ALGEBRAIC CURVES & RIEMANN SURFACES

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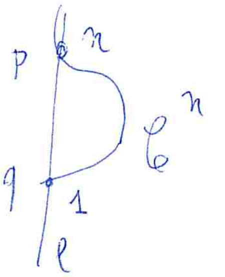
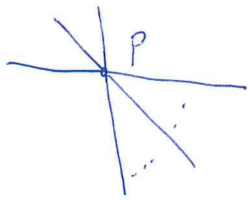
Let $p \in C^n$ with **multiplicity** n.

Then C^n **splits** into n **lines** (through p)



Indeed, a line l joining p with $q \in C^n$ has $n+1$ points in common with C^n, hence $l \subset C^n$, this yielding the conclusion



Therefore, if C^n is **irreducible**, then any **s-ple point** fulfils $s \leq n-1$

A plane irreducible alg. curve C^n has at most $\frac{(n-1)(n-2)}{2}$ double points $\left(g := \frac{(n-1)(n-2)}{2} - \underset{\substack{\#\text{ double}\\ \text{points}}}{\delta} : \text{genus of } C^n \right)$

Proof: Let $n \geq 3$ and C^n with $\frac{(n-1)(n-2)}{2} + 1$ double points. Choose $n-3$ simple points on C^n.

We have then

$$\frac{(n-1)(n-2)}{2} + 1 + n - 3 = \frac{(n-1)(n-2) + 2 + \overbrace{2n-6}^{2n-4 \,=\, 2(n-2)}}{2} =$$

$$= \frac{(n-1)(n-2) + 2(n-2)}{2} = \frac{(n-2)(n-1+2)}{2} = \frac{(n-2)(n+1)}{2}$$

points, through which a C^{n-2} passes

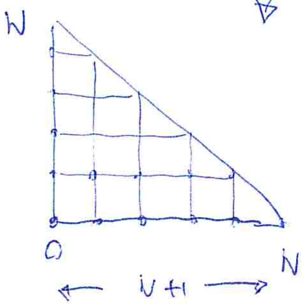
Indeed, C^{n-2} - curves are parametrized by

$$\frac{(n-2)(n+1)}{2} \text{ numbers}$$

monomial $x_0^a x_1^b x_2^c$

Aside: The C^N depend on $\frac{N(N+3)}{2}$ parameters:

$a+b+c=N$
a, b are free and vary from 0 to N

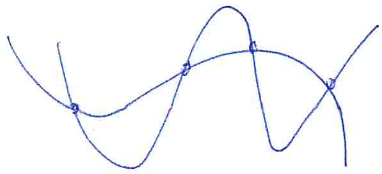


$$\frac{(N+1)(N+2)}{2} - 1 = \frac{N^2+3N+2}{2} - 1$$

$$= \frac{N(N+3)}{2}$$

(ex: $N=1$ (lines) $\# = 2$ $N=2$ (Conics) $\# = 5$

Now



The **total number of intersections** is at least

$$\underset{\substack{\uparrow \\ \text{simple points}}}{n-3} + \left[\overset{\text{double pts}}{\frac{(n-1)(n-2)}{2}} + 1\right] \cdot 2$$

$$= n-3 + (n-1)(n-2) + 2$$

$$= n-1 + (n-1)(n-2)$$

$$= (n-1)(n-1)$$

$$= n(n-2) + 1,$$

Contradicting **Bézout** (# intersections = $n(n-2)$

Notice that the result holds for $n=2$ as well.

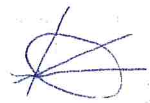
Remark: there exist C^n with exactly $\frac{(n-1)(n-2)}{2}$ double pts.

$$g := \frac{(n-1)(n-2)}{2} - \delta \qquad \text{genus of } C^n$$

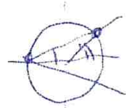
δ double pts (nodes & cusps)

★ **A genus zero C^n is rational**

A conic is rational



Example



$$\begin{cases} x^2+y^2-1=0 \\ y = t(x+1) \end{cases}$$

$$\begin{cases} x = \frac{1-t^2}{1+t^2} \\ y = \frac{2t}{1+t^2} \end{cases}$$

Indeed, let $n \geq 3$ and take its $\frac{(n-1)(n-2)}{2}$ double pts together with $n-3$ additional simple pts. One gets a pencil

$$\underset{C^{n-2}}{f_1} + h \underset{C^{n-2}}{f_2} = 0$$

In the base points,

$\underset{\text{double pts}}{(n-1)(n-2)} + \underset{\text{simple}}{n-3}$ intersections of C^n on C^{n-2} are absorbed ($n(n-2)$ is their total #)

$$= n^2 - 3n + 2 + n - 3$$
$$= n^2 - 2n - 1$$

Then $n(n-2) - (n^2-2n-1) = n^2-2n-(n^2-2n-1)$
$$= 1$$

so there is an extra intersection with a C^n whose coordinates are rational functions of h

The converse is true as well