Theorem Every compact RS T is isomorphic to the RS of an algebraic function

Proof Let f_1 be a non constant meromorphic function on T [f_1 exists by Riemann-Roch], viewed as a branched covering of degree d of $\bar{\mathbb{C}}$

Riemann sphere

Let $\{P_1 \ldots P_d\}$ be a generic fibre.

Let f_2 be a meromorphic function (RR again!) separating $P_1 \ldots P_d$. (f_1, f_2) maps T to $\boxed{C}$, analytic curve in $\bar{\mathbb{C}}_z \times \bar{\mathbb{C}}_w$, which is indeed algebraic. Indeed, $f_1 : T \to \bar{\mathbb{C}}_z$ defines a branched covering, so $f_1^{-1}(z)$ (fibre over $z \in \bar{\mathbb{C}}_z$) contains the same number of elements, except for a finite number of pts $z_1 \ldots z_k \in \mathbb{C}_z$.

Let $z \in \bar{\mathbb{C}}_z \setminus \{z_1 \ldots z_k\}$ and set $f_1^{-1}(z) = \{P_1(z) \ldots P_d(z)\}$

The $P_i = P_i(z)$ are multivalued functions of z:

The set $\{P_1 \ldots P_d\}$ is well-defined but one cannot obtain global holomorphic functions.

Consider the ordinates $w_i(z) = f_2(P_i(z))$ of the points in $C \cap \{z\} \times \bar{\mathbb{C}}_w$ "line". The w_i are also multivalued on $\bar{\mathbb{C}}_z \setminus \{z_1 \ldots z_k\}$.

Lecture XVIII

ALGEBRAIC CURVES & RIEMANN SURFACES

Prof. M. Spera

Then take

$$\boxed{\begin{aligned} s_1(z) &= w_1(z)+\dots+w_d(z) \\ s_2(z) &= w_1w_2+\dots w_{d-1}w_d \\ &\vdots \\ s_d(z) &= w_1\cdots w_d \end{aligned}}$$

(elementary symmetric functions)

(Schur functions)

punch line

↳ The s_i are meromorphic on $\bar{\mathbb{C}}_z$, hence rational

Consider $F(z,w)$ (polynomial) obtained

from $\left[w^d - s_1(z)w^{d-1}+\dots+(-1)^d s_d(z) \right]$ the roots are the w_i

fraction

by clearing denominators (via a polynomial in z)

Then $F(z,w)=0$ precisely on G. Thus

T is the RS of any germ $w(z)$ such that

$F(z,w(z))=0$: indeed these two surfaces

are both compact and coincide up to finitely

many pts.

$\mathbb{C}[x,y]$ polyn. in two variables ($\mathbb{C}$- coeff.)

$\Downarrow$

F irreducible , $\deg_y F \geq 1$

$\mathbb{C}[x,y] \cong \mathbb{C}[x][y] \underset{\text{Gauss lemma}}{\Rightarrow} F$ irr in $\mathbb{C}(x)[y]$

polyn ring over $\mathbb{C}(x)$, rational functions in x

$\downarrow$

unique fact. domain (factorial ring)

" Algebraic function $\rightsquigarrow$ $\underset{\text{irred.}}{F(x,y)} = 0$ "

Implicit f. Theorem ("Holomorphic Dini)

Let f hol. in x, y $\{(x,y) \in \mathbb{C}^2 / |x| < r_1, |y| < r_2, r_i > 0\}$

Assume $f(0,0) = 0$ $\frac{\partial f}{\partial y}(0,0) \neq 0$

Then $\exists$ $\varepsilon > 0$, $\delta > 0$ s.t. $\forall x \in D_\varepsilon = \{|z| < \varepsilon\}$,

$\exists !$ $y = y(x)$ satisfying $f(x, y(x)) \equiv 0$ with

$|y(x)| < \delta$. $y = y(x)$ is holomorphic

<u>Proof</u> From $\frac{\partial f}{\partial y}(0,0) \neq 0$, $\exists \delta > 0$ with $f(0,y) \neq 0$

for $0 < |y| \leq \delta$. Let $\varepsilon > 0$ s.t. $f(x,y) \neq 0$

for $|x| \leq \varepsilon$, $|y| = \delta$ ($f \neq 0$ on $\underbrace{\{0\} \times \{|y| = \delta\}}_{\text{compact}}$

Then, for $|x| < \varepsilon$

crucial! $\longrightarrow$ $n(x) := \frac{1}{2\pi i} \int_{|y|=\delta} \frac{\frac{\partial f}{\partial y}(x,y)}{f} dy$ = integer =

[Ref: Narasimhan]

XVIII-3

$= \#$ zeros of $y \mapsto f(x,y)$ in $|y| < \delta$

(Argument principle)

We have $n(0) = 1$ (choice of δ) ; $n = n(x)$ is continuous (f is such)

$\Rightarrow n(x) = 1$ for $|x| < \varepsilon$ $\Rightarrow$

$\exists !$ zero $y(x)$ of f with $|y(x)| < \varepsilon$

$x \mapsto y(x)$ is holomorphic: $\sim d\log f$

$$y(x) = \frac{1}{2\pi i} \int_{|y| = \delta} y \, \frac{\frac{\partial f}{\partial y}(x,y)}{f(x,y)} \, dy \qquad \frac{1}{2\pi} \int_{\mathcal{C}} f(\xi)\, d\log(\xi - a)$$

$$\shortparallel$$

(residue theorem) $$f(a) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(\xi)}{\xi - a} \, d\xi$$

also $a_{-1} = \frac{1}{2\pi i} \int f(z)\, dz$

— · — ·

Application:

$$F(x,y) = a_0(x) y^n + a_1(x) y^{n-1} + \dots a_n(x)$$

$\in \mathbb{C}[x,y]$ irred., $n \geq 1$. The a_i have no non-constant common factor (F is irreducible)

($F = 0$ algebraic curve) $\longrightarrow$

* **Lemma**: let $a \in \mathbb{C}$ with $a_0(a) \neq 0$

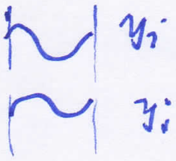
and s.that $\nexists\, b$ with $F(a,b) = \frac{\partial F}{\partial y}(a,b) = 0$

$\Rightarrow$ Then $\exists\, \varepsilon > 0$ and n holom. functions

$y_1(x) \dots y_n(x)$ in $D = \{|x-a| < \varepsilon\}$

possessing the following properties

(i) $y_i(x) \neq y_j(x')$ if $i \neq j$, $|x-a| < \varepsilon$, $|x'-a| < \varepsilon$



& $F(x, y_i(x)) \equiv 0$ $|x-a| < \varepsilon$ $i = 1 \dots n$

(ii) If $\eta \in \mathbb{C}$ & $F(x,\eta) = 0$, $|x-a| < \varepsilon$

$\Rightarrow$ $\eta = y_i(x)$ for a unique $i \in \{1, \dots, n\}$

Proof Since $\frac{\partial F}{\partial y}(a,b) \neq 0$ $\forall\, b$ s.t. $F(a,b) = 0$,

$F(a,y)$ has precisely n roots $b_1 \dots b_n$

For $\varepsilon > 0$ small enough, one gets the $y_i(x)$

($y_i(a) = b_i$) $F(x, y_i(x)) \equiv 0$: the y_i have (i)

Also (ii) is ok: $F(x,\eta)$ has at most n solutions

Prop $F \in \mathbb{C}[x,y]$ irreducible. $\exists$ only f. many $x \in \mathbb{C}$

s.t. $F(x,y) = 0 = \frac{\partial F}{\partial y}(x,y)$ $\quad [F = a_0(x) y^n + \dots a_n(x)$

Proof By Euclid, $\exists\, b_i \in \mathbb{C}[x]$ ($i \geq 0$)

$b_0 = a_0$ & pol. $A_i, Q_j \in \mathbb{C}[x,y]$ ($j \geq 1$)

such that.

Short proof: **Bézout**

$$b_0^n F = A_1 \frac{\partial F}{\partial y} + Q_1 \qquad \deg_y Q_1 < \overbrace{\deg_y \frac{\partial F}{\partial y}}^{n-1}$$

$$b_1 \frac{\partial F}{\partial y} = A_2 Q_1 + Q_2 \qquad \deg_y Q_2 < \deg_y Q_1$$

$\vdots$

$$b_{k-1} Q_{k-2} = A_k Q_{k-1} + Q_k \qquad \deg_y Q_k < \deg_y Q_{k-1}$$

[the b_j's are needed to clear denominators...]

Assume $\deg_y Q_k = 0$ (no loss of generality)

$Q_k \in \mathbb{C}[x]$ Claim $Q_k(x) \not\equiv 0$

Otherwise if $Q_k \equiv 0$ then any prime factor P of Q_{k-1} ($\deg_y P > 0$) would divide $b_{k-1} Q_{k-2}$ and hence Q_{k-2}.

Iterating the process yields $P \mid Q_j$ and $P \mid F$, $P \mid \frac{\partial F}{\partial y}$

absurd! (F is irreducible)

$\Rightarrow Q_k = Q_k(x) \in \mathbb{C}[x] \not\equiv 0$

From $F(a,b) = \frac{\partial F}{\partial y}(a,b) = 0$ we get $Q_1(a,b) = Q_2(a,b)$

$= Q_k(a,b) = Q_k(a) = 0 \Rightarrow \{x \mid \exists y \text{ s.t. } F(x,y) = \frac{\partial F}{\partial y}(x,y) = 0\}$

$\subset \{x \mid Q_k(x) = 0\}$

is a finite set

Topological concepts

(Digression) locally compact

$p: X \to Y$ $\quad X, Y$ l.c. Hausdorff

p **proper**: $K \subset Y$ compact $\Rightarrow p^{-1}(K) \subset X$ compact

p proper $\Rightarrow p$ closed:

let $A \subset X$ closed, $y_0 \in Y$ let $K \ni y_0$ compact

y_0 K $\qquad p(A) \cap K = p(\underbrace{A \cap p^{-1}(K)}_{\text{closed} \; \text{compact}})$ is compact

closed $\subset$ compact $\Rightarrow$ compact $\Rightarrow p(\text{compact}) = \text{compact}$

$p: X \to Y$ is proper $\Leftrightarrow \forall Z$ l.c.

$p \times id_Z \quad X \times Z \to Y \times Z \quad (x, z) \mapsto (p(x), z)$ is closed

[Let $\{x_1 \ldots x_n \ldots\}$ be a sequence of pts in X without limit points but such that $p(x_n)$ converges. then $p(x_n, \frac{1}{n}) \subset X \times \mathbb{R}$ is not closed in $Y \times \mathbb{R}$

$= (p(x_n), \frac{1}{n})$

If $p: \overset{\text{l.c}}{X} \to \overset{\text{l.c}}{Y}$ proper, $\overset{\text{l.c. Cond. top.}}{Z \subset Y}$
$\Rightarrow p|_{p^{-1}(Z)}$ is proper.

($K \subset Z$ compact $\Rightarrow K \subset Y$ compact

Let $c_1 .. c_n \in \mathbb{C}$ and assume that

$$w^n + c_1 w^{n-1} + c_2 w^{n-2} + c_n = 0$$

then $|w| < 2 \max_\nu |c_\nu|^{\frac{1}{\nu}}$

Let $c = \max |c_\nu|^{\frac{1}{\nu}} > 0$

Put $z = \frac{w}{c}$; then $z^n + \frac{c_1}{c} z^{n-1} + .. \frac{c_n}{c_n} = 0$

$\Rightarrow$ $(|c_\nu| < c^\nu)$ $|z|^n \leq |z|^{n-1} + , . . . 1$

But

$$1 \leq \frac{1}{|z|} + \frac{1}{|z|^2} + .. \frac{1}{|z|^n}$$

If $|z| \geq 2$ this is false

$\Rightarrow$ $|z| < 2$ i.e
$|w| < 2c$

Let $F \in \mathbb{C}[x,y]$, $F(x,y) = a_0(x) y^n + .. a_n(x)$, $a_0(x) \not\equiv 0$

Let $V = \{ F(x,y) = 0 \}$ $S_0 = \{ a_0(x) = 0 \}$

Let $\pi : V \to \mathbb{C}$, $(x,y) \to x$. Then $\pi|_{\pi^{-1}(\mathbb{C} \setminus S_0)} \to \mathbb{C} - S_0$ is proper

Proof. Let $K \subset \mathbb{C} \setminus S_0$ be compact. Then $\exists \delta > 0$ with $|a_0(x)| \geq \delta$ & $|a_\nu(x)| \leq \frac{1}{\delta}$ $\forall x \in K$.

If $(x,y) \in V$, $x \in \pi^{-1}(K)$, one has

$$y^n + \frac{a_1(x)}{a_0(x)} y^{n-1} + ... \frac{a_n(x)}{a_0(x)} = 0$$

$\Rightarrow \quad |y| \leq 2 \max\limits_{\nu} \delta^{-\frac{2}{\nu}} \quad \Rightarrow \pi^{-1}(K)$ is bounded

But $\pi^{-1}(K) = (K \times \mathbb{C}) \cap V$ is closed in $\mathbb{C}^2 \Rightarrow$

$\pi^{-1}(K)$ is compact.

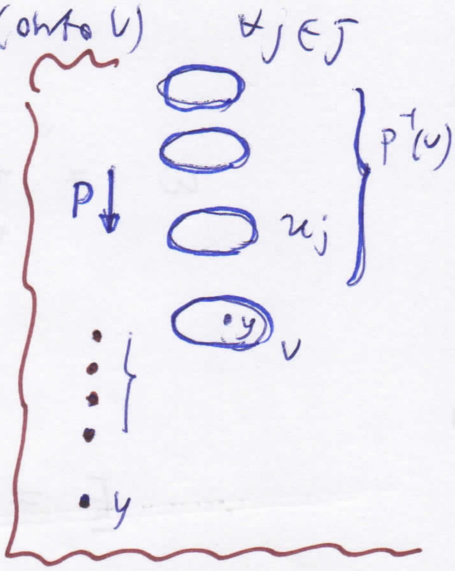
★★ **Covering maps** : $p : X \longrightarrow Y$ continuous (Hausdorff, Hausdorff)

$\forall y_0 \in Y$, $\exists V \ni y_0$ (open) s.t. $p^{-1}(V) = \dot{\bigcup\limits_{j\in J}} U_j$ ← disjoint

s.th. $p|_{U_j} : U_j \to V$ homeom.. (onto V) $\forall j \in J$

(X, Y, p) : (unramified) covering

V : evenly covered by p



$\boxed{\text{Hence } \# p^{-1}(y) \text{ (card. of } p^{-1}(y)) \text{ is locally constant} \Rightarrow \text{if } Y \text{ is connected, } \# p^{-1}(y) \text{ is independent of } y}$

★ n-sheeted covering : $\# p^{-1}(y) = n \quad (\forall y)$

$p : X \to Y$, $p' : X' \to Y$ isomorphic if $\exists \varphi : X' \to X$ omeo s.t. $p \circ \varphi = p'$

$$\begin{tikzcd} X' \arrow[r, "\varphi"] \arrow[dr, "p'"'] & X \arrow[d, "p"] \\ & Y \end{tikzcd}$$

→ Example (typical)

$\Delta : \{ z \mid |z| < 1 \}$ $\quad \Delta^* = \Delta \setminus \{0\}$

let $n \geq 1$ $\quad p_n : \Delta^* \to \Delta^*$ $\quad z \longmapsto z^n$ $\quad$ n-sheeted covering

$p: \mathbb{C} \to \mathbb{C}^* \quad : \quad p(z) = e^z$ infinite covering

$w = e^z$

$z = \log w$!

$\log|w| + i\varphi$

$|e^z| = |e^{x+iy}| = |e^x e^{iy}|$

$= e^x$

$\underset{l.c}{X} \qquad \underset{l.c}{Y}$

$p: X \to Y$ local homeo : $\forall a \in X \ \exists\, U \ni a$ s.th. $V = p(U)$ open and $p|_U : U \to V$ homeo

p finite covering $\Leftrightarrow$ p proper

$(\Rightarrow)$ $y_0 \in Y$, $V \ni y_0$ evenly covered, $p|_{p^{-1}(V)} \to V$ proper

$\Rightarrow p$ proper.

$(\Leftarrow)$ let p be a proper local homeo: let $y_0 \in Y$ &

$p^{-1}(y_0) = \{x_1 .. x_n\}$ let $U'_j \ni x_j$; $p|_{U_j} : U_j - V_j := p(U'_j)$ homeo

$X - \bigcup_{j=1}^{n} U'_j$ closed in X, $E := p(\quad)$ closed in Y

(p is proper). $y_0 \notin E$ $V = Y - E$

$p^{-1}(V) \subset U'_1 \cup .. U'_n$ & $V \subset V_1 \cap .. V_n$

Set $U_j = U'_j \cap p^{-1}(V)$. Then $p^{-1}(V) = \bigcup_{j=1}^{n} U_j$ & $p|_{U_j}$ homeo onto V

UPSHOT

$F \in \mathbb{C}[x,y]$ irreducible

$\|$

$a_0(x)\, y^n + \dots a_m(x)$

$S_0 = \{ x \in \mathbb{C} \ / \ a_0(x) = 0 \}$

$S_1 = \{ x \in \mathbb{C} \ / \ \exists\, y : F(x,y) = 0, \ \frac{\partial F}{\partial y}(x,y) = 0 \}$

Let $V = \{ (x,y) \ / \ F(x,y) = 0 \}$ $\qquad \pi : V \to \mathbb{C}$, $(x,y) \mapsto x$

Then

$$\pi\Big|_{\pi^{-1}(\mathbb{C} \setminus (S_0 \cup S_1))} \longrightarrow \mathbb{C} \setminus (S_0 \cup S_1)$$

is an <u>n-sheeted finite covering</u>