★ Local study of a plane algebraic curve

ALGEBRAIC CURVES & RIEMANN SURFACES

[Lecture VII]

$\mathcal{C}^n: \quad f(x_0,x_1,x_2)=0$ complex plane algebraic curve of order n

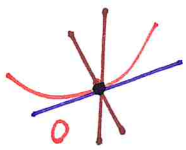
f homogeneous of degree n

let $P \equiv O: [1,0,0]$ use affine coordinates x, y

$$\mathcal{C}^n: \quad \underbrace{(a_0x+a_1y)}_{P_1(x,y)} + \underbrace{(b_0x^2+b_1xy+b_2y^2)}_{P_2(x,y)} + \dots = 0$$
$$+ \underbrace{(c_0x^3+c_1x^2y+c_2xy^2+c_3y^3)}_{P_3(x,y)} + \dots$$

$$P_1(x,y)+P_2(x,y)+P_3(x,y)+\dots = 0$$

P_j homogeneous in x, y, of degree j



line issuing from O: pencil $\quad r: \begin{cases} x = ht \\ y = kt \end{cases} \quad (h,k) \neq (0,0)$

$$kx - hy = 0$$

intersections $\mathcal{C}^n \cap r$: roots of

$$(a_0h+a_1k)t + (b_0h^2+b_1hk+b_2k^2)t^2 + $$
$$+ (c_0h^3+c_1h^2k+c_2hk^2+c_3k^3)t^3 + \dots = 0$$

($t=0$ is always a root)

If $(a_0,a_1) \neq (0,0)$

we have a simple root $t=0$ (corresponding to O)

unless $a_0h+a_1k=0$, i.e. if $r: a_0x+a_1y=0$
$P_1(x,y)=0$

In this case we have O counted at least twice

[$P_1(x,y)=0$ tangent to $\mathcal{C}^n$ at O]

★ O: simple pt.

if $P_2(h,k) = (b_0h^2+b_1hk+b_2k^2) \neq 0$, we have a

2-point contact $\quad$ O is counted twice among $\mathcal{C}^n \cap r$

contact of order 1

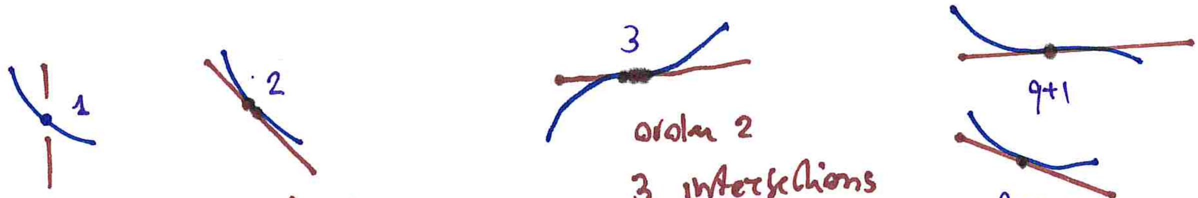
However, if $P_2(h,k) = P_3(h,k) = \dots P_q(h,k) = 0$

but $P_{q+1}(h,k) \neq 0$,

★ $\mathcal{C}^n$ and r have a $\underline{\underline{(q+1)\text{-point contact}}}$ at O — contact of order q

$\Updownarrow$

$q+1$ pts among the intersections of $\mathcal{C}^n$ and r coalesce at O



order 0, 1 intersection

order 1, 2 intersections

order 2, 3 intersections, ★ flex of order 1, ordinary flex

flex of order $q-1$

✦ Approximating parabolas of order n

$O: [1,0,0]$ $\quad r_0: y = 0$

$$\mathcal{C}^n : \quad \overset{q_1=1}{y} + P_2(x,y) + P_3(x,y) + \dots = 0 \qquad (*)$$

$$\frac{\partial f}{\partial y}(0,0) = 1 \neq 0 \implies \mathcal{C}^n : \quad y = y(x)$$
$$= m_1 x + m_2 x^2 + m_3 x^3 + \dots$$

Substitution into $(*)$ yields:

$$0 = m_1 x + m_2 x^2 + m_3 x^3 + \dots$$
$$b_0 x^2 + b_1 x(\underset{\overset{\shortparallel}{0}}{m_1} x + m_2 x^2 + m_3 x^3 + \dots) + b_2(m_1 x + m_2 x^2 + m_3 x^3 + \dots)^2$$
$$\overset{\shortparallel}{0}$$

$$\implies m_1 = 0, \quad m_2 + b_0 = 0, \quad m_3 + b_1 m_2 + c_0 = 0 \quad \dots$$

$\leadsto$ the m_i are obtained recursively

yielding $\underline{\text{parabolas of any order } n}$ $\quad \to$ $(n+1)$-contact at O with C^n

$$y = \sum_{i=2}^{n} m_i x^i \qquad (m_1 = 0)$$

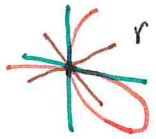
we have a <u>linear branch</u> $y = \sum_{j=1}^{\infty} m_j x^j$

If 0 is a flex of order $q-1$ $(q \geq 2)$ ($q+1$-pt contact)

$y = m_{q+1} x^{q+1} + \dots$

* Multiple pts

Let $a_0 = a_1 = 0$, $(b_0, b_1, b_2) \neq (0,0,0)$



for r generic $r: \begin{cases} x = ht \\ y = kt \end{cases}$ $(h,k) \neq (0,0)$

2 intersections are absorbed in O $\boxed{\text{O : double pt}}$

tangents : the 2 lines having at least $\underline{\underline{3}}$ intersections with $\mathcal{C}^n$ coalescing at O
("tangent cone")

we must have $b_0 h^2 + b_1 hk + b_2 k^2 = 0$ i.e.

$$P_2(x,y) = b_0 x^2 + b_1 xy + b_2 y^2 = 0$$

$$\tau_1 \tau_2 = 0$$

O : ordinary double pt : τ_1, τ_2 distinct ("node")

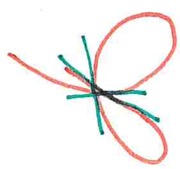
extraordinary double pt : $\tau_1 = \tau_2$

if $\tau_1 = 0$ (say) also divides
$P_3 \dots P_q$, but <u>not</u> P_{q+1} , then
τ_2 will have a $(q+1)$-pt contact with $\mathcal{C}^n$ in O

◆ <u>Triple point</u> $a = 0$, $b = 0$ but $c = (c_0, c_1, c_2, c_3) \neq (0,0,0,0)$

$$P_3(x,y) = c_0 x^3 + c_1 xy + c_2 xy^2 + c_3 y^3 = 0$$

$\rightsquigarrow$ $P_3 = \tau_1 \tau_2 \tau_3 = 0$ τ_i tangents



s-ple point : s intersections of $\mathcal{C}^n$ and r (generic) coalesce at 0

$P_s = r_1 \cdots r_s = 0$ tangents at 0 ("tangent cone")

r_j has $s+1$ intersections in common with C^n at 0

$$\mathcal{C}^n: \quad \underbrace{P_s(x,y)}_{\substack{= \\ 0}} + P_{s+1}(x,y) + P_n(x,y) = 0$$

tangents

In projective coordinates x_1, x_2, x_3 $\quad O: [0,0,1]$ s-ple

$$\mathcal{C}^n: \quad x_3^{n-s} \underbrace{P_s(x_1,x_2)}_{\text{tangents}} + x_3^{n-s-1} P_{s+1}(x_1,x_2) + \cdots P_n(x_1,x_2) = 0$$

P_j: binary forms of degree j
hom polynomials

★ Multiple points: general approach

$P(y_i) = [y_1\ y_2\ y_3]$ $Q(z_j): [z_1\ z_2\ z_3]$

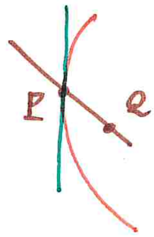
$r_{PQ}: \quad \lambda y_i + \mu z_i = 0 \quad i = 1,2,3$ $\dfrac{\partial f}{\partial y_i} \equiv \dfrac{\partial f}{\partial x_i}(P)$

$r_{PQ} \cap \mathcal{C}^n$

$$0 = f(\lambda y_i + \mu z_i) = \lambda^n f(y_i) + \lambda^{n-1} \mu \sum_j z_j \frac{\partial f}{\partial y_j}$$

$$+ \frac{1}{2} \lambda^{n-2} \mu^2 \sum_{j,k} z_j z_k \frac{\partial^2 f}{\partial y_j \partial y_k} + \dots = 0$$

$P \in \mathcal{C}^n \Rightarrow f(y_i) = 0$ (let $\lambda \neq 0$)



$\mu = 0$ simple root: $\sum_j z_j \dfrac{\partial f}{\partial y_j} \neq 0$

◆ P simple

τ: tangent to $\mathcal{C}^n$ at P: $\left\{ \sum_j x_j \dfrac{\partial f}{\partial y_j} = 0 \right\}$ 2-pt contact with $\mathcal{C}^n$

◆ P double $\dfrac{\partial f}{\partial y_i} = 0$ but

$\sum z_i z_j \dfrac{\partial^2 f}{\partial y_j \partial y_k} = 0$ should not be an identity

$\left\{ \sum x_i x_j \dfrac{\partial^2 f}{\partial y_i \partial y_j} = 0 \quad \tau_1 \tau_2 = 0 \text{ tangents} \right\}$

Remark: by Euler $\dfrac{\partial f}{\partial x_i} = 0 \ \forall i \Rightarrow f = 0$

more generally $\dfrac{\partial^l f}{\partial x_{i_1} \cdots \partial x_{i_l}} = 0 \Rightarrow$ all lower order derivatives vanish up to $f = 0$

In general

A necessary and sufficient condition for the existence of a (at least) s-ple point $P \in \mathbb{C}^n$ is that all the $\frac{s(s+1)}{2}$ $(s-1)$th derivatives of f vanish at P

tangents x_i, $i=1..s$ $\quad P = P(y_i)$

$$\left[\sum_{j,k,l} x_1^j x_2^k x_3^l \frac{\partial^s f}{\partial y_1^j \partial y_2^k \partial y_3^l} = 0 \quad j+k+l = s \right]$$

$$\frac{\partial^s f}{\partial x_1^j \partial x_2^k \partial x_3^l}(P)$$

★ **Crucial observation**

Amplification

F homogeneous $\quad F_i = \dfrac{\partial F}{\partial x^i}$

From Euler: $\quad x^i F_i = n F$

$$F_i = 0 \;\; \forall i \;\; \Rightarrow \;\; F = 0$$

Let $F_{ij} = \dfrac{\partial^2 F}{\partial x^i \partial x^j} = F_{ji}$ then

$$(n-1) F_i = x^j F_{ij}$$

Thus, $F_{ij} = 0$ (at a point, $\forall i, j$)

implies $F_i = 0 \;\; \forall i \;\; \Rightarrow \;\; F = 0$

More generally

$F_I = 0 \;\; \forall I \quad I = (i_1 \dots i_s) \quad i_k = 0,1,2$

$\Rightarrow F_{I'} = 0 \;\; \forall I' = (j_1 .. j_{s'}) \quad s' < s$

$\Rightarrow F = 0$

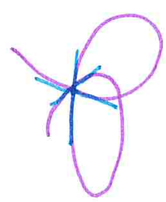
★★ **s-ple point of $F = 0$:**

$F_{I'} = 0 \quad \forall \; I'$ $(s-1)$-multiindex

but $F_{\tilde{I}} \neq 0$ for some $\hat{I}$ s-multiindex

Assume it placed at $O : [1,0,0]$

$$F = \underbrace{\varphi_s}_{\text{homogeneous of degree } s} + \varphi_{s+1} + \dots = 0$$



$\varphi_s = 0$: principal tangents