see (Narasimhan)

Boundary pts .. Algebraic boundary pts

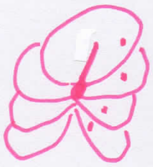
X RS $\quad \{x_\nu\}_{\nu \geq 1}$ sequence in X

$p: X \to \mathbb{C}$ local homeo

with the following properties:

1) $\{x_\nu\}$ discrete (no limit pts in X)

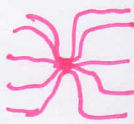
2) $p(x_\nu) \to a \in \mathbb{P}^1 = \bar{\mathbb{C}}$



3) $D_\varepsilon = \{z \mid |z-a| < \varepsilon\} \quad a \in \mathbb{C}$

$D_\varepsilon = \{z \mid |z| > \frac{1}{\varepsilon}\} \cup \{\infty\}$ for $a = \infty$

then for ε small, all but finitely many $\{x_\nu\}$ lie in the same c. comp. of $p^{-1}(D_\varepsilon)$



$\boxed{\{x_\nu\} \sim \{y_\nu\}}$: $z_\nu = \begin{cases} x_{\frac{\nu+1}{2}} & \nu \text{ odd} \\ y_{\frac{\nu}{2}} & \nu \text{ even} \end{cases}$ has the same properties

Boundary pt (rel to p) $\boxed{P = [\{x_\nu\}] \text{ equivalence class}}$

$\hat{X} = X \cup \{\text{b.pts}\}$

Neighbourhood of $P = [\{x_\nu\}]$

Ω_ε = conn. comp of $p^{-1}(D_\varepsilon) \ni$ all but f. many x_ν

$\tilde{\Omega}_\varepsilon = \Omega_\varepsilon \cup \{\underset{\text{bp.}}{Q}\}$ $\quad Q = [\{y_\nu\}]$, $\{\nu \mid y_\nu \notin \Omega_\varepsilon\}$ finite

The $\tilde{\Omega}_\varepsilon$ form a fundam system on neigh of $P \in \hat{X} \setminus X$

Lecture XIX

ALGEBRAIC CURVES & RIEMANN SURFACES

The topology is Hausdorff $\quad P, Q \quad P \neq Q$
$\{x_\nu\} \; \{y_\nu\}$

$\exists\, \varepsilon > 0 \quad \Omega_{\varepsilon,1}, \Omega_{\varepsilon,2}$ of $p^{-1}(D_\varepsilon)$
are distinct and $\tilde{\Omega}_{\varepsilon,1} \cap \tilde{\Omega}_{\varepsilon,2} = \phi$.

p extends to $\tilde{p} : \hat{X} \to \mathbb{P}^1 \quad \tilde{p}(P) = a = \lim p(x_\nu)$

P algebraic: $\quad p(\Omega) \subset D_\varepsilon - \{a\}$

D_ε disc around $\tilde{p}(P)$

Ω = conn. comp. of $p^{-1}(D_\varepsilon)$ containing almost all pts

! $\Omega \not\ni P$

$\tilde{p} : \Omega \to D_\varepsilon \setminus \{a\}$ is a finite covering

$$\Delta_R = \{ z \in \mathbb{C} \,/\, |z| < R \}$$
$$\Delta^*_R = \Delta_R \setminus \{0\} \qquad \exists\, n \geq 1 \text{ s.th.}$$
$$p : \Omega \to D_\varepsilon - \{a\}$$
isom to $\quad p_n : \Delta^*_{\varepsilon^{\frac{1}{n}}} \to D_\varepsilon - \{a\}$

typical example

$$p_n(z) = a + z^n, \; a \in \mathbb{C}$$
or
$$p_n(z) = z^{-n}, \quad a = \infty$$
$$= \left(\tfrac{1}{z}\right)^n = \xi^n$$

$\tilde{\Omega} = \Omega \cup \{P\}$ neigh of P in $\hat{X}$ non containing any other b. pt. of X $\quad p|_\Omega \to D_\varepsilon \setminus \{a\} \sim p_n$, one has

$$\varphi : \tilde{\Omega} \to \Delta_{\varepsilon^{\frac{1}{n}}} \quad \text{homeo}, \; \varphi(P) = 0, \quad p \circ \varphi^{-1} = p_n \text{ on } \Delta^*_{\varepsilon^{\frac{1}{n}}}$$

$\varphi|_\Omega$ is holomorphic

$\hat{X} = X \cup \{\text{algebr. b. pts}\}$

extend the complex structure

chart around $P \in \hat{X} \setminus X$:

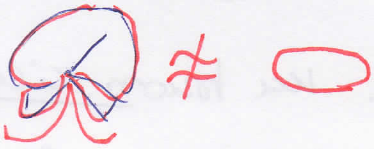
$(\tilde{\Omega}, \varphi)$

$\hat{p} := \tilde{p}\big|_{\hat{X}}$ $\quad$ $(\hat{X}, \hat{p})$: algebraic completion of (X, p)

$\hat{p} : \hat{X} \to \mathbb{P}^1$ is holom., but it is not nec. a homeom.

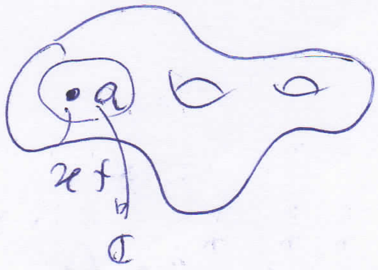
P alg. b. pt. $\quad$ Ω n - sheeted covering of $D_\varepsilon \setminus \{a\}$, $n > 1$

then $\hat{p}$ not a local homeo at P



*** $\hat{p}$: branched covering

X RS $a \in X$



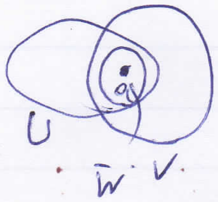
$\mathcal{U} \ni a$

open $f : U \to \mathbb{C}$ ho

$(U, f) \sim (V, g)$

"RS of a holomorphic function"

& define the same germ if $\exists\, W \subset U \cap V$ $f|_W = g|_W$



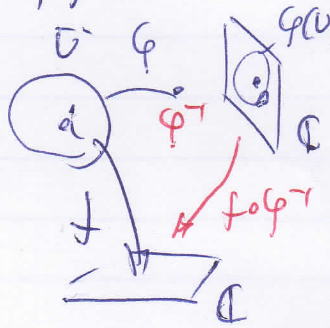
Equivalence class: a germ of a hol f. at a

$[(U, f)] =$ germ of f at a $\equiv \underline{f}_a$

$\underline{f}(a) = f(a)$ $\forall\ (U, f)$ defining the same germ

[morally: germ = power series]

(U, φ): chart centred at a: $\varphi(a) = 0$



give sense to derivatives

$$\underline{f}^{(k)}(a) = \left(\frac{d}{dz}\right)^k f \circ \varphi^{-1}\Big|_{z=0}$$ $\forall$ pair (V, f) defining $\underline{f}_a$

$\mathcal{O}_a = \{\text{germes at } a\}$: ring or better, an algebra ($\mathbb{C}$-algebra)

m_a: $\{\text{germs} \,/\, \underline{f}(a) = 0\}$ ideal unique maximal ideal

$\mathcal{O}_a \setminus m_a$ = units of $\mathcal{O}_a$ ($\underline{f}_a$ invertible iff $\underline{f}_a(a) \neq 0$)

Therefore, via a chart

$$\underline{f}_a \longmapsto \underbrace{\sum_{n=0}^{\infty} \frac{1}{n!} \underline{f}_a^{(n)}(a)\, z^n}_{\mathbb{C}(z)}$$ $\mathbb{C}$-algebra isomorphism

$\mathbb{C}(z)$ power series with non zero radius of convergence

XIX-4

$\sigma \equiv \mathcal{O}_X = \bigsqcup_{a \in X} \mathcal{O}_a$ (disj union)

$$\begin{array}{ll} p: \mathcal{O}_X \to X & p(\underset{\mathcal{O}_X}{\underbrace{f}}) = a \\ \quad f \mapsto f(a) & \end{array}$$

$\underline{f}_a \in \mathcal{O}_X$ (U, f) pair defining $\underline{f}_a$.

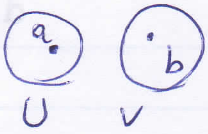
$N(U,f) = \{ \underline{f}_a \,/\, a \in U \}$

topology: make the $N(U,f)$ a fund. system of neighbourhoods of $\underline{f}_a$, when (U,f) runs over all pairs defining $\underline{f}_a$

Lemma: $\mathcal{O}_X$ is Hausdorff and $p: \mathcal{O}_X \to X$ is a local homeo

Proof $\underline{f}_a, \underline{g}_b \in \mathcal{O}_X$ $\underline{f}_a \neq \underline{g}_b$. They can be separated: indeed

① $a \neq b$ $(U,f), (V,g)$ pairs.



find $U \cap V = \emptyset$
$\Rightarrow N(U,f) \cap N(V,g) = \emptyset$

② $a = b$ U connected, open $U \ni a$, f, g hol.
$N(U,f) \cap N(U,g) = \emptyset$. Otherwise
If h_a is in the inters., f & g define $h_a \Rightarrow$ coincide in a neigh of a. But U is connected so,

by analytic continuation implies $f \equiv g$ so that $\underline{f}_a = \underline{g}_a$, a contradiction

$\mathcal{O}_X$ is Hausdorff ② XIX-5

Moreover, if $U \subset X$ open f def on U

then $p(N(U,f)) = U$ and $p|_{N(U,f)}$ is injective

$a \mapsto f_a$ inverse, whence the lemma follows

[If X has a countable base, then any connected comp of $\mathcal{O}_X$ has a countable base (Poincaré - Volterra)]

*** Riemann surface of an analytic function

$X = \mathbb{C}$ $\mathcal{O}_{\mathbb{C}}$ M a conn. comp of $\mathcal{O}_{\mathbb{C}}$

$p : M \to \mathbb{C} \subset \mathbb{P}^1 = \bar{\mathbb{C}}$ = restriction to M of

$f_a \mapsto a$; $p : M \to \mathbb{C}$ local homeo $\Rightarrow$

$\exists!$ structure of RS on M making p holo

$\Rightarrow p$ locally analytic : any $a \in M$ has $U \ni a$ with $p|_U$ anal. iso of U onto $p(U)$

Define h, holo on M, by $h(f_a) = f(a)$, $a \in U$

$\Rightarrow h$ is holo.

h ("universal") describes all germs obtained via analytic continuation of a fixed germ $f_a \in M$.

$\hat{M} = M \cup \{$algebraic boundary pts$\}$

$\hat{p} : \hat{M} \to \mathbb{P}^1 = \bar{\mathbb{C}}$ the map extending p

Let U open, connected [illegible] assume f def a germ $\underline{f_a}$, $a \in U$ (=> ok on all of U) connected

$$E = \hat{M} \setminus M \quad \text{discrete}$$

X_f = $M \cup$ pts which are not essential singularities of h ($h \to$ hol ex or a pole)

$\Rightarrow \exists\ h_f$ on X_f merom. such that

$h_f\big|_M = h$ let $p_f : X_f \to \mathbb{P}^1 = \hat{p}\big/_{X_f}$

$$\boxed{(X_f, p_f, h_f) : \text{RS of } f \text{ on } \bar{U}}$$

$P \in X_f - M$ $p_f(P) = a$ then near P,

$p_f \sim z \mapsto a + z^n$ $n \geq 1$ or $z \mapsto z^{-n}$

z local coor. at P on X_f w l. coord at a on $\mathbb{P}^1$

we have the following (local description)

$$\begin{cases} w = z^n \\ h_f = \sum_{-k}^{\infty} a_\nu w^{\nu/n} = \sum_{-k}^{\infty} a_\nu z^\nu \end{cases}$$

XIX-7

The RS of an algebraic function

$$F(x,y) = a_0(x)\, y^n + a_1(x)\, y^{n-1} + \dots\ a_n(x) = 0$$

irred. $V = \{ F(x,y) = 0 \}$

$S_0 = \{ a_0(x) = 0 \}$

$S_1 = \{ x \,/\, \exists y : F(x,y) = \frac{\partial F}{\partial y}(x,y) \} = 0$

$S = S_0 \cup S_1 \cup \{\infty\} \subset \mathbb{P}^1 \qquad \pi : V \to \mathbb{C}$, $(x,y) \mapsto x$

$V' = V - \pi^{-1}(S) = V - \pi^{-1}(S_0 \cup S_1)$

$\pi' = \pi|_{V'}$

Then $\left[\pi'|_{\pi^{-1}(D_\varepsilon - \{a\})} \to D_\varepsilon - \{a\} \quad \text{finite covering (}n\text{-sheeted)} \right]$

$\pi^{-1}(D_\varepsilon \setminus \{a\}$: finitely many connected components.

Let W = c. comp of V', then $\pi'|_W : W \to \mathbb{P}^1 \setminus S$

covering. Let $W_1 \dots W_r$ be the comp of V'.

$\pi_j = \pi|_{W_j} : W_j \to \mathbb{P}^1 \setminus S$ finite covering

$\Rightarrow$ every boundary pt of W_j is algebraic.

$\hat{\pi}_j : \hat{W}_j \to \mathbb{P}^1$ alg. completion of π_j

if $p \in \hat{W}_j \setminus W_j$, $a = \hat{\pi}_j(p)$, $\exists\, U \ni p$ and $\varepsilon > 0$ st.

$\hat{\pi}_j|_U \to D_\varepsilon$ isom to $z \mapsto a + z^m$ / z^{-m} for some $m > 0$

$\Rightarrow \hat{\pi}_j|_U \to D_\varepsilon$ proper.

Then $\forall a \in S$, $\exists \epsilon > 0$ s.t.

$$\hat{\pi}_j \Big|_{\hat{\pi}_j^{-1}(D_\epsilon)} \to D_\epsilon \quad \text{is proper}$$

But $\hat{\pi}_j\big|_{W_j} \to \mathbb{P}^1 \setminus S$ is proper, $\hat{\pi}_j : \hat{W}_j \to \mathbb{P}^1$

proper $\Rightarrow$ $\hat{W}_j$ is compact.

$P_2 : V \to \mathbb{C}$ $(x, y) \mapsto y$. $\quad \eta = P_2\big|_{V'}$ holo on V'

$\Rightarrow$ $\eta_j = \eta\big|_{W_j}$ holo.

Claim: η_j extends to a meromorphic f. on $\hat{W}_j$

Let $a \in S$. $\quad \overset{\vee}{\underline{P}} = \hat{\pi}_j(P) = a$: Locally $\left(\begin{matrix} z \sim P \\ w \sim a \end{matrix}\right)$

$\hat{\pi}_j : z \mapsto z^m = w$.

Then $\quad \eta_j^n + \dfrac{a_1(w)}{a_0(w)} \eta_{j-1}^{n-1}(z) + \cdots \dfrac{a_n(w)}{a_0(w)} = 0 \qquad w = \hat{\pi}_j(z)$

$\dfrac{a_\nu}{a_0}$ mer at $w = 0$ $\Rightarrow$ $\exists\, C > 0$ $N > 0$ with.

$$\frac{a_\nu(w)}{a_0(w)} \leq \frac{C}{|w|^N} \quad \text{near } w = 0.$$

Then $|\eta_j(z)| \leq 2 \max \dfrac{C^{1/\nu}}{|w|^{N/\nu}} \leq \dfrac{C_1}{|z|^k}$ for constants C_1, k.

$\Rightarrow$ η_j extends meromorphically to $\hat{W}_j$.

Crucial fact: V' is connected } reason F is irreducible

If not, $\pi_1 : W_1 \to \mathbb{P}^1 \setminus S$ r-sheeted $1 \le r < n$

Let $a \in \mathbb{P}^1 \setminus S$, $b_\nu(a)$ $\nu = 1 \dots r$ the ν^{th} el. symm function of $y_1 \dots y_r$ $y_j = \eta_1(\pi_1^{-1}(a))$

$\shortparallel$

values of p_2 at $(x,y) \in \bar{V}$

$\Rightarrow F(a, y_j) = 0 \quad j = 1 \dots r$

the b_ν extend to mer. f. on $\mathbb{P}^1$: the y_j are the values of η_1, merom, so near $a \in S$ we have

$$|b_\nu(a)| = \Big| \sum_{j_1 \dots j_\nu} \eta_1(P_{j_1}) \dots \eta_1(P_{j_\nu}) \Big| \le C|z|^{-\ell} \le C_1 |a - a|^{-\ell'}$$

$$(C_1 |a|^{-\ell'} \quad a = \infty)$$

$$\{P_1 \dots P_r\} = \pi_1^{-1}(a)$$

$\Rightarrow b_\nu$ is merom on $\mathbb{P}^1$, hence rational

Let $G(a,y) = y^r + b_1(a) y^{r-1} + \dots b_r(a)$. If $a \in \mathbb{P}^1 \setminus S$ the roots $y_1 \dots y_r$ are roots of $F(a,y) = 0$.

G divides F in $\mathbb{C}(a)[y] \Rightarrow$ G divides F in $\mathbb{C}[a][y] = \mathbb{C}[a,y]$

$\deg_y G \ge 1$

(Gauss lemma) $\Rightarrow F$ is not irreducible.

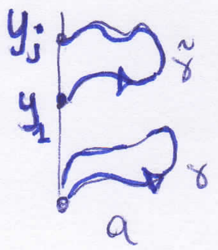
V' is then connected and $\hat{W}$ is a comp. RS.

$\hat{W}$ carries η (merom) and, if $\hat{\pi} : \hat{W} \to \mathbb{P}^1$ extends $\pi' : V' \to \mathbb{P}^1 \setminus S$, $F(\hat{\pi}(P), \eta(P)) \equiv 0$ on $\hat{W}$

XIX-10

Remark: The connectedness of $\bar{V}'$ is equivalent to the following.

Let $a \in \mathbb{P}^1 \setminus S$. Let $y_1 \ldots y_n$ the germs at a fulfilling $F(x, y_j(x)) = 0$. Then $\forall j$ $\exists$ γ closed in $\mathbb{P}^1 \setminus S$ starting at a such that the analytic continuation of y_1 along γ leads to y_j



compare the example provided by cusps

XIX-11

Riemann surface associated to a germ of a holomorphic function recap

$f : (\mathbb{C}, x) \to \mathbb{C}$ power series centred at x, with a non zero radius of convergence

$\mathcal{G} = \{$ germs of hol functions $\}$

$U \subset \mathbb{C}$ open $\quad f : U \to \mathbb{C} \quad \mathcal{U}(U,f) = \{$ germs $f_x \mid x \in U \}$

take these as a base for a topology on $\mathcal{G}$

The map
$$\pi : \mathcal{G} \to \mathbb{C}, \quad f_x \mapsto x$$

is continuous (and it induces local homeomorphisms, upon restriction to $\mathcal{U}(U,f)$)

The above topology is Hausdorff. This is clear for $x \neq y$

Now take f_x, g_x, germs of f and g on U

If there exists $h \in \mathcal{U}(U,f) \cap \mathcal{U}(U,g)$ then (suitably restricted) $f = g \Rightarrow f_x = g_x$.

Otherwise $f_x \neq g_x$. This proves our assertion.

Let $f_x : (\mathbb{C}, x) \to \mathbb{C}$ a germ.

Riemann surface of f_x : $\mathcal{S}(f_x) :=$ connected component of $\mathcal{G}$ containing f_x

$f : U \to \mathbb{C}$

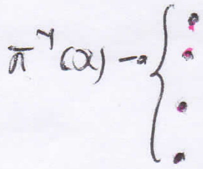
$$\mathcal{S}(f) = \bigcup_{x \in U} \mathcal{S}(f_x)$$

XIX-12

Concretely: the germs g_y are obtained by analytic continuation along a path γ joining x to y

[as long as one acts in a simply connected open set, the result is independent of γ (monodromy theorem)

If f is "multivalued" $\pi^{-1}(\{x\})$ consists of more than one pt (determinations of f); one gets



$\tilde{f} : S(f) \to \mathbb{C}$ holomorphic and one-valued

Explicitly: $\tilde{f} : x \mapsto f_x \leftarrow$ a branch of f

Poincaré – Volterra: the base is countable
(idea: use polygonal paths with vertices having rational coordinates, together with the monodromy theorem

XIX-13