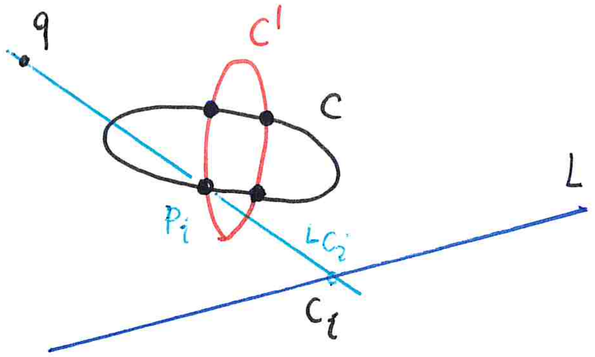
★ Bézout's theorem



informal proof: via "conservation of intersections" (continuity)

$q \notin L$

Basic idea: (Brieskorn - Knörrer)

$C \cap C'$

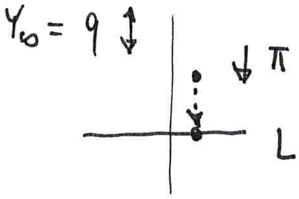
$\Downarrow$

on each line L_{c_i} (connecting q to P_i and intersecting L in c_i) there is at most a finite number of points in $C \cap C'$ (C and C' do not share a common component). Then one has to show that a finite set of lines L_{c_i} exists.

Let us determine these lines L_{c_i}

$[x_0, x_1, x_2]$: homogeneous coordinates on $\mathbb{P}^2(\mathbb{C})$ $\quad q : [0,0,1]$

$L: \quad x_2 = 0 \qquad \pi([x_0, x_1, x_2]) = [x_0, x_1]$



$C: \quad F(x_0, x_1, x_2) = 0 \quad$ deg m

$C': \quad G(x_0, x_1, x_2) = 0 \quad$ deg n

★ polynomials in x_2 with coefficients in $\mathbb{C}[x_0, x_1]$

$F = A_0 x_2^m + A_1 x_2^{m-1} + \ldots A_m \qquad A_i$ homogeneous of degree i

$G = B_0 x_2^n + B_1 x_2^{n-1} + \ldots B_n \qquad B_j$ homogeneous of degree j

Also, $A_0 \neq 0$, $B_0 \neq 0$ (constant)

(for $q : [0,0,1] \notin C \cup C'$)

C and C' do not have a common component.

Consequently F and G have no common factor, whence their resultant $R_{F,G}(x_0,x_1)$ is a non-zero, homogeneous polynomial of degree $m\cdot n$

Let then $p:[c_0,c_1,c_2]\in C\cap C'$. Therefore

$$\begin{cases} F(c_0\, c_1\, c_2) = 0 \\ G(c_0, c_1\, c_2) = 0 \end{cases} \Rightarrow c_2 \text{ solves}$$

$$(*)\ \begin{cases} A_0(c_0,c_1)x_2^m + \dots + A_m(c_0,c_1) = 0 \\ B_0(c_0,c_1)x_2^n + \dots + B_n(c_0,c_1) = 0 \end{cases}$$

Thus $R_{F,G}(c_0,c_1) = 0$ (and conversely, $R_{F,G}(c_0,c_1)=0$ yields a common solution c_2 to $(*)$)

* Upshot: The zeros of $R_{F,G}$ are precisely the $c\in L$ such that L_c contains points in $C\cap C'$. This yields the lines L_{c_i} (and the intersection points as the roots of a polynomial)

* Weak Bézout theorem

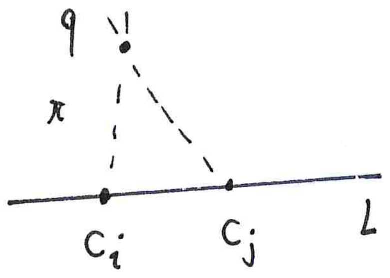
$\underset{m}{C}, \underset{n}{C'}$ have at most $m\cdot n$ intersection points

(without a common component)

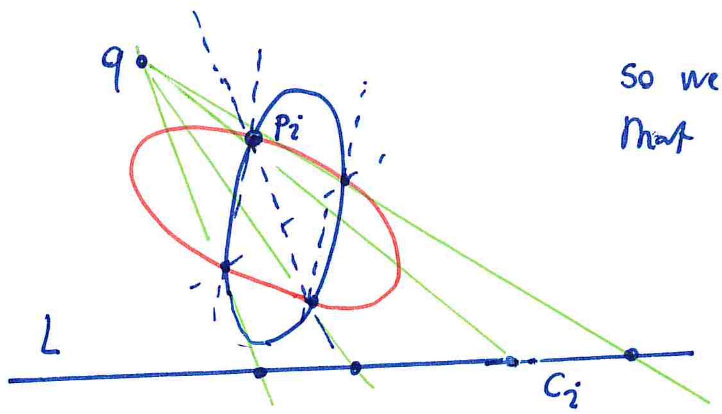
Proof. By contradiction, assume that $C \cap C'$ consists of at least $m \cdot n + 1$ points

$p_1 \dots p_{mn+1}$. Let $q \notin L_{ij}$ $i \neq j$

line connecting p_i to p_j



Then $c_i \neq c_j$. So we get $c_1 \dots c_{mn+1}$, distinct pts on L yielding zeroes of $R_{F,G}$. But $R_{F,G}$ has degree $m \cdot n$, so we get a contradiction, thus achieving the conclusion. □



So we can assume that $q \notin \bigcup_{i,j} L_{ij}$ and get a bijection $p_i \leftrightarrow c_i$
$$c_i = \pi(p_i)$$

$p_i : (c_0^i, c_1^i, c_2^i)$ $\quad c_i : (c_0^i \; c_1^i)$

$\nu_{p_i}(C, C') :=$ multiplicity of the zero c_i of $R_{F,G}$

intersection multiplicity of C and C' at p_i

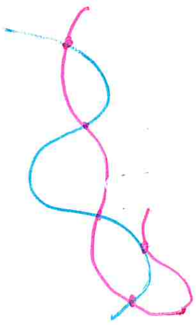
★★★ Bézout's Theorem

$$\mathcal{C} = C^m \qquad \mathcal{C}' = C^n$$

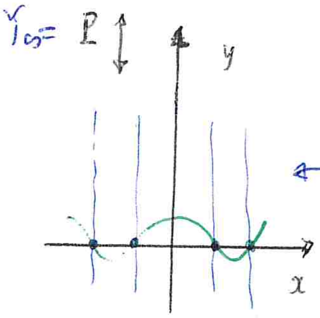
Any two plane (algebraic) curves C^m and C^n, not sharing a common component, have $m \cdot n$ points in common
(complex)

<u>Proof</u> The above argument shows that if we define the multiplicity of intersection $m_p(C^m, C^n)$ of two curves at a point as the multiplicity of the corresponding root of $R(x_0, x_1) = 0$, since the latter has degree mn, by the FTA we are done.

R: resultant



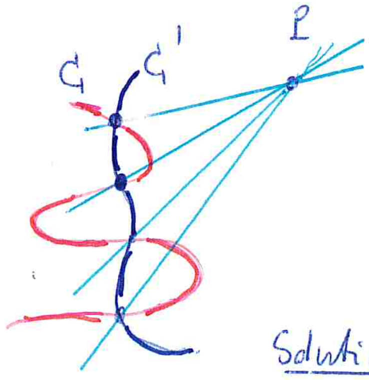
$$\boxed{mn = \sum_{P \in C \cap C'} m_p(C, C')}$$



roots of R

$\leftarrow R(x_0, x_1) = 0$: lines connecting $P : [0,0,1] = Y_\infty$ with the roots of R

projective situation



★ **Problem:**

One should prove that m_p is independent of the choice of a reference frame

<u>Solution</u> Let $P : [0,0,1] \notin C \cup C' \cup \bigcup_{i,j} L_{ij}$

L_{ij} : line joining intersection points p_i and p_j, $i \neq j$

Equation $H(x_0, x_1, x_2) = 0$

Admissible coordinate changes:

$$X = \{ A \in GL(3, \mathbb{C}) \mid H(a_{02}, a_{12}, a_{22}) \neq 0 \}$$

AP (third column of A)

That is: $X = \{ (a_{ij}) \in \underset{\mathbb{C}^9}{\underset{\cong}{M(3, \mathbb{C})}} \mid \mathcal{P} \neq 0 \}$

where $\mathcal{P}\{a_{ij}\} := D\{a_{ij}\} \cdot H\{a_{ij}\}$

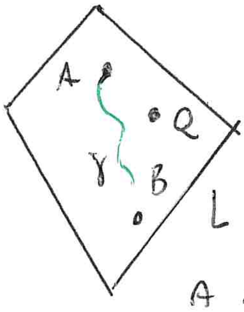
D: determinant

Let $\bar{W} \subset \mathbb{C}^9$ $\quad \bar{W} : \mathbb{C}^9 \setminus X$

$\bar{W} : \{ P = 0 \}$ $\quad \bar{W}$ has real codimension 2 in $\mathbb{C}^9 \cong \mathbb{R}^{18}$

whence X is connected (hence pathwise connected)

explicitly



take L the complex line connecting A, B in X (a plane). Then one finds an arc γ connecting A and B and avoiding $Q \in \bar{W}$ (such points yield a finite set).

Let ν_R, ν'_R the intersection multiplicity defined in the two coordinate systems. Given a continuous curve $[0,1] \ni t \longmapsto A_t \in X$ ← connected $A(0) = I$, $A(1) = A$ one has admissible coordinate changes A_t.

From $\underset{C}{F(x) = 0}$, $\underset{C'}{G(x) = 0}$ in the initial system one obtains obvious equations

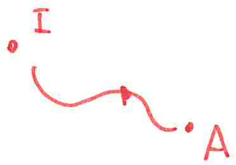
$F_t(x_t) = 0$, $G_t(x_t) = 0$

together with resultants $R_t(x_0, x_1) \equiv$

$R_{F_t, G_t}(x_{t,0}, x_{t,1})$: homogeneous polynomials of degree mn.

The number of distinct zeros $c_i(t)$ does not change with t. Hence, the multiplicity of a root also does not vary, since it is a continuously varying integer (hence constant).

□



Consequences of Bézout

★ **Theorem** If $\mathcal{C}$ and $\mathcal{C}'$ (plane complex curves) are both of order n, and if mn of these lie on an irreducible curve C'' of order m $(< n)$ then the remaining $n(n-m)$ lie on a curve C''' of order $n-m$

Proof $\mathcal{C} : F_1 = 0$, $\mathcal{C}' : F_2 = 0$

intersect in n^2 points. Let $G = 0$ be the equation of C''. Consider the pencil $\lambda_1 F_1 + \lambda_2 F_2 = 0$

Let a curve $\tilde{\mathcal{C}}$ of this pencil pass through a point in G, different from the previous ones.

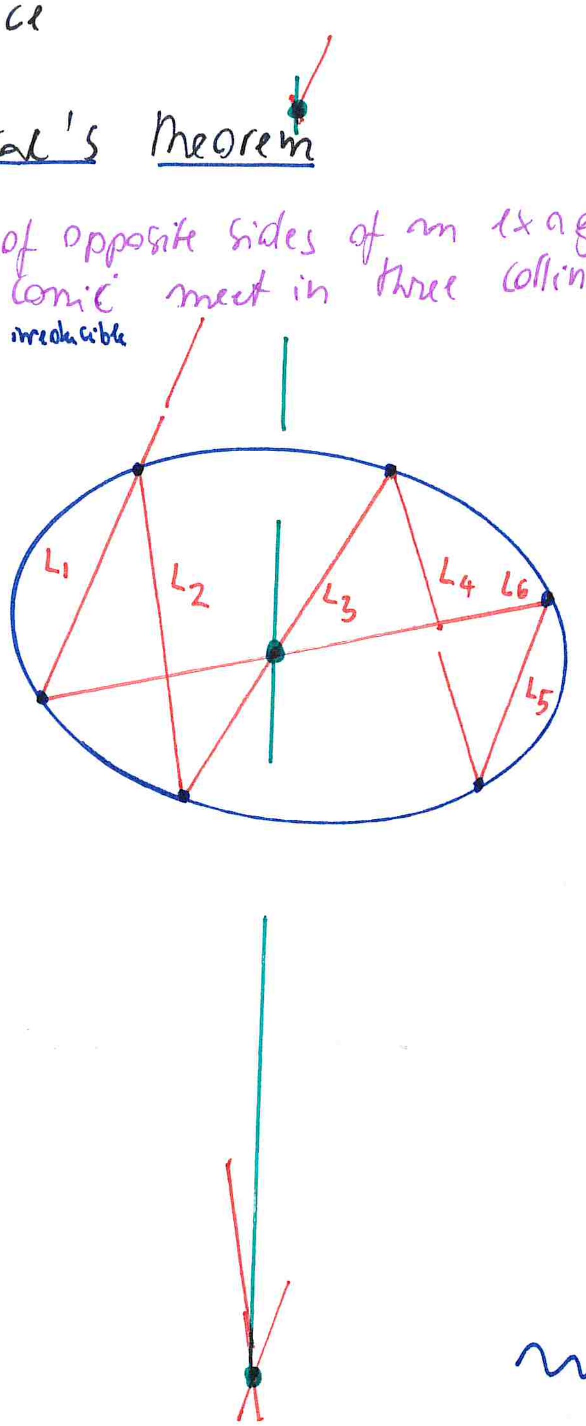
Then G and $\tilde{\mathcal{C}}$ have $mn+1$ points in common, whence they have a common component, which is G, being irreducible. Therefore

$\lambda_1 F_1 + \lambda_2 F_2 = G \cdot H$ passes through the initial n^2 points and H (of order $n-m$) contains the extra $n-m$ points not on G.

Consequence

Pascal's theorem

Pairs of opposite sides of an exagon inscribed in a conic meet in three collinear points
irreducible



1 - 4
2 - 5
3 - 6

Proof

Consider the cubics $L_1L_3L_5$, $L_2L_4L_6$

They intersect in 9 points, of which 6 lie on a conic (irreducible). Then the remaining 3 must be collinear.

□

⇝ Brianchon (via duality)

— . — . — . —

⇛ We shall use the previous result in the proof of the group law on an elliptic curve.