Singularities of plane curves: examples

ALGEBRAIC CURVES & RIEMANN SURFACES
Prof. M. Spera UCSC

- $\boxed{y^2 = 2x^2y + x^5}$ $\quad \mathscr{C}$

$O: (0,0)$ double pt

$\mathscr{C} \cap \underset{y=0}{\overset{\shortparallel}{r}} = \{O\}$. (5-fold)

$\begin{cases} y^2 = 2x^2y + x^5 \\ y = 0 \end{cases}$

$\begin{cases} x^5 = 0 \\ y = 0 \end{cases}$

neither a node nor an ordinary cusp

Lecture XIV

linear branches: $\quad p \quad y = ax^2 + bx^3 + \dots$

$$(ax^2 + bx^3 + cx^4 + \dots)^2 = 2x^2(ax^2 + bx^3 + cx^4 + \dots) + x^5$$

$$a^2x^4 + \dots = 2a\,x^4 + \dots$$

$\longrightarrow$ $[a(a-2) = 0]$ $\quad \nearrow a = 0 \quad \searrow a = 2$ $\quad$ 5-fold int. $p \cap \mathscr{C}$

- $\boxed{a = 0}$

$$(bx^3 + cx^4 + \dots)^2 = 2x^2(bx^3 + cx^4) + x^5$$

$$0 = (2b+1)x^5 \qquad b = -\frac{1}{2}$$

$\rightarrow$ 6 intersections $\quad \left(-\frac{1}{2}x^3 + cx^4 + \dots\right)^2 = 2cx^6 + \dots$

$$\frac{1}{4}x^6 + \dots = 2c\,x^6 + \dots \qquad c = \frac{1}{2}$$

$\longrightarrow$ $\left\{ y = -\frac{1}{2}x^3 + \frac{1}{2}x^4 + \dots \right\}$

- $\boxed{a = 2}$

$$(2x^2 + bx^3 + cx^4 + \dots)^2 = 2x^2(2x^2 + bx^3 + cx^4 + \dots) + x^5$$

$$4x^2 = 4x^2$$

$$4b\,x^5 + \dots \quad (2b+1)x^5 \qquad 4b = 2b+1$$

$$2b = 1 \quad b = \frac{1}{2}$$

$$(2x^2 + \tfrac{1}{2}x^3 + cx^4 + \dots)^2 = 2x^2(2x^2 + \tfrac{1}{2}x^3 + cx^4 + \dots) + x^5$$

$$\vdots$$

$$\frac{1}{4}x^6 + 4cx^6 = 2cx^6$$

$$\frac{1}{4} + 4c = 2c \qquad 2c = -\frac{1}{4}$$

$$c = -\frac{1}{8}$$

$$\rightarrow \quad \boxed{y = 2x^2\left(+\tfrac{1}{2}x^3 - \tfrac{1}{8}x^4 + \dots\right)}$$

double ★ **tacnode**

- double tangent
- 5-fold intersections with the
- distinct parabolas

$$\boxed{y^2 = x^3 + x^2 y}$$

$O : (0,0)$ double pt : $y^2 = 0$ tangents

try $y = ax^2 + bx^3 + \dots$

$(ax^2+bx^3+\dots)^2 = x^3 + x^2(ax^2+bx^3+\dots)$

cannot be cancelled

no thouroughfare

new ansatz: $x = t^2 \quad y = at^3 + bt^4 + ct^5 + \dots$

$(at^3+bt^4+ct^5)^2 = t^6 + t^4(at^3+bt^4+ct^5+\dots)$

Newton-Puiseux

$a^2 t^6 + \dots = t^6 + \dots \longrightarrow \boxed{a = \pm 1}$

$y^2 = x^3$

$y = \pm x^{3/2}$

not distinguishable

however

$x = e^{i\varphi} \qquad x^{\frac{3}{2}} = e^{\frac{3}{2} i\varphi} =: y_1$

$\varphi \mapsto \varphi + 2\pi \qquad e^{\frac{3}{2} i(\varphi + 2\pi)}$

$= e^{\frac{3}{2} i\varphi} \cdot \underbrace{e^{3i\pi}}_{-1} = -y_1 = y_2$

the two roots are exchanged : branch point!

★ ordinary cusp

$$\boxed{(y-x^2)^2 - x^5 = 0}$$

O : $(0,0)$ double pt $y^2 = 0$ coinciding tangents

$$\begin{cases}(y-x^2)^2 - x^5 = 0\\ y = 0\end{cases} \qquad \begin{cases}x^4 - x^5 = 0\\ y = 0\end{cases}$$

in O: $(0,0)$ 4-fold intersection

Ansatz: $y = ax^2 + bx^3 + \dots$

$$\big((a-1)x^2 + bx^3 + \dots\big)^2 - x^5 = 0$$

$$\Rightarrow \quad a = 1 \qquad (bx^3 + \dots)^2 - x^5 = 0$$

? no avail!

Try then
$$x = t^2$$
$$y = t^4 + at^5 + \dots$$

set
$$(at^5 + \dots)^2 - t^{10} = 0$$
$$a^2 t^{10} - t^{10} = 0 \Rightarrow a^2 = 1$$
$$a = \pm 1$$

$$\begin{cases}x = t^2\\ y = t^4 \pm t^5\end{cases}$$

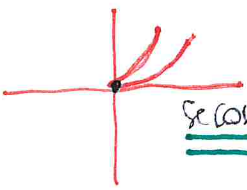
not distinguishable:

$x = \rho e^{i\varphi}$ $\rho > 0$ small

$t = \rho^{\frac12} e^{i\frac{\varphi}{2}}$ $y_1 = \rho^2 e^{i2\varphi} + \rho^{5/2} e^{i\frac52\varphi}$

Let $\varphi \mapsto \varphi + 2\pi$ $\underset{t^4+t^5}{y_1} \mapsto \rho^2 \underbrace{e^{i2(\varphi+2\pi)}}_{= e^{2i\varphi}} + \rho^{5/2} e^{i\frac52\varphi} \underbrace{e^{5i\pi}}_{= -1}$

$$\underbrace{\rho^2 e^{2i\varphi} - \rho^{5/2} e^{i\frac52\varphi}}_{= t^4 - t^5}$$



second order cusp