ALGEBRAIC CURVES & RIEMANN SURFACES

Lecture XI

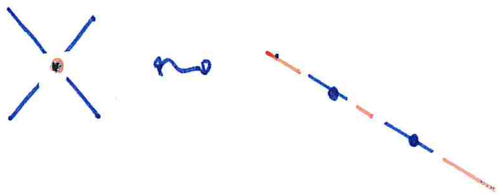
✶✶✶ Duality

(Gergonne)

$$\mathbb{P}^2 \longleftrightarrow \mathbb{P}^2$$

$[x_0, x_1, x_2]$ point coord. $\qquad$ $[u_0, u_1, u_2]$ line coordinates

points $\leftrightarrow$ lines



$\mathcal{C}$ viewed as the envelope of its tangents

curve $\qquad\qquad$ dual curve

$\mathcal{C}: \; F(x_0, x_1, x_2) = 0 \qquad \mathcal{C}^*: \; F^*(u_0, u_1, u_2) = 0$

✶ task: find F^* from F $\qquad \mathcal{C}^{**} = \mathcal{C}$

work affinely: $F(1, x, y) \equiv f(x, y) = 0$

tangent to $\mathcal{C}$ at P_0

$$\tau_{P_0} : \quad f_x^0 (x - x_0) + f_y^0 (y - y_0) = 0$$

$P_0 = (x_0\ y_0) \in \mathcal{C}$, $\quad f_x^0 = f_x(P_0)$

$f(x_0, y_0) = 0$

$$\Rightarrow \quad \underbrace{\frac{-f_x^0}{f_x^0 x_0 + f_y^0 y_0}}_{u} x - \underbrace{\frac{f_y^0}{(\ \ )}}_{v} y + 1 = 0$$

inhomogeneous line coordinates

Then find $\quad x_0 = x_0(u, v)$ $\quad$ from $f(x_0, y_0) = 0$
$\qquad\qquad\quad y_0 = y_0(u, v)$

$$\begin{cases} u = -\dfrac{f_x^0}{f_x^0 x_0 + f_y^0 y_0} \\ v = \dfrac{-f_y^0}{(\ \ )} \end{cases}$$

$\longrightarrow$ get $\boxed{\varphi(u, v) = 0}$

non-homogeneous

✶ homogenize and get $\boxed{F^*(u_0, u_1, u_2) = 0}$

$u = \dfrac{u_1}{u_0} \quad v = \dfrac{u_2}{u_0}$

and clear denominators

★ Intersection multiplicities

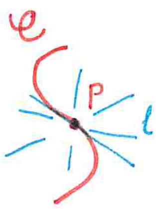
$C : F = 0$

$\boxed{m_p(C)}$: intersection multiplicity of C at p

$p \notin C$: $m_p(C) = 0$

$p \in C$: $\boxed{\min \left\{ \begin{array}{l} \text{intersection multiplicity} \\ \text{of } C \text{ with a line } \ell \text{ through } p \end{array} \right\}}$

$\equiv$ " multiplicity of 0 as a root of the polynomial obtained from $\begin{cases} \ell = 0 \\ F = 0 \end{cases}$



affine parameter

$x = p + tv$

$v \neq 0$

$p \underset{\mathcal{C}}{\in}$ simple (or non singular) : $m_p(C) = 1$

$\boxed{m_p(\mathcal{C}, \mathcal{C}')}$: intersection multiplicity of $\mathcal{C}$ and $\mathcal{C}'$ at a common point p

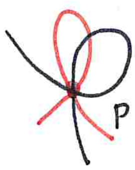
fully satisfactory definition : via elimination theory

intuitive, but vivid and reasonable definition :

$m_p(\mathcal{C}, t) = 2$

at a simple point

two common points

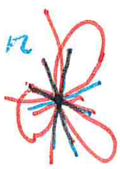


$\mathcal{C}$: p is an r-ple pt

$\mathcal{C}'$: p is an s-ple pt

no tangent in common :

r principal tangents at p



$\left[m_p(\mathcal{C}, \mathcal{C}') = r \cdot s \right]$

p absorbes all intersections of the $r+s$ lines (all tangents are part of the line pencil centred at p)

If $\mathcal{C}$ & $\mathcal{C}'$ have t tangents are in common at p

then $\left[m_p(\mathcal{C}, \mathcal{C}') = r \cdot s + t \right]$

p absorbs t extra intersections

★ inflexion points (flexes)

$m_p(\mathscr{C}, \tau) = 3$

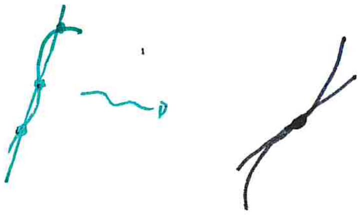
↓

simple pt

$r = 1$

$s = 1$

$t = 2$



$\mathscr{C}: \quad y = x^3$

$\tau_0: \quad y = 0$

$\mathscr{C} \cap \tau_0: \begin{cases} y = x^3 \\ y = 0 \end{cases}$

$x^3 = 0 \qquad x = 0$

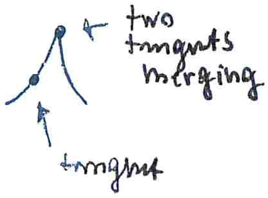
thrice

$\mathscr{C}$ and τ have two tangents in common

★ explain this

$\mathscr{C}$		$\mathscr{C}^*$
d : order		d^* : class
double point	↔	bitangent
cusp	↔	flex

(distinct tangent)



* **Polarity**

$$\mathscr{C}^n : \quad F(x_1, x_2, x_3) = 0 \qquad n > 1$$

↓ notice

(first) polar $\mathscr{P}$ of $P : [p_1\ p_2\ p_3]$

(*) $\mathscr{P}$: $\dfrac{\partial F}{\partial x_1} p_1 + \dfrac{\partial F}{\partial x_2} p_2 + \dfrac{\partial F}{\partial x_3} p_3 = 0$ $\qquad \dfrac{\partial F}{\partial x^i} p^i = 0$

$\mathscr{P}$ is of order $n-1$ $\qquad \mathscr{P} \equiv \mathscr{P}^{n-1}$

If $\mathscr{C}^n$ has no multiple points (thus it is irreducible), then $\mathscr{P}$ and $\mathscr{C}^n$ have no common components $\Rightarrow$ they have $n(n-1)$ points in common. If $Q : [q_1\ q_2\ q_3] \in \mathscr{C}^n \cap \mathscr{P}^{n-1}$

we have
$$\left[\begin{aligned} &\frac{\partial F}{\partial x_1}(Q)\, p_1 + \frac{\partial F}{\partial x_2}(Q)\, p_2 + \frac{\partial F}{\partial x_3}(Q)\, p_3 = 0 \\ &F(q_1, q_2, q_3) = 0 \end{aligned}\right]$$

But τ_Q : $\dfrac{\partial F}{\partial x_1}(Q)\, x_1 + \dfrac{\partial F}{\partial x_2}(Q)\, x_2 + \dfrac{\partial F}{\partial x_3} x_3 = 0$

tangent to $\mathscr{C}^n$ in $Q \in \mathscr{C}^n$ $\qquad \Rightarrow$ the tangents to $\mathscr{C}^n$ at the intersection points with $\mathscr{P}$ pass through P (cf the result for conics)

Conversely, the tangents to $\mathscr{C}^n$ issuing from P touch $\mathscr{C}^n$ in the intersection points with $\mathscr{P}$

It is easily verified that $\mathscr{P}$ is covariant with respect to collineations

Hint: $x^i = a^i{}_k \hat{x}^k \quad \det(a^i{}_k) \neq 0$ → Then (*) becomes

$$\frac{\partial \hat{F}}{\partial \hat{x}^k} = \frac{\partial \hat{F}}{\partial x^i} \frac{\partial x^i}{\partial \hat{x}^k} = \frac{\partial \hat{F}}{\partial x^i} a^i{}_k \qquad \frac{\partial \hat{F}}{\partial \hat{x}^i} \hat{p}^i = 0$$

★ Class of a curve $\mathcal{C}^n$

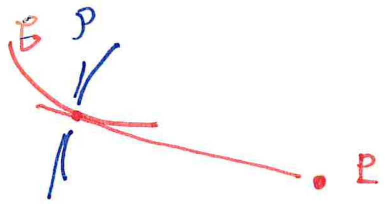
class of $\mathcal{C}^n$: # tangents to the curve drawn from a *generic* point in the plane of the curve

$\parallel$

(order of the dual curve)

If $\mathcal{C}^n$ has no multiple points ($\mathcal{C}^n$ smooth) then its class is $n(n-1)$

Tacit assumption: every pt in $\mathcal{C}^n \cap \mathcal{P}^{n-1}$ has multiplicity 1 (this is true if these points are simple for $\mathcal{P}^{n-1}$ and the respective tangents are distinct)



However, if $\mathcal{C}^n$ (irreducible) has singular points, then $\mathcal{P}^{n-1}$ passes through them: this is clear by (★).

$\Rightarrow$ the class of $\mathcal{C}^n$ decreases

→ we wish to arrive at the ★ first Plücker formula

$$\text{class} \equiv m = n(n-1) - 2\delta - 3\kappa$$

δ: ordinary double points: distinct tangents

$\updownarrow$ (bitangents of $\mathcal{C}^*$)

κ: ordinary cusps

$\updownarrow$ (flexes of the dual curve $\mathcal{C}^*$)

Proof Let C^n have a node (at $O:[0,0,1]$)

with distinct tangents $x_1=0$, $x_2=0$ 

Then

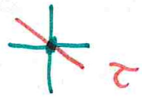
$$F(x_1,x_2,x_3) = x_1 x_2 x_3^{n-2} + \underbrace{\varphi_3(x_1x_2)}\, x_3^{n-3} + \underbrace{\varphi_n(x_1,x_2)} = 0$$

φ_i : forms of degree i (i.e hom. polyn)

$$\frac{\partial F}{\partial x_1} \equiv F_1 = x_2 x_3^{n-2}, \quad F_2 = x_1 x_3^{n-2} \quad F_3 = (n-2)\, x_1 x_2 x_3^{n-3} + \ldots$$

Polar of P^* $\mathscr{C}^{n-1}$: $(x_1^* x_2 + x_2^* x_1)\, x_3^{n-2} + \ldots = 0$

$O:[0,0,1]$ is then a simple point thereof, with tangent τ: $x_1^* x_2 + x_2^* x_1 = 0$: for generic P^*

$\tau \neq$ principal tangents 

$\Rightarrow$ O absorbs 2 intersections of $C^n \cap \mathscr{C}^{n-1}$

(the same holds for an isolated pt 

actually, since we work in $\mathbb{C}$, no essential distinction between a node and an isolated point exists)

Now, let O be a cusp, with tangent $x_2=0$ (actually it is to be counted twice). Then

$$\mathscr{C}^n : x_2^2 x_3^{n-2} + \varphi_3(x_1,x_2)\, x_3^{n-3} + \ldots$$

$$F_1 = \underset{\substack{\uparrow\, >2 \\ \text{in } x_1 x_2}}{\ldots}, \quad F_2 = 2x_2 x_3^{n-2} + \ldots \quad F_3 = (n-2)\, x_2^2 x_3^{n-3}$$

C^{n-1} polar of P^* :

$$2\, x_2^* x_2 x_3^{n-2} + \dots = 0$$

$\Rightarrow$ O is simple for C^{n-1}, with tangent $x_2 = 0$

$\Rightarrow$ O absorbs 3 intersections of $C^n \cap C^{n-1}$

(cf $m = rs + t$ here : $t = 1$ $\overset{C^n}{r = 2}$, $\overset{C^{n-1}}{s = 1}$

$m = 2 + 1 = 3$)

hypshot :

$$m = n(n-1) - 2 \cdot \delta - 3 \cdot \kappa$$

δ: double pts w. distinct tangents

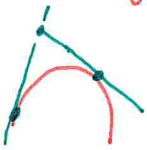
κ: cusps

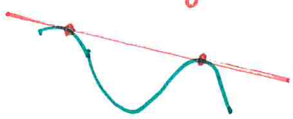
i.e the first Plücker formula

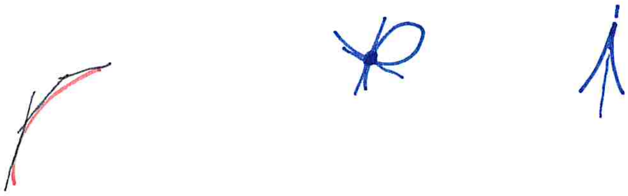
The <u>second Plücker formula</u> follows by <u>duality</u>

$$n = m(m-1) - 2\tau - 3\iota$$

degree — class — bitangents — flexes





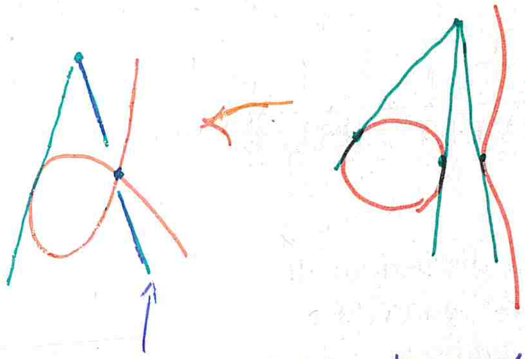


degree ↔ class
(order)

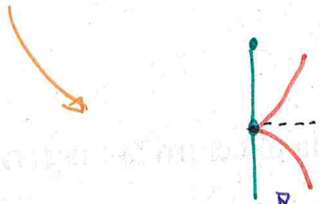
ordinary double pt ↔ bitangent

flex ↔ cusp

* Intuitive derivation of the first Plücker formula



two coalescing tangents to be removed for each ordinary double pt.



remove the extra tangent (three tangents to be removed) for each cusp

$$\Rightarrow \quad m = n(n-1) - 2\delta - 3\kappa$$