |u|^{p^*} \, dx \right)^{\frac{n-1}{n}} = \beta_\delta \end{aligned} \tag{17}$$

with $\beta_\delta \rightarrow 0$ as $\delta \rightarrow 0$. Then, taking into account (6) and (7) one has

$$\begin{aligned} 0 &\geq -\beta_\delta + \lim_{\varepsilon \rightarrow 0} \nu \int_\Omega |\nabla u_\varepsilon|^p \psi_\delta \, dx - \lim_{\varepsilon \rightarrow 0} \mathcal{L}^n(\Omega)^{\frac{\varepsilon}{p^*}} \left(\int_\Omega |u_\varepsilon|^{p^*} \psi_\delta \, dx \right)^{\frac{p^* - \varepsilon}{p^*}} \\ &\geq -\beta_\delta + \nu \int_\Omega \psi_\delta \, d\mu - \int_\Omega \psi_\delta \, d\sigma. \end{aligned}$$

Letting $\delta \rightarrow 0$, it results $\nu \mu_j \leq \sigma_j$. By means of (13) the proof is complete \blacksquare

In the next result we obtain uniform lower bounds for the non-zero weak limits.

Lemma 3.3. *If $u \neq 0$, then $\int_\Omega |\nabla u|^p \, dx > \nu^{\frac{n}{p^*}} S^{\frac{n}{p^*}}$ and $\int_\Omega |u|^{p^*} \, dx > \nu^{\frac{n}{p}} S^{\frac{n}{p}}$.*

Proof. By Lemma 3.2 we may assume that μ has at most r Dirac masses μ_1, \dots, μ_r at x_1, \dots, x_r , respectively. Let now $0 < \delta < \frac{1}{4} \min\{|x_i - x_j| : i \neq j\}$

and $\psi_\delta \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \leq \psi_\delta \leq 1$, $|\nabla\psi_\delta| \leq \frac{2}{\delta}$, $\psi_\delta(x) = 1$ if $x \in B(x_j, \delta)$ and $\psi_\delta(x) = 0$ if $x \notin B(x_j, 2\delta)$. Taking into account (7), for each $\varepsilon, \delta > 0$ we have

$$\int_{\Omega} D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon (1 - \psi_\delta) dx \geq 0.$$

Then, since one can choose $(1 - \psi_\delta)u_\varepsilon$ as test, by (6) one obtains

$$\begin{aligned} 0 &= f'_\varepsilon(u_\varepsilon)((1 - \psi_\delta)u_\varepsilon) \\ &= \int_{\Omega} \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon (1 - \psi_\delta) dx \\ &\quad - \int_{\Omega} u_\varepsilon \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi_\delta dx \\ &\quad + \int_{\Omega} D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon (1 - \psi_\delta) dx \\ &\quad - \int_{\Omega} |u_\varepsilon|^{p^* - \varepsilon} (1 - \psi_\delta) dx \\ &\geq \nu \int_{\Omega} |\nabla u_\varepsilon|^p (1 - \psi_\delta) dx \\ &\quad - \int_{\Omega} u_\varepsilon \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi_\delta dx \\ &\quad - \mathcal{L}^n(\Omega)^{\frac{\varepsilon}{p^*}} \left(\int_{\Omega} |u_\varepsilon|^{p^*} (1 - \psi_\delta) dx \right)^{\frac{p^* - \varepsilon}{p^*}}. \end{aligned} \tag{18}$$

On the other hand, arguing as for (17), one obtains

$$\lim_{\varepsilon \rightarrow 0} \left| \int_{\Omega} u_\varepsilon \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi_\delta dx \right| \leq \beta_\delta \tag{19}$$

for each $\delta > 0$. Now, it results

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla u_\varepsilon|^p (1 - \psi_\delta) dx &= \int_{\Omega} (1 - \psi_\delta) d\mu \\ &\geq \int_{\Omega} |\nabla u|^p (1 - \psi_\delta) dx + \sum_{j=1}^r \mu_j (1 - \psi_\delta(x_j)) \\ &= \int_{\Omega} |\nabla u|^p dx + o(1) \end{aligned} \tag{20}$$

as $\delta \rightarrow 0$ and

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} |u_\varepsilon|^{p^*} (1 - \psi_\delta) dx &= \int_{\Omega} (1 - \psi_\delta) d\sigma \\ &= \int_{\Omega} |u|^{p^*} (1 - \psi_\delta) dx + \sum_{j=1}^r \sigma_j (1 - \psi_\delta(x_j)) \\ &= \int_{\Omega} |u|^{p^*} dx + o(1) \end{aligned} \tag{21}$$

as $\delta \rightarrow 0$. Therefore, in view of (19) - (21), by letting $\delta \rightarrow 0$ and $\varepsilon \rightarrow 0$ in (18) one concludes that

$$\nu \int_{\Omega} |\nabla u|^p dx \leq \int_{\Omega} |u|^{p^*} dx. \tag{22}$$

As Ω is bounded, by Proposition 3.1/(b) one has $\int_{\Omega} |\nabla u|^p dx > S \left(\int_{\Omega} |u|^{p^*} dx \right)^{\frac{p}{p^*}}$ which combined with (22) yields the assertion ■

Lemma 3.4. *Let $(u_{\varepsilon})_{\varepsilon>0} \subset W_0^{1,p}(\Omega)$ be a sequence of solutions of problem $(\mathcal{P}_{\varepsilon})$ and let u be its weak limit. Then u is a solution of problem (\mathcal{P}_0) .*

Proof. For each $\varepsilon > 0$ and $\varphi \in C_c^{\infty}(\Omega)$,

$$\int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla \varphi dx + \int_{\Omega} D_s \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \varphi dx = \int_{\Omega} |u_{\varepsilon}|^{p^*-2-\varepsilon} u_{\varepsilon} \varphi dx. \tag{23}$$

Since $(u_{\varepsilon})_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$, up to a subsequence, u satisfies

$$\left. \begin{aligned} \nabla u_{\varepsilon} &\rightharpoonup \nabla u && \text{in } L^p(\Omega) \\ u_{\varepsilon} &\rightarrow u && \text{in } L^p(\Omega) \\ u_{\varepsilon}(x) &\rightarrow u(x) && \text{for a.e. } x \in \Omega \end{aligned} \right\}$$

as $\varepsilon \rightarrow 0$. Moreover, by [7: Theorem 1], up to a further subsequence, we have $\nabla u_{\varepsilon}(x) \rightarrow \nabla u(x)$ for a.e. $x \in \Omega$. Therefore, in view of (5) one deduces that

$$\nabla_{\xi} \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \rightharpoonup \nabla_{\xi} \mathcal{L}(x, u, \nabla u) \quad \text{in } L^{p'}(\Omega, \mathbb{R}^n). \tag{24}$$

By (4) - (6) one finds a constant $M > 0$ such that for each $\delta > 0$

$$|D_s \mathcal{L}(x, s, \xi)| \leq M \nabla_{\xi} \mathcal{L}(x, s, \xi) \cdot \xi + a_{\delta}(x) + \delta |s|^{p^*} \tag{25}$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$. If we test equation (23) with the functions

$$\varphi_{\varepsilon} = \varphi \exp\{-Mu_{\varepsilon}^+\} \quad (\varepsilon > 0)$$

where $0 \leq \varphi \in W_0^{1,p} \cap L^{\infty}(\Omega)$, we obtain

$$\begin{aligned} &\int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla \varphi \exp\{-Mu_{\varepsilon}^+\} dx \\ &\quad - \int_{\Omega} |u_{\varepsilon}|^{p^*-2-\varepsilon} u_{\varepsilon} \varphi \exp\{-Mu_{\varepsilon}^+\} dx \\ &+ \int_{\Omega} \left[D_s \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) - M \nabla_{\xi} \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon}^+ \right] \varphi \exp\{-Mu_{\varepsilon}^+\} dx = 0. \end{aligned}$$

Since by inequalities (7) and (25) for each $\varepsilon > 0$ and $\delta > 0$ we have

$$\left[D_s \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) - M \nabla_{\xi} \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon}^+ \right] \varphi \exp\{-Mu_{\varepsilon}^+\} - \delta |u_{\varepsilon}|^{p^*} \leq a_{\delta}(x),$$

arguing as in [18: Theorem 3.4] one obtains

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} \left[D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) - M \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon^+ \right] \varphi \exp\{-Mu_\varepsilon^+\} dx \\ & \leq \int_{\Omega} \left[D_s \mathcal{L}(x, u, \nabla u) - M \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u^+ \right] \varphi \exp\{-Mu^+\} dx. \end{aligned}$$

Therefore, taking into account (24) and since as $\varepsilon \rightarrow 0$

$$\int_{\Omega} |u_\varepsilon|^{p^* - 2 - \varepsilon} u_\varepsilon \varphi dx \rightarrow \int_{\Omega} |u|^{p^* - 2} u \varphi dx$$

for each $0 \leq \varphi \in W_0^{1,p} \cap L^\infty(\Omega)$, one may conclude that

$$\begin{aligned} & \int_{\Omega} \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi \exp\{-Mu^+\} dx \\ & \quad - \int_{\Omega} |u|^{p^* - 2} u \varphi \exp\{-Mu^+\} dx \tag{26} \\ & + \int_{\Omega} \left[D_s \mathcal{L}(x, u, \nabla u) - M \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u^+ \right] \varphi \exp\{-Mu^+\} dx \geq 0 \end{aligned}$$

for each $0 \leq \varphi \in W_0^{1,p} \cap L^\infty(\Omega)$. Testing now (26) with $\varphi_k = \varphi \vartheta \left(\frac{u}{k}\right) \exp\{Mu^+\}$ where $0 \leq \varphi \in C_c^\infty(\Omega)$ and ϑ is smooth, $\vartheta = 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and $\vartheta = 0$ in $(-\infty, -1] \cup [1, +\infty)$, it follows that

$$\begin{aligned} & \int_{\Omega} \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi_k \exp\{-Mu^+\} dx \\ & \quad - \int_{\Omega} |u|^{p^* - 2} u \varphi \vartheta \left(\frac{u}{k}\right) dx \\ & + \int_{\Omega} \left[D_s \mathcal{L}(x, u, \nabla u) - M \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u^+ \right] \varphi \vartheta \left(\frac{u}{k}\right) dx \geq 0 \end{aligned}$$

which, arguing again as in [18: Theorem 3.4], yields as $k \rightarrow +\infty$

$$\int_{\Omega} \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla \varphi dx + \int_{\Omega} D_s \mathcal{L}(x, u, \nabla u) \varphi dx \geq \int_{\Omega} |u|^{p^* - 2} u \varphi dx$$

for each $0 \leq \varphi \in C_c^\infty(\Omega)$. Analogously, testing with $\varphi_\varepsilon = \varphi \exp\{-Mu_\varepsilon^-\}$, one obtains the opposite inequality, i.e. u is a solution of problem (\mathcal{P}_0) ■

4. Proofs of the main results

Let now $(u_\varepsilon)_{\varepsilon>0}$ be a sequence of solutions of problem $(\mathcal{P}_\varepsilon)$ with $f_\varepsilon(u_\varepsilon) \rightarrow c$ and

$$\frac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}} < c < 2\frac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}}. \tag{27}$$

Then there exist a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ and two bounded positive measures μ and σ verifying (10) - (13).

Proof of Theorem 2.7. Let us first show that there exists at most one j such that $\mu_j \neq 0$. Suppose that $\mu_j \neq 0$ for $j = 1, \dots, r$; in view of Lemma 3.2 one has $\mu_j \geq \nu \frac{n}{p^*} S^{\frac{n}{p}}$. Following the proof of Lemma 2.5, we obtain

$$\begin{aligned} c &= \lim_{\varepsilon \rightarrow 0} f_\varepsilon(u_\varepsilon) \\ &\geq \frac{p^*-p-\gamma}{pp^*} \nu \lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla u_\varepsilon|^p dx \\ &\geq \frac{p^*-p-\gamma}{pp^*} \nu \int_{\Omega} d\mu \\ &\geq \frac{p^*-p-\gamma}{pp^*} \nu \sum_{j=1}^r \mu_j \\ &\geq r \frac{p^*-p-\gamma}{pp^*} (\nu S)^{\frac{n}{p}}. \end{aligned}$$

Taking into account (27) one has

$$2\frac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}} > c \geq r \frac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}},$$

hence $r \leq 1$. Now, arguing again as in Lemma 2.5 one obtains

$$\begin{aligned} 2\frac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}} > c &= \lim_{\varepsilon \rightarrow 0} f_\varepsilon(u_\varepsilon) \\ &\geq \frac{p^*-p-\gamma}{pp^*} \nu \lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla u_\varepsilon|^p dx \\ &\geq \frac{p^*-p-\gamma}{pp^*} \left(\nu \int_{\Omega} |\nabla u|^p dx + \nu \mu_1 \right). \end{aligned}$$

If both summands were non-zero, by Lemmas 3.2 and 3.3 we would obtain

$$\begin{aligned} \nu \int_{\Omega} |\nabla u|^p dx &> (\nu S)^{\frac{n}{p}}, \\ \nu \mu_1 &\geq (\nu S)^{\frac{n}{p}} \end{aligned}$$

and thus a contradiction. Vice versa, let us assume that $u = 0$ and $\mu_1 = 0$. Let $0 \leq \psi \in C_c^1(\Omega)$. By testing our equation with ψu_ε and using Hölder inequality, one

gets

$$\begin{aligned}
 & \int_{\Omega} u_{\varepsilon} \nabla_{\xi} \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla \psi \, dx \\
 & + \int_{\Omega} \psi \nabla_{\xi} \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon} \, dx \\
 & + \int_{\Omega} D_s \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \psi u_{\varepsilon} \, dx = \int_{\Omega} |u_{\varepsilon}|^{p^* - \varepsilon} \psi \, dx \\
 & \leq \left(\int_{\Omega} |u_{\varepsilon}|^p \, dx \right)^{\frac{p^* - \varepsilon}{p^*}} \mathcal{L}^n(\Omega)^{\frac{\varepsilon}{p^*}}
 \end{aligned} \tag{28}$$

Since $(u_{\varepsilon})_{\varepsilon > 0}$ is bounded in $W_0^{1,p}(\Omega)$, by (5) there exists a constant $C > 0$ such that

$$\left| \int_{\Omega} u_{\varepsilon} \nabla_{\xi} \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla \psi \, dx \right| \leq C \|u_{\varepsilon}\|_p$$

which by $u_{\varepsilon} \rightarrow 0$ in $L^p(\Omega)$ yields

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_{\varepsilon} \nabla_{\xi} \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla \psi \, dx = 0.$$

Moreover, since by (7) we get

$$\int_{\Omega} D_s \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \psi u_{\varepsilon} \, dx \geq 0,$$

taking into account (6) and passing to the limit in (28) we get

$$\forall \psi \in C_c(\Omega) : \psi \geq 0 \implies \nu \int_{\Omega} \psi \, d\mu \leq \int_{\Omega} \psi \, d\sigma. \tag{29}$$

On the other hand, $\mu_1 = 0$ and $u = 0$ imply $\sigma = 0$. Then, since $\mu \geq 0$, by (29) we get $\mu = 0$. In particular, by (3), (6) and (7) one gets

$$\begin{aligned}
 c &= \lim_{\varepsilon \rightarrow 0} f_{\varepsilon}(u_{\varepsilon}) \\
 &= \lim_{\varepsilon \rightarrow 0} \left[\int_{\Omega} \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \, dx \right. \\
 &\quad \left. - \frac{1}{p^* - \varepsilon} \int_{\Omega} \nabla_{\xi} \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon} \, dx \right. \\
 &\quad \left. - \frac{1}{p^* - \varepsilon} \int_{\Omega} D_s \mathcal{L}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) u_{\varepsilon} \, dx \right] \\
 &\leq b_0 \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega} |u_{\varepsilon}|^p \, dx + \int_{\Omega} |\nabla u_{\varepsilon}|^p \, dx \right) \\
 &= b_0 \int_{\Omega} d\mu \\
 &= 0,
 \end{aligned}$$

which is not possible. Therefore, either $\mu_1 = 0$ and $u \neq 0$, or $\mu_1 \neq 0$ and $u = 0$ ■

Remark 4.1. If (27) is replaced by the $(k + 1)$ -th critical energy range

$$k \frac{p^* - p - \gamma}{pp^*} (\nu S)^{\frac{n}{p}} < c < (k + 1) \frac{p^* - p - \gamma}{pp^*} (\nu S)^{\frac{n}{p}}$$

for $k \in \mathbb{N}$, in a similar way one proves that $\mu_j = 0$ for any $j \geq k + 1$ and there holds:

- (a) If $\mu_j = 0$ for every $j \geq 1$, then u is a non-trivial solution of problem (\mathcal{P}_0) .
- (b) If $\mu_j \neq 0$ for every $1 \leq j \leq k$, then $u = 0$.

Remark 4.2. Let $f_0 : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ be the functional associated with problem (\mathcal{P}_0) and let $0 \neq u \in W_0^{1,p}(\Omega)$ be a solution of problem (\mathcal{P}_0) (obtained as weak limit of $(u_\varepsilon)_{\varepsilon > 0}$). Then

$$f_0(u) > \frac{p^* - p - \gamma}{pp^*} (\nu S)^{\frac{n}{p}}. \tag{30}$$

Indeed,

$$\begin{aligned} f_0(u) &= f_0(u) - \frac{1}{p^*} f_0'(u)(u) \\ &\geq \frac{p^* - p - \gamma}{p^*} \int_{\Omega} \mathcal{L}(x, u, \nabla u) \, dx \\ &\geq \frac{p^* - p - \gamma}{pp^*} \nu \int_{\Omega} |\nabla u|^p \, dx \end{aligned}$$

which yields (30) in view of Lemma 3.3. This, in some sense, explains why one chooses c greater than $\frac{p^* - p - \gamma}{pp^*} (\nu S)^{\frac{n}{p}}$ in Theorem 2.7.

Let now $(u_\varepsilon)_{\varepsilon > 0}$ be a sequence of solutions of problem $(\mathcal{P}_\varepsilon)$ with $f_\varepsilon(u_\varepsilon) \rightarrow c$ and

$$\lim_{\varepsilon \rightarrow 0} f_\varepsilon(u_\varepsilon) = \frac{p^* - p - \gamma}{pp^*} (\nu S)^{\frac{n}{p}}.$$

Proof of Theorem 2.8. Let us first note that

$$f_0(u) \leq \lim_{\varepsilon \rightarrow 0} f_\varepsilon(u_\varepsilon) + \frac{1}{p^*} \sum_{j=1}^{\infty} \sigma_j. \tag{31}$$

Indeed, taking into account that by [6: Theorem 3.4]

$$\int_{\Omega} \mathcal{L}(x, u, \nabla u) \, dx \leq \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \, dx,$$

(31) follows by combining Hölder inequality with (12).

Now assume by contradiction that $u \neq 0$. Then there exists $j_0 \in \mathbb{N}$ such that $\mu_{j_0} \neq 0$ and $\sigma_{j_0} \neq 0$, otherwise by Remark 4.2 and (31) we would get

$$\frac{p^* - p - \gamma}{pp^*} (\nu S)^{\frac{n}{p}} < f_0(u) \leq \lim_{\varepsilon \rightarrow 0} f_\varepsilon(u_\varepsilon) = \frac{p^* - p - \gamma}{pp^*} (\nu S)^{\frac{n}{p}}.$$

Arguing as in Lemma 2.5 and applying Lemma 3.2, we obtain

$$\begin{aligned} \frac{p^* - p - \gamma}{pp^*} (\nu S)^{\frac{n}{p}} &= \lim_{\varepsilon \rightarrow 0} f_\varepsilon(u_\varepsilon) \\ &\geq \frac{p^* - p - \gamma}{pp^*} \left(\nu \int_{\Omega} |\nabla u|^p \, dx + \nu \mu_{j_0} \right) \\ &\geq \frac{p^* - p - \gamma}{pp^*} \nu \int_{\Omega} |\nabla u|^p \, dx + \frac{p^* - p - \gamma}{pp^*} (\nu S)^{\frac{n}{p}} \end{aligned}$$

which implies $u = 0$ – a contradiction ■

5. Mountain-pass critical values

In this section, we shall investigate the asymptotics of (u_ε) in the case of critical levels of min-max type. We assume that \mathcal{L} is p -homogeneous with respect to ξ and satisfies a stronger assumption, i.e.

$$\mathcal{L}(x, s, \xi) \leq \frac{1}{p} |\xi|^p \tag{32}$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$. In particular, it results that $\nu \leq 1$. Let u_ε be a critical point of f_ε associated with the mountain pass level

$$c_\varepsilon = \inf_{\eta \in \mathcal{C}_\varepsilon} \max_{t \in [0,1]} f_\varepsilon(\eta(t)) \tag{33}$$

where

$$\mathcal{C}_\varepsilon = \left\{ \eta \in C([0, 1], W_0^{1,p}(\Omega)) : \eta(0) = 0 \text{ and } \eta(1) = w_\varepsilon \right\}$$

and $w_\varepsilon \in W_0^{1,p}(\Omega)$ is chosen in such a way that $f_\varepsilon(w_\varepsilon) < 0$.

Lemma 5.1. *The inequality $\lim_{\varepsilon \rightarrow 0} f_\varepsilon(u_\varepsilon) \leq \frac{1}{n} S^{\frac{n}{p}}$ holds.*

Proof. Let $x_0 \in \Omega$ and $\delta > 0$, and consider the functions T_{δ,x_0} as in (15). By Proposition 3.1/(c) one has

$$\|\nabla T_{\delta,x_0}\|_{p,\mathbb{R}^n}^p = \|T_{\delta,x_0}\|_{p^*,\mathbb{R}^n}^{p^*} = S^{\frac{n}{p}}.$$

Moreover, taking a function $\phi \in C_c^\infty(\Omega)$ with $0 \leq \phi \leq 1$ and $\phi = 1$ in a neighbourhood of x_0 and setting $v_\delta = \phi T_{\delta,x_0}$, it results

$$\left. \begin{aligned} \|\nabla v_\delta\|_p^p &= S^{\frac{n}{p}} + o(1) \\ \|v_\delta\|_{p^*}^{p^*} &= S^{\frac{n}{p}} + o(1) \end{aligned} \right\} \quad (\delta \rightarrow 0) \tag{34}$$

(see [10: Lemma 3.2]).

We want to prove that, for any $t \geq 0$,

$$\lim_{\varepsilon \rightarrow 0} f_\varepsilon(tv_\delta) \leq \frac{1}{n} S^{\frac{n}{p}} + o(1) \quad (\delta \rightarrow 0).$$

By (32) one has

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} f_\varepsilon(tv_\delta) &= t^p \int_\Omega \mathcal{L}(x, tv_\delta, \nabla v_\delta) dx - \lim_{\varepsilon \rightarrow 0} \frac{t^{p^*-\varepsilon}}{p^*-\varepsilon} \int_\Omega |v_\delta|^{p^*-\varepsilon} dx \\ &\leq \frac{t^p}{p} \int_\Omega |\nabla v_\delta|^p dx - \frac{t^{p^*}}{p^*} \int_\Omega |v_\delta|^{p^*} dx. \end{aligned}$$

Keeping into account (34) and the fact that $\frac{t^p}{p} - \frac{t^{p^*}}{p^*} \leq \frac{1}{n}$ for every $t \geq 0$, one gets

$$\lim_{\varepsilon \rightarrow 0} f_\varepsilon(tv_\delta) \leq \frac{t^p}{p} S^{\frac{n}{p}} - \frac{t^{p^*}}{p^*} S^{\frac{n}{p}} + o(1) \leq \frac{1}{n} S^{\frac{n}{p}} + o(1) \quad (\delta \rightarrow 0).$$

Now choose $t_0 > 0$ such that $f_\varepsilon(t_0 v_\delta) < 0$; by (33) we have

$$\lim_{\varepsilon \rightarrow 0} f_\varepsilon(u_\varepsilon) \leq \lim_{\varepsilon \rightarrow 0} \max_{s \in [0,1]} f_\varepsilon(st_0 v_\delta) \leq \frac{1}{n} S^{\frac{n}{p}} + o(1)$$

and this, by letting $\delta \rightarrow 0$, ends up the proof ■

Theorem 5.2. *Suppose that the number of non-zero Dirac masses is*

$$\left[\frac{pp^*}{(p^* - p - \gamma)n\nu^{\frac{n}{p}}} \right]$$

where $[x]$ denotes the integer part of x . Then $u = 0$.

Proof. Keeping into account the previous lemma and arguing as in Lemma 2.5,

$$\begin{aligned} \frac{1}{n} S^{\frac{n}{p}} &\geq \lim_{\varepsilon \rightarrow 0} f_\varepsilon(u_\varepsilon) \\ &\geq \frac{p^* - p - \gamma}{pp^*} \nu \left(\int_\Omega |\nabla u|^p dx + \sum_{j=1}^r \mu_j \right) \\ &\geq \frac{p^* - p - \gamma}{pp^*} \nu \int_\Omega |\nabla u|^p dx + r \frac{p^* - p - \gamma}{pp^*} \nu^{\frac{n}{p}} S^{\frac{n}{p}} \end{aligned}$$

where r denotes the number of non-vanishing masses. Hence it must be

$$0 \leq r \leq \left[\frac{pp^*}{(p^* - p - \gamma)n\nu^{\frac{n}{p}}} \right].$$

In particular, if r is maximum and $u \neq 0$, by virtue of Lemma 3.3 one obtains

$$\frac{p^* - p - \gamma}{pp^*} \nu^{\frac{n}{p}} S^{\frac{n}{p}} > \frac{p^* - p - \gamma}{pp^*} \nu \int_\Omega |\nabla u|^p dx > \frac{p^* - p - \gamma}{pp^*} \nu^{\frac{n}{p}} S^{\frac{n}{p}}$$

which is a contradiction ■

6. Final remarks

Assume that $\mathcal{L}(x, s, \xi)$ is of class C^1 in $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ and, additionally, that the vector-valued function

$$\nabla_\xi \mathcal{L}(x, s, \xi) = \left(\frac{\partial \mathcal{L}}{\partial \xi_1}(x, s, \xi), \dots, \frac{\partial \mathcal{L}}{\partial \xi_n}(x, s, \xi) \right)$$

is of class C^1 in $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n$.

Theorem 6.1. *Let Ω be star-shaped with respect to the origin and assume that*

$$p^* \nabla_x \mathcal{L}(x, s, \xi) \cdot x - n D_s \mathcal{L}(x, s, \xi) s \geq 0$$

for $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. Then (\mathcal{P}_0) has no non-trivial solution u in $C^2(\Omega) \cap C^1(\bar{\Omega})$.

Proof. Let $\widehat{\mathcal{L}} : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ by defined by setting

$$\widehat{\mathcal{L}}(x, s, \xi) = \mathcal{L}(x, s, \xi) - \frac{1}{p^*} |s|^{p^*}$$

for all $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. Then apply the Pucci-Serrin inequality [17]

$$n \widehat{\mathcal{L}} + \nabla_x \widehat{\mathcal{L}} \cdot x - a D_s \widehat{\mathcal{L}} s - (a + 1) \nabla_\xi \widehat{\mathcal{L}} \cdot \xi \geq 0$$

with the choice $a = \frac{n-p}{p}$ ■

Remark 6.2. If Ω is star-shaped and \mathcal{L} does not depend on x , then problem (\mathcal{P}_0) admits no non-trivial solution in $C^2(\Omega) \cap C^1(\overline{\Omega})$ when $D_s \mathcal{L}(s, \xi)s \leq 0$, which is the opposite of (7). In particular, (7) seems to be a natural assumption.

Remark 6.3. As noted in the introduction, if Ω is star-shaped and $\mathcal{L}(\xi) = |\xi|^p/p$, in [10] it is proven that problem (\mathcal{P}_0) has no non-trivial solution in $W_0^{1,p}(\Omega)$. In particular, by Theorem 2.7 one has $\mu_1 \neq 0$.

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