ALGEBRAIC CURVES AND RIEMANN SURFACES

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Lecture VIII

Linear branches

$$F(x_0, x_1, x_2) = 0$$
$$f(x, y) = 0$$

For simplicity, let us work at O

- O: simple $\quad (f_x^0, f_y^0) \neq (0,0)$

Let $y = a_1 x + a_2 x^2 + \dots a_h x^h$ → substitute in

$$0 = f(x,y) = \quad a_{01} x + \underset{\neq 0}{a_{10}} y + (a_{20} x^2 + a_{11} xy + a_{02} y^2) + \dots$$

$$a_{01} x + a_{10}\underbrace{(a_1 x + a_2 x^2 + \dots)}_{y} + (a_{20} x^2 + a_{11} x \cdot \underbrace{(a_1 x + a_2 x^2 + \dots)}_{y} + a_{02} \underbrace{(a_1 x + a_2 x^2 \dots)^2}_{y^2} =$$

$$= \underbrace{(a_{01} + a_{10} a_1)}_{=0} x + (\underbrace{a_{10} a_2 + a_{20} + a_{11} a_1 + a_{02} a_1^2}_{=0}) x^2 + (\quad)$$

get $a_1 = -\dfrac{a_{01}}{a_{10}}$ $\qquad$ get a_2 $\qquad$ get $a_3, a_4 \dots a_h$

→ get a linear branch $y = y(x)$

$$f(x, y(x)) \equiv 0$$

→ osculating parabolas of degree h, arbitrary

- O double point with distinct tangents (ordinary double pt)

get **two** linear branches $\quad y_1 = y_1(x), \; y_2 = y_2(x)$

originating from the two tangents (similar procedure)

⚠ A double point causes no branching in the RS sense!
with distinct tangents

$\boxed{\bullet \quad 0 \quad \underline{\text{cusp}}}$: no linear branches in general, but a branching in the RS sense (of index 1)

$y^2 = 0$ (w.l.o.g.)

$f(x,y) = 0$ $\qquad y^2 = (a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3) + \dots$

substitute

Ansatz: $y = a_2x^2 + a_3x^3 + \dots$

horizontal tangent

$$(a_2x^2 + a_3x^3 + \dots)^2 = (a_{30}x^3 + a_{21}x^2(a_2x^2 + a_3x^3 + \dots) + \dots) + \dots$$

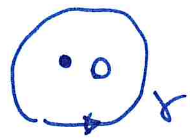
$$a_2^2x^4 + \dots = a_{30}x^3 + a_{21}a_2x^4 + \dots$$

$a_{30} \neq 0$ for an ordinary cusp

$\Rightarrow$ no linear branch exists in general

$\rightarrow$ 0 is a branch point in the RS sense

model: $\quad y^2 = x^3 \qquad y = x^{3/2}$



two values for $x \neq 0$: upon tracing γ counterclockwise the two roots get exchanged

Linear branches in the case of multiple points

* P s-ple point with distinct tangents
  → get s linear branches

◆ Let P double pt with coinciding tangents ($P \equiv 0$)
(with the x-axis)

$$\mathcal{C}^n: \quad y^2 + P_3(x,y) + P_4(x,y) + P_n(x,y) = 0$$

$$y^2 + (c_0 x^3 + \dots) + (d_0 x^4 + \dots) + \dots = 0 \qquad \text{Let } c_0 \neq 0$$

If we try $y = \underset{\substack{\| \\ 0}}{m_1} x + m_2 x^2 + m_3 x^3 + \dots$
$= m_2 x^2 + m_3 x^3 + \dots$

! We cannot annihilate the coefficient of x^3
($c_0 \neq 0$) ⇒ we cannot construct a linear branch

→ Try a different ansatz: $x = t^2$, $y = m_3 t^3$, get

$$m_3^2 t^6 + t^6 P_3(1, m_3 t) + t^8 P_4(1, m_3 t) + \dots$$

$t = 0$ 6-ple root. Forcing a higher multiplicity
we find $m_3^2 + c_0 = 0$ $\quad m_3 = (-c_0)^{\frac{1}{2}}$ ← multivalued

We just have $y^2 = -c_0 x^3$ for both $\pm t$

branching in the RS task

In general $x = t^2$, $y = m_3 t^3 + \dots$
yield two expansions which actually coincide ($c_0 \neq 0$) O: ordinary cusp
→ a single branch (of second order)
since a generic line meets in in two points coalesceing at O

If $c_0 = 0$, try $y = m_2 x^2$, arriving at

$$m_2^2 x^4 + m_2 x^4 (c_1 + c_2 m_2 x + c_3 m_2^2 x^2 + \dots) + (d_0 x^4 + \dots) = 0$$

$\Rightarrow$ $x = 0$ quadruple root. If we require a 5-root we must set $m_2^2 + c_1 m_2 + d_0 = 0$

If $c_1^2 \neq 4 d_0$ we have two distinct parabolas $y = m_2 x^2$, each sharing 5 points with $\mathcal{C}$ falling in 0 $\longrightarrow$ we have two distinct branches, getting

↖ same tangent

a tacnode ★

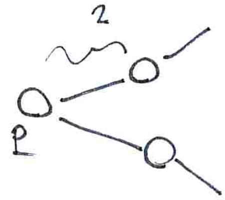
If $c_1^2 = 4 d_0$, the two parabolas coincide

Try then $x = t^2$ $y = m_4 t^4 + m_5 t^5 + \dots$

$\rightsquigarrow$ a single branch second order cusp ★

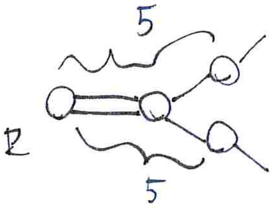
but there is still an exception, wherein one may separate two branches again: oscnode ★ et cetera

∗ Enriques graphs for linear branches

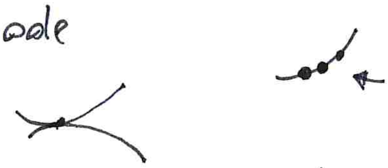


node

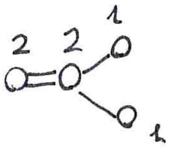
absorbs 2 intersections



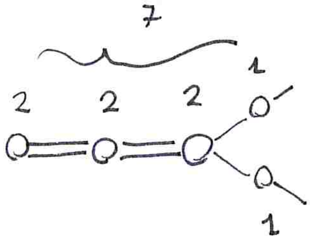
tacnode



5 intersections absorbed in P for each branch



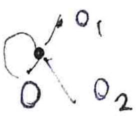
2+2+1 = 5



oscnode

[cf resolution of sing. via quadratic transformation]

∗ infinitesimally near pts (Noether 1871) ⇝ blow-up



$O_{1,2}$ on the principal tangents

$O_{1\,1}, O_{1\,2} \dots$ ← 2nd order neigh. of O

$O_{2\,1} \dots O_{2,2} \dots$ ← first order neigh. of O_1

lie on parabolas

{we shall possibly further deal with this issue later on}