ALGEBRAIC CURVES & RIEMANN SURFACES

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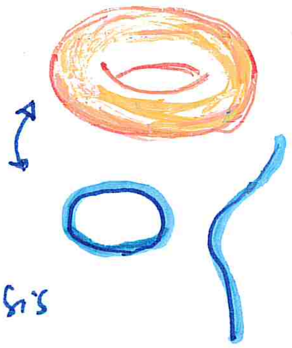

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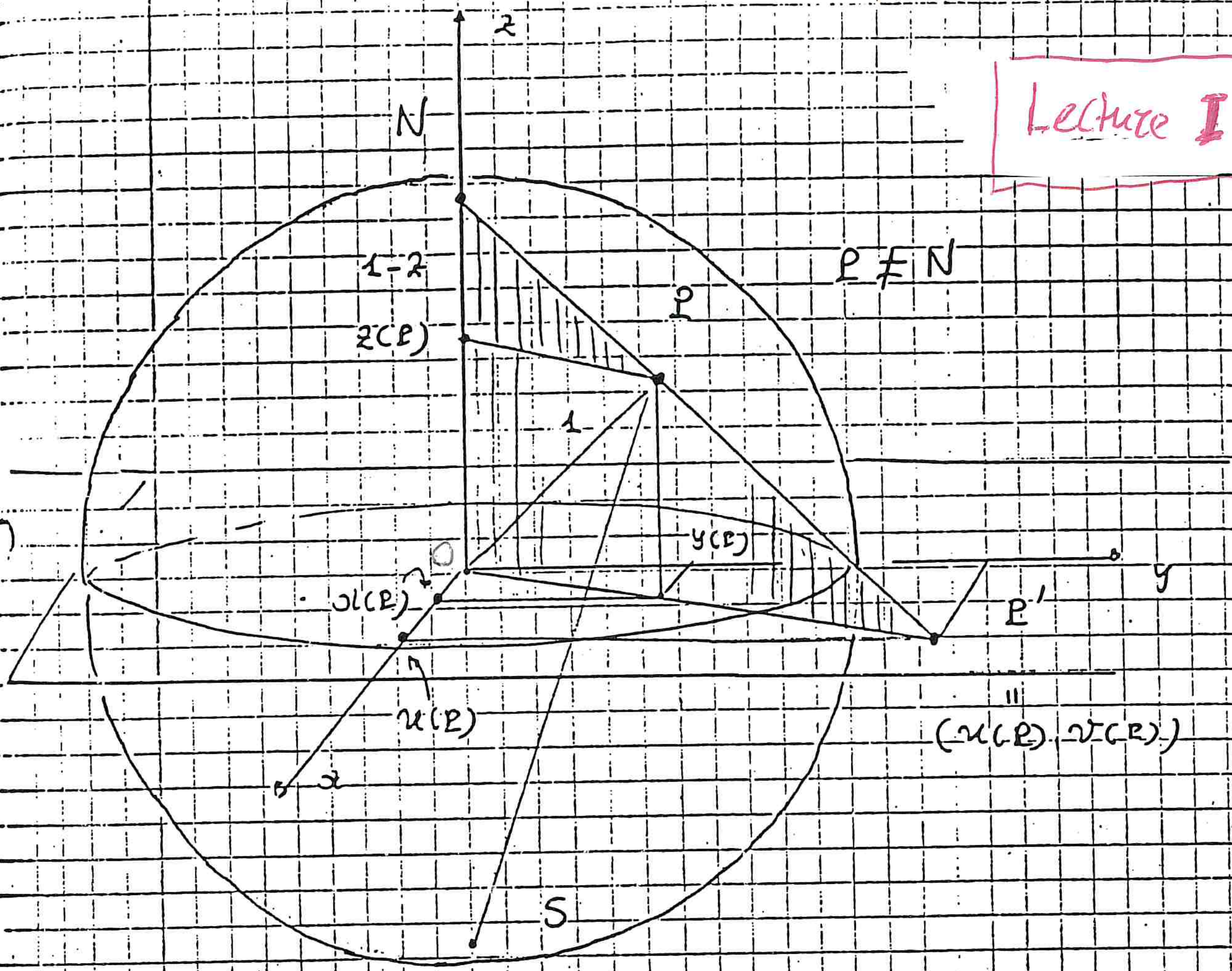
Riemann's sphere

☆ Sfera unitaria (proiezione stereografica)

unit sphere — stereographic - projection

Lecture I



$$\begin{cases} u = \dfrac{x}{1-z} \\ v = \dfrac{y}{1-z} \end{cases} \qquad \begin{aligned} x &= u(1-z) \\ y &= v(1-z) \end{aligned}$$

$$x^2 + y^2 + z^2 = 1 \quad \Rightarrow$$

$$z = \frac{u^2+v^2-1}{1+u^2+v^2} \quad \Rightarrow$$

$$\begin{cases} x = \dfrac{2u}{1+u^2+v^2} \\ y = \dfrac{2v}{1+u^2+v^2} \\ z = \dfrac{u^2+v^2-1}{1+u^2+v^2} \end{cases}$$

In coordinate complesse ... complex coordinates

$\zeta = u + iv$

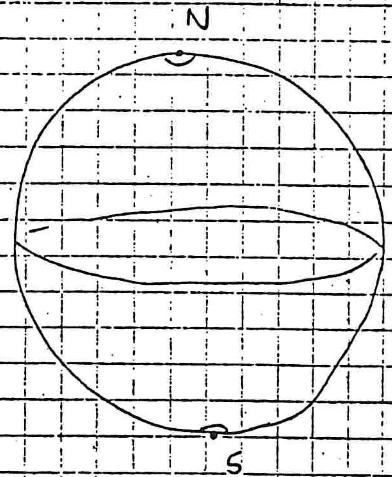
N : non è rappresentabile not representable

$S = (0,0)$

vice versa projecting from S

viceversa, proiettando da S, $P \neq S$

$\Rightarrow$ è $\zeta' = u' + iv' = \dfrac{1}{\zeta}$ per $P \in S^2 - \{N,S\}$



Il piano equatoriale riceve l'orientamento opposto

The equatorial plane is endowed with the opposite orientation

complex projective line

$\Rightarrow$ ☆ $S^2 \simeq \mathbb{CP}^1$: retta proiettiva complessa

"Sfera di Riemann" Riemann sphere

homogeneous coordinates coordinate omogenee

$$\mathbb{CP}^1 = \left\{ (z_0, z_1)_{/\sim} \;/\; (z_0, z_1) \in \mathbb{C}^2 - \{(0,0)\} \right.$$

$$(z_0, z_1) \sim (z_0', z_1') \iff \begin{array}{l} z_0' = \lambda z_0 \\ z_1' = \lambda z_1 \end{array}$$

$\mathbb{C} \ni \lambda \neq 0$ ζ, ζ' coordinate non omogenee

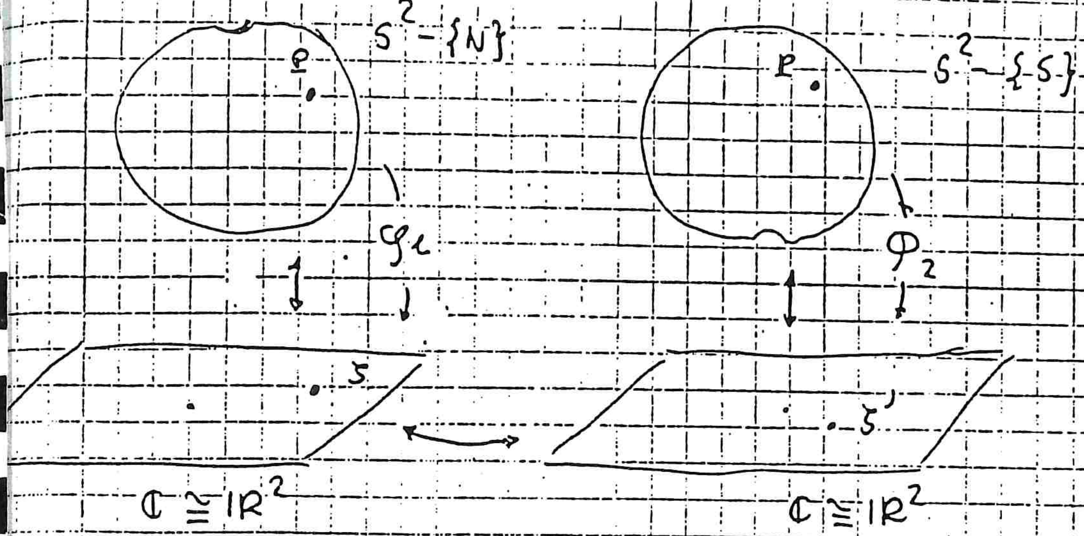
non-homogeneous

$\zeta = \dfrac{z_1}{z_0}$

$[z_0, z_1]$

☆ Commento — Comment

☆ S^2 è descritta da due "carte"
is described via two charts



legate da una legge di trasformazione
related by the following transformation law

$$\zeta' = \frac{1}{\zeta}$$

☆☆ questo conduce al concetto di varietà differenziabile

S^2 becomes a differentiable manifold, actually a complex manifold

The transformation law being biholomorphic

Riemannian metric

☆ Metrica $\propto \dfrac{du^2 + dv^2}{(1 + u^2 + v^2)^2}$

$$\frac{dz\,d\bar{z}}{(1 + |z|^2)^2}$$

(vedi oltre ...)

★ Riemann surface

Holomorphic functions

FUNZIONI OLOMORFE

Let a region open connected set

Sia $\mathcal{U} \subset \mathbb{C}$ una regione (aperto connesso)

$(\mathbb{C} \cong \mathbb{R}^2)$ $z = x + iy$

Let

Sia $f : \mathcal{U} \to \mathbb{C}$

$$f(z) := u(z) + i\,v(z) \equiv u(x,y) + i\,v(x,y)$$

real part of

u parte reale di f $(=: \mathrm{Re} f$

v = immaginaria di f $(=: \mathrm{Im} f)$

imaginary part of

Any such function induces a plane transformation in a natural way:

Una tale funzione induce in modo naturale una trasformazione piana

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases} \qquad (x,y) \in \mathcal{U}$$

Let

Sia $f \in \mathcal{C}^1(\mathcal{U})$ (ovvero: $u, v \in \mathcal{C}^1(\mathcal{U})$)

(Ciò semplifica moltissimo la trattazione ..)

This highly simplifies the treatment

[Actually, a holomorphic function will be automatically C^∞]

let
Sia $z_0 \in \mathcal{U}$

is said to be holomorphic

Def. $f : \mathcal{U} \to \mathbb{C}$ si dice <u>olomorfa</u> in z_0 se

if it is $\mathbb{C}$ - differentiable: namely

è derivabile nel senso <u>complesso</u>, ovvero

if

se $\exists$ $\quad f'(z_0) := \lim_{h \to 0} \dfrac{f(z_0+h) - f(z_0)}{h}$

$(h \in \mathbb{C}) \qquad (f'(z_0) \in \mathbb{C})$

is holomorphic if exists

f si dice olomorfa in $\mathcal{U}$ se f' esiste

at every pt of $\mathcal{U}$

in ogni punto di $\mathcal{U}$. $\quad \mathcal{H}(\mathcal{U}) := \{ f : \mathcal{U} \to \mathbb{C},$
$(f \in \mathcal{C}^1(\mathcal{U}))$ f olomorfa in $\mathcal{U}$ $\}$

Theorem

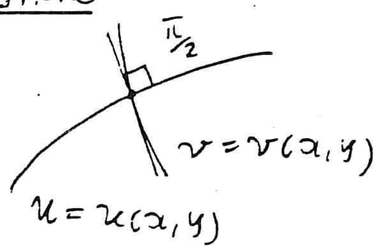
<u>Teorema</u> $\quad f \in \mathcal{H}(\mathcal{U}) \iff$

valgono le condizioni di Cauchy-Riemann

the Cauchy-Riemann conditions hold

$$\text{CR}: \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

Osserviamo che le condizioni di CR ci dicono che la trasformazione piana indotta da f è <u>ortogonale</u>



the CR imply that the transformation law induced by f is orthogonal

Proof

Dim. ($\Rightarrow$) If Se $f \in \mathcal{H}(\mathcal{U})$ then at P_0 si ha in

un dato punto $P_0 \equiv z_0 \equiv (x_0, y_0)$

$$f'(z_0) = u_x^0 + i\, v_x^0 = \ldots = v_y^0 - i\, u_y^0$$

(putting in the def ponendo nella def. prima $h = \Delta x$ and then e poi $h = i\,\Delta y$ ..) $\Rightarrow$ seguono le CR
the CR follow

observe that
Osserviamo poi che

$$\boxed{f'(z_0) = f_x^0 = -i\, f_y^0}$$

write down the incremental ratio and use CR
$\Leftarrow$ Si scrive il rapporto incrementale complesso e si usano le CR ... $\square$

Define
Definiamo
$$dz = dx + i\,dy$$
$$d\bar{z} = dx - i\,dy$$
complex differen
(diff. complesso)

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \left(\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\ldots\right)$$

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

One immediately checks that
Si constata facilmente che

$$df := du + i\,dv = \ldots = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$$

Equivalent formulation of the CR

Le condizioni di Cauchy-Riemann divengono equivalenti a $\frac{\partial f}{\partial \bar{z}} = 0$ ((calco[illegible]

$\boxed{\text{inoltre (Moreover)} \quad f' = f_z \ (\text{f olo})}$

combinazioni lineari, prodotti, composte di f. olomorfe sono olomorfe

linear combinations, products, composition of hol functions are holomorphic

es: $f(z) = \bar{z}$ non è olomorfa $\quad \frac{\partial \bar{z}}{\partial \bar{z}} = 1$

$f(z) = \sum_{n=0}^{N} a_n z^n$ olomorfa e $f'(z) = \sum_{n=1}^{N} n a$

Altro es: $f = |z|^2 = \bar{z} z$ non è olom

$\frac{\partial f}{\partial \bar{z}} = z \ldots$

per $z \neq$
for $z \neq$

Theorem
Teorema

Sia (Let) $f : \mathcal{U} \to \mathbb{C}$, $f \in \mathcal{H}(\mathcal{U})$

$(z = x + iy \qquad w = u + iv = f(z))$

Sia (Let) $f' \neq 0$ in $\mathcal{U}$.

then f induces a conformal transformation

Allora f induce una trasformazione conformale

Dim (Proof): $dx^2 + dy^2 = (dx + i\,dy)(dx - i\,dy)$
$= dz\, d\bar{z}$

$du^2 + dv^2 =$
$= dw\, d\bar{w} = df\, d\bar{f} = (f_z\, dz + f_{\bar{z}}\, d\bar{z})\overline{(f_z\, dz}$

$(d\bar{f} = \overline{df}) \qquad = f_z\, dz \cdot \bar{f}_z\, d\bar{z} =$

$= |f_z|^2\, dz\, d\bar{z} = \underbrace{|f'|^2}_{>0}\, dz\, d\bar{z}$

Conversely one can show that a conformal map

Viceversa si può mostrare che una is induced by

trasformazione conforme $(x,y) \to (u,v)$ an f

è indotta da una f $(= u + iv)$ which is either holomorph

olomorfa o antiolomorfa$^{(*)}$ (se la trasformazione or anti-holomorph

conserva o cambia l'orientamento, rispettiv.) according to whether

the map preserves orientation or not

Theorem

Teorema (Riemann). Date due regioni (given two) simply connected regions semplicemente connesse, (there exists a) esiste sempre una funzione biolomorfa (bi holomorphic function (†)) che manda l'una sull'altra (relating the two). (†): olomorfa, biunivoca, (holomorphic, bijective,) con inversa olomorfa) (f^{-1} holomorphic)

uniformization theorem

(*) ovvero "olomorfa in $\bar{z}$..." (that is, holomorphic in $\bar{z}$):

$$\frac{\partial f}{\partial z} = 0$$

complex structure $\leftrightarrow$ conformal structure

struttura complessa $\leftrightarrow$ $\{\lambda g\}$ $[g] = \{\lambda g \,/$

$$g = ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

$\lambda \in C^\infty(M, \mathbb{R}^+)$: $f$rs

$f = u + iv$ holomorphic :

u, v isothermal coordinates

i.e. $E = G$, $F = 0$

$$g = \lambda(du^2 + dv^2)$$

$\Rightarrow$ locally, isothermal coordinates exist on any 2d Riemannian manifold

Details

$\nsim$ Dettagli

$f : \mathbb{C} \to \mathbb{C} \qquad f(x, y) \equiv f(z, \bar{z})$

$$d\bar{f} = \bar{f}_z \, dz + \bar{f}_{\bar{z}} \, d\bar{z}$$

$$= \frac{\partial \bar{f}}{\partial z} dz + \frac{\partial \bar{f}}{\partial \bar{z}} d\bar{z} =$$

$$= \overline{\frac{\partial f}{\partial \bar{z}}} dz + \overline{\frac{\partial f}{\partial z}} d\bar{z}$$

$$= \overline{\frac{\partial f}{\partial \bar{z}} d\bar{z}} + \overline{\frac{\partial f}{\partial z} dz}$$

$$= \overline{\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}} = \overline{df}$$

(calcolo diretto , o anche

direct calculation, or observe that .

Si osservi che

rapp. incrementale $\dfrac{\Delta \bar{F}}{\Delta Z} = \overline{\dfrac{\Delta F}{\Delta \bar{Z}}}$