★ Hessian of a plane curve

$F(x_1, x_2, x_3) = 0$

$$H(x_1, x_2, x_3) = \begin{vmatrix} F_{x_1x_1} & F_{x_1x_2} & F_{x_1x_3} \\ F_{x_2x_1} & F_{x_2x_2} & F_{x_2x_3} \\ F_{x_3x_1} & F_{x_3x_2} & F_{x_3x_3} \end{vmatrix} = 0 \qquad F_{x_ix_j} = \frac{\partial^2 F}{\partial x_i \partial x_j} = F_{x_jx_i}$$

$$\equiv (F_{ij}) \qquad \text{degree} = n-2$$

$$\deg H = 3(n-2) \quad \text{(clear)}$$

$\boxed{\text{Lecture XII}}$

$H = 0$ is covariant with respect to collineations

$$\frac{\partial \bar F}{\partial \bar x_i} \equiv \bar F_i \qquad \frac{\partial \bar F}{\partial \bar x^k} = \frac{\partial F}{\partial x^i}\frac{\partial x^i}{\partial \bar x^k} \equiv F_i\, a^i_k \qquad x^i = a^i_k \bar x^k$$

Einstein

$$|a^i_k| \neq 0$$

$$\bar F_i = F_\ell\, a^\ell_i$$

$$\bar F_{ik} = F_{\ell p}\, a^\ell_i\, a^p_k \quad \Rightarrow \quad H = 0 \iff \bar H = 0$$

★ Crucial property of H:

The simple points of C^n belonging to H $(C^{3(n-2)})$ are flexes of C^n

Proof. Let C^n have a flex in $O: [0,0,1]$ with tangent $t: x_1 = 0$

$$F = x_1 x_3^{n-1} + (a_1 x_1^2 + 2a_2 x_1 x_2 + a_3 x_2^2)\, x_3^{n-2}$$
$$+ (b_1 x_1^3 + 3b_2 x_1^2 x_2 + 3b_3 x_1 x_2^2 + b_4 x_2^3)\, x_3^{n-3}$$
$$+ \dots \varphi_n(x_1, x_2) = 0$$

general formula

$x_1 (a_1 x + 2a_2 x_2)\, x_3^{n-2}$

Then $a_3 = 0$ $(m_O(t, \mathscr{C}) \geq 3)$ for a flex

and also $b_4 \neq 0$ for a genuine one $(m_O(t, \mathscr{C}) = 3)$

Compute its Hessian

$$H = \begin{vmatrix} 2a_1 x_3^{n-2} + .. & 2a_2 x_3^{n-2} + .. & (n-1) x_3^{n-2} + .. \\ 2a_2 x_3^{n-2} + .. & 6(b_3 x_1 + b_4 x_2) x_3^{n-3} & 2a_1 x_1 (n-2) x_3^{n-3} \\ (n-1) x_3^{n-2} + . & 2a_2 x_1 (n-2) x_3^{n-3} & (n-1)(n-2) x_1 x_3^{n-3} \end{vmatrix} = 0$$

higher order in x_3

First, observe that $\mathcal{H}$ passes through 0 (e.g. by Sarrus)

tangent τ to $\mathcal{H}$ in 0 : lower order terms in x_3
i.e. those in x_3^{3n-7}

$n \geq 3$

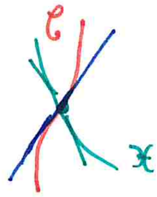
$$\cancel{4}^{\,2}\cancel{(n-1)}(n-2) a_1 a_2 x_1 + \cancel{4}^{\,2} a_2^2 \cancel{(n-1)} x_1 - \cancel{6}^{\,3}(n-1)^{\cancel{2}}(b_3 x_1 + b_4 x_2) - \cancel{4}^{\,2} a_2^2 \cancel{(n-1)}(n-2) x_1 \doteq 0$$

$$\tau: \quad 2(n-2) a_1 a_2 x_1 + 2 \cdot a_2^2 x_1 - 3(n-1)(b_3 x_1 + b_4 x_2) - 2 a_2^2 (n-2) x_1 = 0$$

$$\{2(n-2) a_1 a_2 x_1 - 2a_2^2 (\overbrace{n-2-1}^{n-3})\} x_1 - 3(n-1)(b_3 x_1 + b_4 x_2) = 0$$

distinct from $x_1 = 0$ $\iff$ $b_4 \neq 0$

upshot : 0 simple point for C^n
is an ordinary flex $\iff$ $\mathcal{H}$ passes through 0
with a simple intersection



4 Third Plücker formula

$\mathcal{C}^n$ smooth flexes : $i = \mathcal{C}^n \cap \mathcal{H}$

$\Rightarrow \ i = 3n(n-2)$

Now let $\mathcal{C}^n$ have multiple pts

assume they are double points (distinct tangents) and (ordinary) cusps

Let us show that $\mathcal{H}$ contains these pts

$\mathcal{C}^n$: $\quad x_1 x_2 x_3^{n-2} + (b_1 x_1^3 + 3b_2 x_1^2 x_2 + 3b_3 x_1 x_2^2 + b_4 x_2^3) x_3^{n-3}$

$+ \dots \varphi_i(x_1x_2) = 0$

work in $O : [0,0,1]$

double pt with distinct tangents

tangents: $x_1 = 0$, $x_2 = 0$

$$\mathcal{H}: \quad 0 = \begin{vmatrix} 6(b_1x_1 + b_2x_2)x_3^{n-3} + \dots & x_3^{n-2} + \dots & (n-2)x_2x_3^{n-3} \\ x_3^{n-2} + \dots & 6(b_1x_1 + b_2x_2)x_3^{n-3} & (n-2)x_1x_3^{n-3} \\ (n-2)x_2x_3^{n-3} + \dots & (n-2)x_1x_3^{n-3} & (n-1)(n-2)x_1x_2x_3^{n-4} + \dots \end{vmatrix}$$

$\mathcal{H}$ goes through O (clear)
double pt.

tangents $\quad x_1 x_2 = 0$

(coefficient of x_3^{3n-8} $= 0$)

same tangents as $\mathcal{C}^n$

(2 tangents in common)

$m_O(\mathcal{C}, \mathcal{H}) = 2 \cdot 2 + 2 = 6$

$\quad\quad\quad\quad\quad\quad\quad\quad r \cdot s + t$

Now let $O:[0,0,1]$ be an ordinary cusp

$$\mathcal{C}^n : \quad x_1^2 x_3^{n-2} + (b_1 x_1^3 + 3b_2 x_1^2 x_2 + 3b_3 x_1 x_2^2 + b_4 x_2^3) x_3^{n-3} + \dots \varphi_n(x_1,x_2) = 0$$

$$\mathcal{H} : \quad 0 = \begin{vmatrix} 2x_3^{n-2} + \dots & 6(b_2 x_1 + b_3 x_2) x_3^{n-3} + \dots & 2(n-2) x_1 x_3^{n-3} \\ 6(b_2 x_1 + b_3 x_2) x_3^{n-3} & 6(b_3 x_1 + b_4 x_2) x_3^{n-3} & (n-3)(*) x_3^{n-4} \\ 2(n-2) x_1 x_3^{n-3} & (n-3)(*) x_3^{n-4} & (n-2)(n-3) x_1^2 x_3^{n-4} \end{vmatrix}$$

$$(*) \to \frac{\partial}{\partial x_2}(b_1 x_1^3 + \dots)$$

Highest degree terms in x_3 : x_3^{3n-9}

$$\cdot \; 12(n-2)(n-3) x_1^2 (b_3 x_1 + b_4 x_2)$$

$$- 24(n-2)^2 x_1^2 (b_3 x_1 + b_4 x_2) \qquad = 0$$

$\rightsquigarrow$

$$x_1^2 (b_3 x_1 + b_4 x_2) = 0$$

$\Rightarrow$ O is a triple pt for $\mathcal{H}$, 2 of the tangents coincide with $x_1 = 0$ unless $b_4 = 0$ but in this case we will not have an ordinary cusp

$\Rightarrow$ in each cusp, 8 intersections with $\mathcal{H}$ are absorbed :

$$\underset{\text{cusp of } \mathcal{C}}{2} \cdot \underset{O \text{ triple pt for } \mathcal{H}}{3} + \underset{\text{tangents in common}}{2} = 8$$

$$r \cdot s + t$$

Hence, the number of flexes is given by

$$\boxed{i_f = \underbrace{3n(n-2)}_{\mathscr{C} \cap \mathscr{H}} - 6\delta - 8\kappa}$$

ordinary double pts   ordinary cusps

dual formula:

$$\boxed{\kappa^* = 3m(m-2) - 6b - 8f}$$

cusps of the dual curve $\mathscr{C}^*$ class   bitangents   flexes

eventually we have two Plücker formulae + their duals

★ The genera of $\mathscr{C}$ and $\mathscr{C}^*$, in turn, coincide

Let us deduce that $g = g^*$ from Plücker formulae

Scheme:

$$m = n(n-1) - 2\delta - 3\kappa \qquad (1)$$

$$n = m(m-1) - 2b - 3f \qquad (2)$$

$$f = 3n(n-2) - 6\delta - 8\kappa \qquad (3)$$

$\Rightarrow f = f(n, \delta, \kappa)$

Compute

$$g^* = \frac{(m-1)(m-2)}{2} - \underset{\underset{b}{\parallel}}{\delta^*} - \underset{\underset{f}{\parallel}}{\kappa^*}$$

From (1): $m = m(n, \delta, \kappa)$

 (2) $b = b(m, n, f) \underset{(1)\,\&\,(3)}{=} b(n, \delta, \kappa)$

Then compute $g^* = \ldots\ldots = \dfrac{(n-1)(n-2)}{2} - \delta - \kappa$

$= g$

★ $g = g^*$ (fifth Plücker formula)

$$g = \frac{(n-1)(n-2)}{2} - \delta - \kappa = \frac{(m-1)(m-2)}{2} - b - f = g^*$$

Proof:

$$m = (n-1)n - 2\delta - 3\kappa$$

$$f = 3n(n-2) - 6\delta - 8\kappa$$

$$3m = 3n(n-1) - 6\delta - 9\kappa$$

$$3m - f = \underbrace{3n(n-1) - 3n(n-2)}_{3n} - \kappa$$

$$3m - 3n = f - \kappa$$

$$m + (-2n+2) = (n-1)n - 2\delta - 3\kappa + (-2n+2)$$

$$n + (-2m+2) = m(m-1) - 2b - 3f + (-2m+2)$$

$$\Downarrow$$

$$(m-2n+2) - (n-2m+2) = n^2 - n - 2\delta - 3\kappa - 2n + 2 - (m^2 - m - 2b - 3f - 2m + 2)$$

$$3m - 3n = n^2 - 3n + 2 - 2\delta - 3\kappa - (m^2 - 3m + 2) + 2b + 3f$$

$$f - \kappa = (n-1)(n-2) - 2\delta - 3\kappa - (m-1)(m-2) + 2b + 3f$$

$$\Rightarrow \quad (n-1)(n-2) - 2\delta - 2\kappa$$

$$= (m-1)(m-2) - 2b - 2f$$

$$\Updownarrow$$

$$\underbrace{\frac{(n-1)(n-2)}{2} - \delta - \kappa}_{g} = \underbrace{\frac{(m-1)(m-2)}{2} - b - f}_{g^*}$$

$\bigstar$ A smooth plane quartic has 24 flexes and 28 bitangents

Proof Recall Plücker's formulae: (d, $\mathcal{C}$, $\mathcal{C}^*$ dual)

$$\begin{cases} d^* = d(d-1) - 2\delta - 3\kappa \\ d = d^*(d^*-1) - 2b - 3f \\ g = \dfrac{(d-1)(d-2)}{2} - \delta - \kappa \\ g = \dfrac{(d^*-1)(d^*-2)}{2} - b - f \end{cases}$$

C: smooth plane quartic:

$d = 4 \quad \delta = \kappa = 0$

$d^* = 4 \cdot 3 = 12$

$g = \dfrac{3 \cdot 2}{2} = 3 \qquad \overset{3}{\underset{\|}{}} g = \dfrac{11 \cdot 10}{2} - b - f = 55 - b - f$

$$\boxed{b + f = 52}$$

Also $4 = 12 \cdot 11 - 2b - 3f$

$2b + 3f = 12 \cdot 11 - 4 = 4 \cdot 33 - 4 = 4 \cdot 32 = 128$

$$\boxed{2b + 3f = 128}$$

$3b + 3f = 52 \cdot 3 = 156$

$2b + 3f = 128 \qquad \Rightarrow \quad b = 156 - 128 = 28$

$f = 52 - 28 = 24$

$$\boxed{\begin{matrix} f = 24 \\ b = 28 \end{matrix}}$$

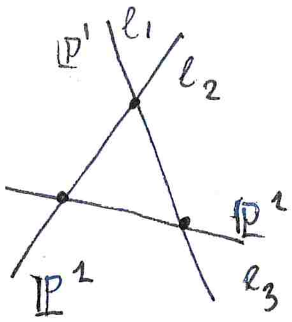
★ On Riemann's formula $\mathcal{C}^n$ smooth

$$(\lozenge) \quad \boxed{g = \frac{(n-1)(n-2)}{2}}$$

(V. Arnold, Catastrophe theory)

Illustration for $n=3$, $g=1$

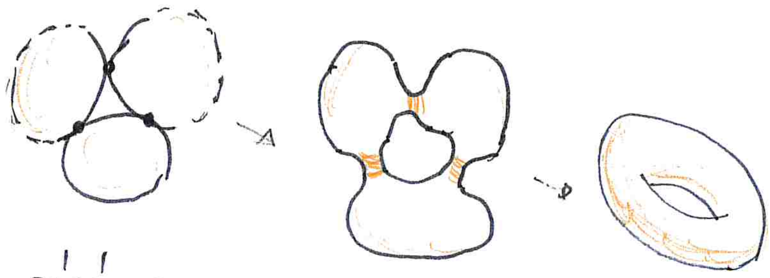
the general case follows easily, see next page



$\simeq S^2$

$\tilde{\mathcal{C}} = l_1 l_2 l_3$ (reducible) $f=0$

perturb: $f = \varepsilon$

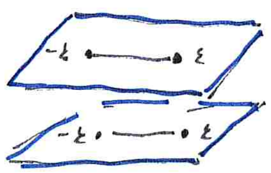


Explanation:  $\sim x'y'=0$

$x' = x - iy$
$y' = x + iy$

$x^2+y^2=0 \rightsquigarrow x^2+y^2=\varepsilon^2$

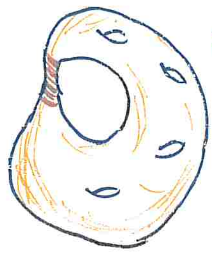
$y^2 = \varepsilon^2 - x^2$ $\pm\varepsilon$: branch points



glue appropriately $\rightsquigarrow$  cylinder

More generally, from

single double point

 $\rightsquigarrow$  smooth

pinched Riemann surface

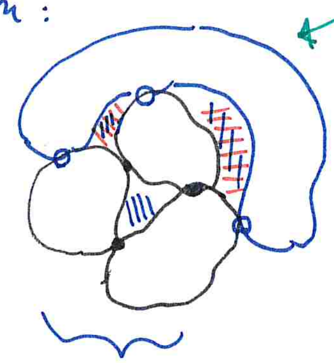
i.e. one

gets an extra handle, this easily leading

to Plücker's formula $g = \frac{(n-1)(n-2)}{2} - \delta$ ← ordinary double points & cusps

* Completion of the argument

Induction:
step



An additional sphere touches the existing ones (n) in n points, with the consequent formation of $n-1$ extra handles.

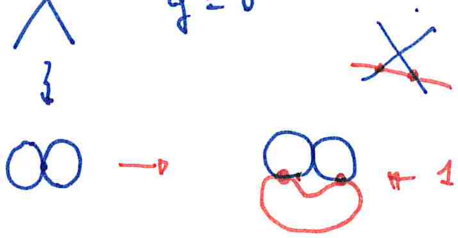
$$g_n = \frac{(n-1)(n-2)}{2}$$

Thus
$$g_{n+1} = \overbrace{\frac{(n-1)(n-2)}{2}}^{g_n} + (n-1) =$$

$$= \frac{(n-1)(n-2) + 2(n-1)}{2} = \frac{(n-1)(n-\cancel{2}+\cancel{2})}{2}$$

$$= \frac{n(n-1)}{2}$$

Start from X $g=0$



+ 1 handle $g=1$

□