Klein's approach

ALGEBRAIC CURVES AND RIEMANN SURFACES

Lecture V

Preliminaries

$(X, Y) \equiv X + iY$

Review: $f = u + iv$ holomorphic

$$\bar{V} = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

streamlines

$u = c$

$\parallel$ CR

$$\left(\frac{\partial u}{\partial x}, -\frac{\partial v}{\partial x}\right) \equiv \overline{\frac{\partial f}{\partial z}} = \overline{\frac{\partial f}{\partial x}}$$

$\operatorname{div} V = u_{xx} + u_{yy} = 0$

(P, Q)

$\operatorname{curl} V = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k} = 0$

V solenoidal, irrotational

But

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]$$

$$= \frac{1}{\cancel{2}}\left[\cancel{2}\frac{\partial u}{\partial x} + \cancel{2}i\frac{\partial v}{\partial x}\right] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}$$

Check: $f = z$

$1 = \frac{\partial(x+iy)}{\partial x}$ ✓

$$\Rightarrow \quad V \leftrightarrow \overline{f'} \qquad ' = \frac{d}{dz}$$

conjugate flow

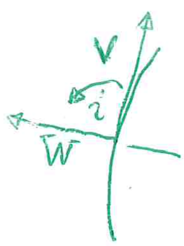
$$\bar{W} = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = \left(-\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right)$$

$$-\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = -\left[\frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}\right] = -\overline{\frac{\partial f}{\partial y}}$$

$\bar{W} \perp V$

$$\frac{\partial f}{\partial z} = \frac{1}{\cancel{2}}\left[\cancel{2}\frac{\partial v}{\partial y} + \cancel{2}i\left(-\frac{\partial u}{\partial y}\right)\right] = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= -i\frac{\partial f}{\partial y} \qquad \overline{\frac{\partial f}{\partial z}} = i\overline{\frac{\partial f}{\partial y}}$$



$V = -i\bar{W}$

$\bar{W} = i\bar{V}$

Check $f = z \quad -i\cdot\frac{\partial(x+iy)}{\partial y} = -i\cdot i = +1$ ✓

V-I

* On Klein's approach

logarithmic singularities

$$w = \underbrace{A \log(z - z_0)} + C_0 + C_1 (z - z_0) + \dots$$

$\boxed{A \in \mathbb{R}}$ $\quad C_0 = a + ib$ $\qquad w = u + iv$ $\qquad \log(z - z_0) =$

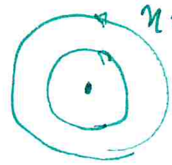
$z - z_0 = r e^{i\phi}$ $\qquad \log r + i\phi$

first approximation

$$\begin{cases} u = A \log r + a \\ v = a\phi + b \end{cases}$$

$u - a = A \log r$

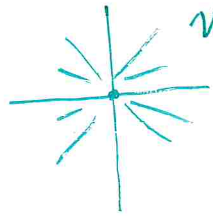
$u = a + \log r = c$ (constant)

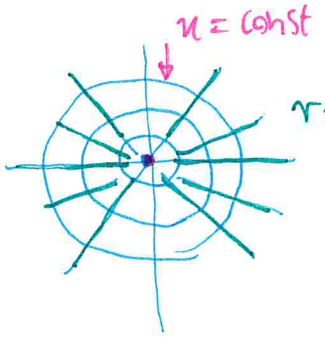


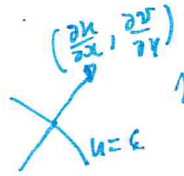
$\log r =$ constant

$r =$ constant

$v = a\phi + b =$ const







$u = c$ equipotential lines

$v = c$ stream lines

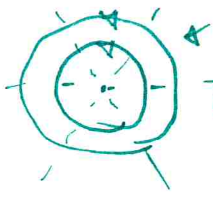
Source / Sink

Strength $= 2\pi A$

$A \equiv iA$, $A \in \mathbb{R}$ $\qquad \begin{cases} u = -A\phi + a \\ v = A \log r + b \end{cases}$

$$iA(\log r + i\phi) + a + ib$$
$$= \underbrace{(-A\phi + a)}_{u} + i\underbrace{(A \log r + b)}_{v}$$

→ get a vortex of strength $2\pi A$



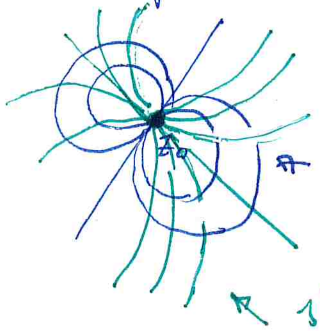
4 algebraic singularities

consider $w = \frac{A_1}{z-z_0} + C_0 + \dots$ $\quad z - z_0 = r e^{i\phi}$, $\quad A_1 = p e^{i\psi}$

$$w - C_0 = \frac{p}{r}\left\{\cos(\psi-\phi) + i \sin(\psi-\phi)\right\}$$

$$\frac{p}{r}\cos(\psi-\phi) = \text{const}$$

pencil of circles tangent to $\varphi = \psi + \frac{\pi}{2}$



general picture $\quad$ ν-ple sing $\sim \frac{1}{(z-z_0)^\nu}$

$\nu = 2$