ALGEBRAIC CURVES & RIEMANN SURFACES

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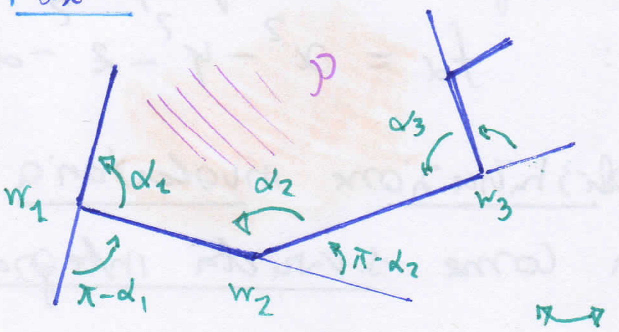
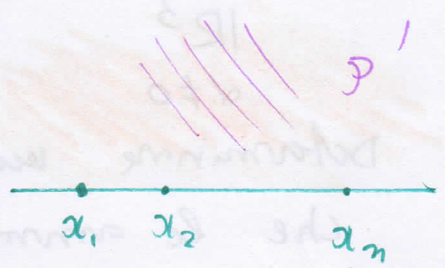
Lecture XXI - add

# * The Schwarz - Christoffel transformation

$$\boxed{\frac{dw}{dz} = A\,(z-x_1)^{\frac{\alpha_1}{\pi}-1}\cdot(z-x_2)^{\frac{\alpha_2}{\pi}-1}\cdots(z-x_n)^{\frac{\alpha_n}{\pi}-1}}$$




$w = u + iv$

$w_i \leftrightarrow x_i$

Three images can be fixed arbitrarily

$$w = A\int (z-x_1)^{\frac{\alpha_1}{\pi}-1}\cdots(z-x_n)^{\frac{\alpha_n}{\pi}-1}\,dz + B$$

If $x_n = \infty$, the corresponding factor disappears

A & B govern position & shape of P

if P is closed,

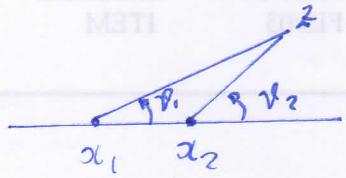
$$(\pi-\alpha_1)+(\pi-\alpha_2)+\dots(\pi-\alpha_n) = 2\pi$$

$$\Rightarrow \boxed{\sum_{i=1}^{n}\left(1-\frac{\alpha_i}{\pi}\right) = 2}$$

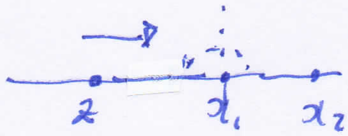
(proof)

Proof (sketch)

$$\arg(dw) = \arg dz + \arg A + \left(\frac{\alpha_1}{\pi}-1\right)\arg(z-x_1) + \dots + \left(\frac{\alpha_n}{\pi}-1\right)\arg(z-x_n)$$

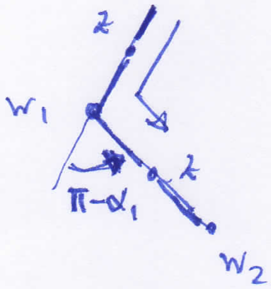



Let $z \in \mathbb{R}$



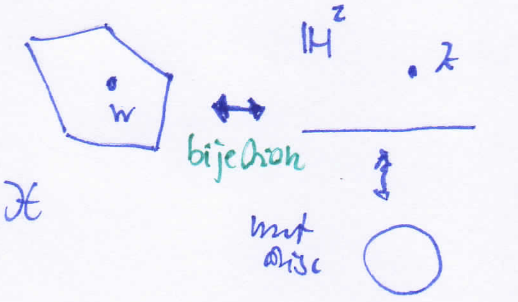

On crossing $x_1$, $\arg(z-x_1)$ changes abruptly from $\pi$ to $0$, and all remaining terms are constant. Therefore, the variation of $dw$ is

$$-\left(\frac{\alpha_1}{\pi}-1\right)\cdot\pi = \pi-\alpha_1$$



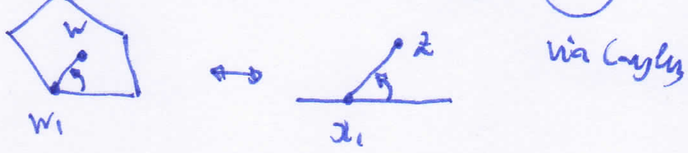

and so on...

Also, if $P$ is closed and $w \in$ its interior then $z \in \mathbb{H}^2$




This is clear since $w = f(z)$, $f \in \mathcal{H}$ preserves orientations

This is confirmed by observing that, from




$$\frac{1}{2\pi i}\int_M \frac{dw}{z-a} = 1, \quad \text{we have} \quad \frac{1}{2\pi i}\int_C \frac{df}{f(z)-a} = 1$$


(Cauchy)

$\Rightarrow$ by the argument principle, $\exists!\, z_0$ such that $f(z_0)=a$

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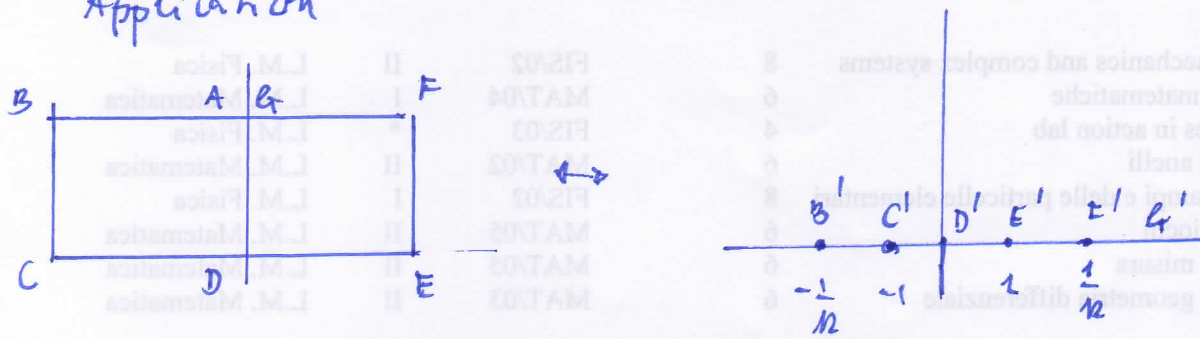
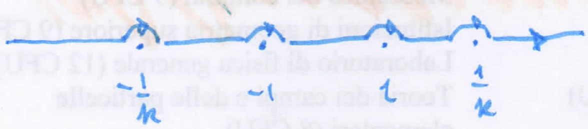
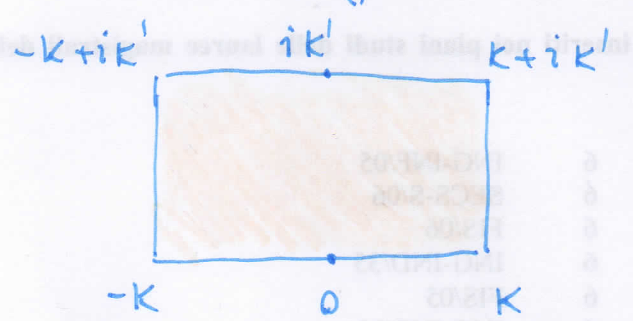
## Application




$$\boxed{w = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}}$$

$0 < k < 1$




$K(k)$ ///

$$\boxed{K = \int_0^1 \left((1-x^2)(1-k^2x^2)\right)^{-\frac{1}{2}}\, dx}$$

$$K' = \int_1^{\frac{1}{k}} \left((x^2-1)(1-k^2x^2)\right)^{-\frac{1}{2}}\, dx$$

$K' = K(k')$

$k' = \sqrt{1-k^2}$

Complementary modulus

!!!

Jacobi's complete elliptic integral of the first kind

! Can be tabulated

### Big & Small Circle theorems


$zf(z) \to A$ as $z \to \infty$ $\Rightarrow$ $\int_{\Gamma_R} f(z)\,dz \xrightarrow{R\to\infty} i \cdot A \cdot \Delta\alpha$


$zf(z) \to a$ as $z \to 0$ $\Rightarrow$ $\int_{\Gamma_r} f(z)\,dz \xrightarrow{r\to 0} i \cdot a \cdot \Delta\alpha$

Cf. Delidne theorem

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