★ Agnesi's Curve (also: Cauchy's curve)

$$\boxed{y(x^2+1)-1=0} \qquad y=\frac{1}{1+x^2}$$

Lecture XIII

rational $\Rightarrow g=0$ 3rd order

double points: $F = x_2(x_1^2+x_0^2)-x_0^3=0$

$$g=\frac{(n-1)(n-2)}{2}-\delta=0 \qquad x_2x_1^2+x_2x_0^2-x_0^3=0$$

$\Downarrow$

$\delta=1$

$$F_0 = 2x_2x_0-3x_0^2$$
$$F_1 = 2x_2x_1$$
$$F_2 = x_1^2+x_0^2$$

$F_i=0$
$$\begin{cases}(2x_2-3x_0)x_0=0\\ x_1x_2=0 \\ x_1^2+x_0^2=0\end{cases} \to \text{ either } x_1=0 \text{ or } x_2=0$$

Let $x_1=0$ Then $x_0=0$ $\Rightarrow$ get $\begin{array}{c}x_0\ \ x_1\ \ x_2\\ [0,0,1]\end{array}$

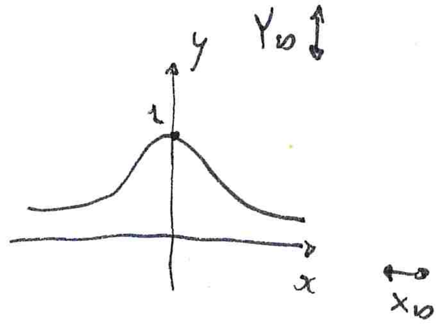
If $x_2=0$, then $x_0=0$ and $x_1=0$, not acceptable.

$\Rightarrow$ [$Y_\infty : [0,0,1]$ is the (unique) multiple pt.]

Near Y_∞ we have $\underbrace{x_2(x_1^2+x_0^2)}_{\text{term of highest degree in } x_2} + \dots = 0$

$$\underbrace{x_1^2+x_0^2}=0$$

$x_1^2+x_0^2=0$ principal tangents at Y_∞ : complex conjugate (isolated point)



* Inflection pts

* Plücker: $f = \overset{3}{3n}(\underset{\sim}{n-2}) - 6\delta - 8\kappa$ ($3-2=1$)

so $f = 9 - 6 \cdot 1 = 3$

Notice  that, by elementary methods, $(+)$ we get two of them (proper pts.)

$$\begin{cases} F = x_2 x_1^2 + x_2 x_0^2 - x_0^3 = 0 \\ F_0 = 2x_0x_2 - 3x_0^2 \\ F_1 = 2x_1x_2 \qquad F_2 = x_1^2 + x_0^2 \end{cases}$$

$F_{00} = 2x_2 - 6x_0$

$F_{01} = 0$

$F_{02} = 2x_0$

$F_{11} = 2x_2$

$F_{12} = 2x_1$

$F_{22} = 0$

$$H = \begin{vmatrix} 2x_2 - 6x_0 & 0 & 2x_0 \\ 0 & 2x_2 & 2x_1 \\ 2x_0 & 2x_1 & 0 \end{vmatrix} = 0$$

$$-8x_0^2x_2 - 4x_1^2(2x_2 - 6x_0) = 0$$

$$-8x_0^2x_2 - 8x_1^2x_2 + 24x_1^2x_0 = 0$$

$$\begin{cases} H = 0 \\ F = 0 \end{cases} \rightsquigarrow \begin{cases} -(x_0^2 + x_1^2)x_2 + 3x_1^2x_0 = 0 \\ x_2(x_1^2 + x_0^2) - x_0^3 = 0 \end{cases}$$

flexes:
$X_\infty : [0,1,0]$
$X_\pm : [1, \pm\frac{1}{\sqrt{3}}, \frac{3}{4}]$

pts at ∞ $x_0 = 0 \Rightarrow \begin{cases} -x_1^2x_2 = 0 \\ x_1x_2^2 = 0 \end{cases}$ $x_1 = 0 \Rightarrow x_2 \neq 0$ $\to Y_\infty : [0,0,1]$ double pt

$x_1 \neq 0$ $x_2 = 0$ $\to X_\infty : [0,1,0]$ genuine flex 

Let then $x_0 = 1$
$$\begin{cases} -(1+x^2)y + 3x^2 = 0 \\ y(x^2+1) - 1 = 0 \end{cases} \to y = \frac{1}{1+x^2}$$

$$\left[-1 + 3x^2 = 0\right] \quad x = \pm\frac{1}{\sqrt{3}}, \quad y = \frac{1}{1+\frac{1}{3}} = \frac{3}{4}$$

(+) check $y = \frac{1}{1+x^2}$ $y' = -\frac{2x}{(1+x^2)^2}$

$$y'' = \frac{-2(1+x^2)^2 + 2x \cdot 2(1+x^2) \cdot 2x}{(1+x^2)^4} = 0$$

$-2(1+x^2) + 8x^2 = 0$

$-1 - x^2 + 4x^2 = 0$

$3x^2 = 1 \quad x = \pm\frac{1}{\sqrt{3}}$

Nicomedes' concoid

$$(x-1)^2(x^2+y^2)-\ell^2x^2=0 \qquad \ell>0$$

multiple pts.

$$F= (x_1-x_0)^2(x_1^2+x_2^2)-\ell^2x_1^2=0$$

$$F_0 = -2(x_1-x_0)(x_1^2+x_2^2) = 0$$

$$F_1 = 2(x_1-x_0)(x_1^2+x_2^2)+2(x_1-x_0)^2x_1 - 2\ell x_1 = 0$$

$$F_2 = 2x_2(x_1-x_0)^2 = 0$$

- proper pts: $x_0 \neq 0 \rightsquigarrow x_0=1$

$$F_2=0 \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 0 \end{cases}$$

$x_1=1 \Rightarrow -2\ell=0$!

$\Rightarrow x_2=0 \longrightarrow O: [1,0,0]$

tangents $(1-\ell^2)x^2+y^2=0$

$0<\ell<1$	isolated pt	$\cdot$
$\ell=1$	cusp	$\prec$
$\ell>1$	node	α

- points at infinity $x_0=0$

$2x_1x_2=0 \Rightarrow x_1=0 \longrightarrow Y_\infty: [0,0,1]$

$x_2=0 \quad x_1^2\cdot x_1^2-\ell^2x_1^2=0$

$x_1^2(x_1^2-\ell)=0 \to x_1=0$ NO

$x_1^2=\ell$

but $F_0= -2\cdot x_1\cdot x_2^2 = -2x_1^3 = -2\ell^3=0$!

$Y_\infty:[0,0,1]$

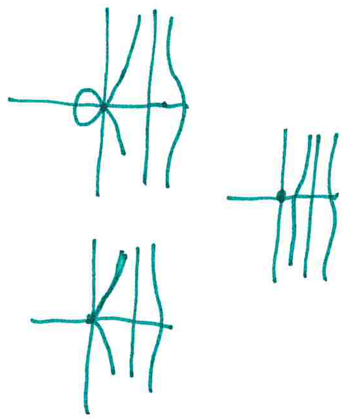
* Y_∞ is a double pt

$F_{00} = 2(x_1^2 + x_2^2)$ $\qquad F_{00}(Y_\infty) = 2 \neq 0$

* tangents: $\to$ terms of highest degree in x_2 set $=0$

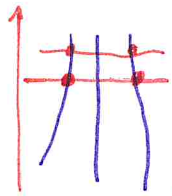
$$x_2^2 \underbrace{(x_1 - x_0)^2}_{=0} + \dots$$

$x_1 = x_0 \quad \rightsquigarrow \quad x = 1$ (double)



* $\boxed{\text{tacnode}}$

(two distinct branches)



$y = y_0$

$$(x-1)^2 \frac{x^2 + y_0^2}{y_0^2} - \ell^2 \frac{x^2}{y_0} = 0 \qquad y_0 \to \infty$$

$(x-1)^2 \sim 0 \qquad x \sim 1$

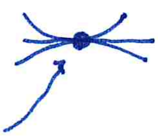
$$\begin{cases} F = 0 & 4 \\ x_1 = x_0 & 1 \end{cases}$$

$x_1 = 0 \Rightarrow x_0 = 0$

$\Rightarrow Y_\infty : [0, 0, 1]$

* 4 intersections coalescing at Y_∞

$y^2 = x^4 + \dots \qquad y = \pm x^2 + \dots$



two branches can be identified

$\begin{cases} y^2 = x^4 \\ y = 0 \end{cases}$ $\quad x^4 = 0 \quad$ all four pts coalesce

★ Una quartica piana Γ con 3 pti doppi è razionale

Infatti $g = 0$:
$$g = \frac{(4-1)(4-2)}{2} - \delta = 3 - \delta = 3 - 3 = 0$$

In linea di principio, la si può parametrizzare così:

Siano $A, B, C \in \Gamma$ i pti doppi, e $D \in \Gamma$ distinto da questi. Si consideri il fascio di coniche $\mathcal{F}$ avente A, B, C, D come pti base. Se $\mathcal{C} \in \mathcal{F}$
$$\underset{2}{\mathcal{C}} \cap \underset{4}{\Gamma} = \{8 \text{ pti}\},$$
di cui 6 in tutto cadono in A, B, C e 1 in D. Rimane allora un ulteriore punto P, che dipende razionalmente da un parametro affine che descrive $\mathcal{F}$

◆ ◆ ◆

A plane quartic Γ with 3 double pts is rational

Indeed, let $A, B, C \in \Gamma$ be its double points, and D another point on Γ. Let $\mathcal{F}$ be the pencil of conics with base pts A, B, C, D. By Bézout, every conic $\mathcal{C} \in \mathcal{F}$ intersects Γ in 8 points: A, B, C (double) D (simple intersection) and in an extra point P, depending rationally on an affine parameter on $\mathcal{F}$.

Example: Pascal's limaçon

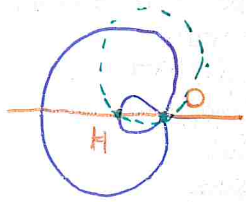
★ Double pts: O, $C_\pm$ (cyclic points)

G. Ferrarese
Curve algebriche
2004

4 On Pascal's limaçon

$$\mathcal{C}: \quad (x^2+y^2)^2 + 4x(x^2+y^2) + 3x^2 - y^2 = 0$$

Let us ascertain its rational character



double points: O, $C_\pm$
cyclic pts

consider the circles (passing through $C_\pm$) and O

$$\mathcal{X}: \quad x^2 + y^2 + \alpha x + \beta y = 0$$

$H = (-1, 0) \in \mathcal{C}$: $1 - 4 + 3 = 0$

if $H \in \mathcal{X}$ $\quad 1 - \alpha = 0 \Rightarrow \alpha = 1$

$\leadsto$ get $\quad x^2 + y^2 + x + \beta y = 0 \quad$ set $t = -\beta$

$$x^2 + y^2 = ty - x$$

$t = 2y_c \qquad y_c = \frac{t}{2}$

$x_c = -\frac{1}{2}$

Subsequently,

$$(ty - x)^2 + 4x(ty - x) + 3x^2 - y^2 = 0$$

$$t^2y^2 - 2txy + x^2 + 4txy - 4x^2 + 3x^2 - y^2 = 0$$

$$(t^2-1)y^2 + 2txy = 0 \qquad y\left((t^2-1)y + 2tx\right) = 0$$

$y = 0 \quad x^2 + x = 0 \quad x(x+1) = 0 \quad x = 0 \quad x = -1$

$$\begin{cases} (t^2-1)y + 2tx = 0 \\ x^2 + y^2 = ty - x \end{cases} \qquad x = \frac{1-t^2}{2t} \cdot y$$

$$\left(\left(\frac{t^2-1}{2t}\right)^2 + 1\right) y^2 = ty - \frac{1-t^2}{2t} y$$

$$4t^2 \quad \frac{(t^2-1)^2+4t^2}{4t^2}\, y^2 = \frac{2t^2+t^2-1}{2t}\, y \cdot 4t^2 \quad \cdots \; y = 0$$

$$\left((t^2-1)^2+4t^2\right) y = (3t^2-1)\,2t$$

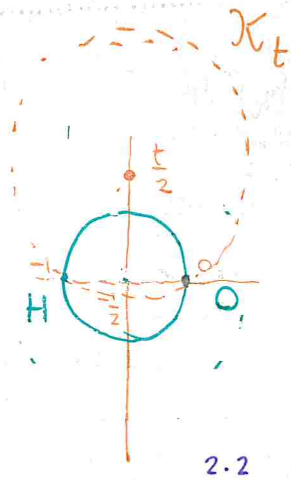
$$\begin{array}{c} \| \\ t^4-2t^2+1+4t^2 \\ \| \\ (t^2+1)^2 \end{array} \qquad (t^2+1)^2\, y = 2t\,(3t^2-1)$$

$$y = \frac{2t(3t^2-1)}{(t^2+1)^2}$$

$$x = \frac{1-t^2}{\cancel{2t}} \cdot \cancel{2t} \cdot \frac{3t^2-1}{(t^2+1)^2}$$

$$= \frac{(3t^2-1)(1-t^2)}{(t^2+1)^2}$$

$$\begin{cases} x = \dfrac{(3t^2-1)(1-t^2)}{(1+t^2)^2} \\[2ex] y = \dfrac{2t\,(3t^2-1)}{(t^2+1)^2} \end{cases}$$



2.2

$$X_t \cap \mathcal{C} = \{C_\pm, O, H, P\}$$

4 2 1

$$4 \times 2 = 8 \; = \; \underbrace{4+2+1}_{7} + 1$$

$$m(\mathcal{C}, \mathcal{B}') = r.s + t \quad (t=0)$$

XIII-7

* Calculations for Pascal's limaçon

$$F(x_1,x_2,x_3) = (x_1^2+x_2^2 - a\, x_1 x_3)^2 - \ell^2 x_3^2 (x_1+x_2^2) = 0$$

$a > 0$
$\ell > 0$

- points at ∞ $\quad x_1^2+x_2^2 = 0$

$\Rightarrow \quad C_\pm : [\,1, \pm i, 0\,]$ cyclic pts

$$F_1 = 2\left[(2x_1 - a x_3)(x_1^2+x_2^2 - a x_1 x_3) - \ell^2 x_1 x_3^2\right]$$
$$F_2 = 2\left[2x_2\,(x_1^2+x_2^2 - a x_1 x_3) - \ell^2 x_2 x_3^2\right]$$
$$F_3 = 2\left[-a x_1\,(x_1^2+x_2^2 - a x_1 x_3) - \ell^2 (x_1^2+x_2^2) x_3\right]$$

$F_i = 0 \ \forall i \Rightarrow \quad O : [0,0,1], \quad C_\pm : [1, \pm i, 0]$

$$F_{11} = 2\left[2(x_1^2+x_2^2 - a x_1 x_3) + (2x_1 - a x_3)^2 - \ell^2 x_3^2\right]$$
$$F_{12} = 4 x_2\,(2x_1 - a x_3)$$
$$F_{13} = -2\left[a\,(x_1^2+x_2^2 - a x_1 x_3) + a x_1 (2x_1 - a x_3) + \ell^2 x_1\right]$$
$$F_{22} = 2\left[2\,(x_1^2+x_2^2 - a x_1 x_3) + 4x_2^2 - \ell^2 x_3^2\right]$$
$$F_{23} = -4\left[a\, x_1 x_2 + \ell^2 x_2 x_3\right]$$
$$F_{33} = 2\left[a^2 x_1^2 - \ell^2 (x_1^2+x_2^2)\right] \quad \Rightarrow O, C_\pm \text{ double pts}$$

$\forall a, \ell \qquad F_{22}(0,0,1) = -2\ell^2 \neq 0$
$\qquad\qquad F_{22}(1, \pm i, 0) = -8 \neq 0$

tangents: $\quad C_+ \quad \frac{1}{2}\left(8x_1^2 + 16 i\, x_1 x_2 - 8a\, x_1 x_3 - 8x_2^2 - 8a i\, x_2 x_3 + 2a^2 x_3^2\right) = 0$

degenerate conic

$$A_+ = \begin{bmatrix} 4 & 4i & -2a \\ 4i & -4 & -2ai \\ -2a & -2ai & a^2 \end{bmatrix}$$ has rank 1 $\Rightarrow \parallel$

$\Rightarrow \left\{ \begin{array}{ll} C_+ : & \text{first order cusp} \\ C_- : & = \end{array} \right\}$

$O : [0,0,1]$

$$(a^2-\ell^2)x_1^2 - \ell^2 x_2^2 = 0$$ tangents

$\ell < a$ real & distinct  node

$\ell = a$  cusp (of first order

$\ell > a$  isolated pt.

$\rightsquigarrow$ O either a node or a cusp

From Plücker

	n	δ	κ	$m = n^*$	$b = \delta^*$	$f = \kappa^*$
$\ell \neq a$	4	1	2	4	1	2
$\ell = a$	4	0	3	3	1	0

quartic, with the same structure

$g = 0$

cubic with a double pt.