ALGEBRAIC CURVES & RIEMANN SURFACES

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★★ Plücker formulae via Riemann-Hurwitz ★

Lecture XXIII

Terminology

point coordinates

$f : S \to \mathbb{P}^2$ algebraic curve

line coordinates

$f^* : S \to \mathbb{P}^{2^*}$ dual curve

$(f^*)^* = f$

envelope of tangents

- **regular point**:

  smooth point of f and f^*

  $x_0 x_1 x_2$

  $p : [1,0,0]$

  local expression $f(z) = [1, z + \dots, z^2 + \dots]$

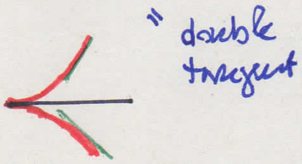
- **ordinary flex**

  smooth point whose tangent line has contact of order 3

$f(z) = [1, z + \dots, z^3 + \dots]$

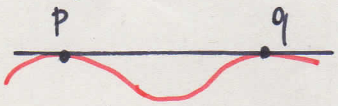
$f^*(z) = [1, z^2 + \dots, z^3 + \dots]$ → cusp of f^*

- **cusp of f** $f(z) = [1, z^2 +, \dots z^3 \dots]$

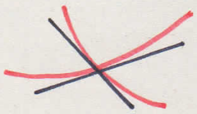


p is a flex of f ⟺ p is a cusp for f^*

- **bitangent of f** p, not a flex, such that its tangent simply touches f in another point $q \neq p$



- **ordinary double point** (node) : two branches of f cross transversally



p is a bitangent for f ⟺ p is an ordinary double point for f^*

Notation

g: genus of S $\qquad C = f(S) \quad C^* = f^*(S)$

d, d^*: degrees of C, C^* resp. $\qquad d^*$: class of C (# of tangents emanating from a generic pt)

b, b^*: # bitangents (of C, C^*)

f, f^*: # flexes (of C, C^*)

κ, κ^*: # cusps (of C, C^*)

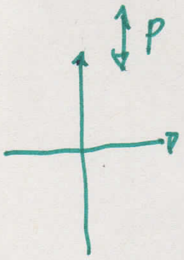
δ, δ^*: # double points of C, C^*

Clearly:

$$\begin{bmatrix} b = \delta^* , & b^* = \delta \\ f = \kappa^* , & f^* = \kappa \end{bmatrix}$$

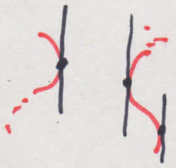
Assumptions: p generic, not lying on tangents issuing from the singular points

local coordinates $[x_0, x_1, x_2]$ on $\mathbb{P}^2$, $p : [0,0,1] = Y_\infty$



$C : \quad g(x_0, x_1, x_2) = 0 \qquad \deg C = d$

tangent lines to C from p ↔ smooth points of C such that $\dfrac{\partial g}{\partial x_2}(q) = 0$



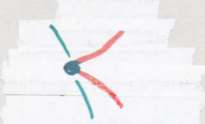
Let $C' : \quad \dfrac{\partial g}{\partial x_2} = 0 \qquad \deg C = d-1$

bitangent of C^* $\qquad$ flex of C^*

passes through each double point & cusp of C

with intersection multiplicity 2 & 3 resp.

at a sing. pt: $\dfrac{\partial g}{\partial x_i} = 0$



There

points of intersection of C' with $C \setminus \{\text{singular pts}\}$

$= (C.C') - 2\delta - 3\kappa$, namely

$$\boxed{\underbrace{d^*}_{\text{class of } C} = d(d-1) - 2\delta - 3\kappa}$$

class of C
= # tangents emanating from a generic pt
in our case p

Subsequently, project C onto a line $\mathbb{P}^1$:

$\pi : C \to \mathbb{P}^1$

↗ d-sheeted covering of $\mathbb{P}^1$

Therefore, by Riemann-Hurwitz

$$\chi(S) = 2 - 2g = 2d - b$$

↑ # branch points of $\pi \circ f : S \to \mathbb{P}^1$

$$S \xrightarrow{f} \mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^1$$

$f(S)$
$\parallel$
C

Now, a smooth pt of C, say q

is a branch pt iff $\frac{\partial g}{\partial z_2}(q) = 0$

$\Rightarrow$ among the smooth pts of C one has

$d(d-1) - 2\delta - 3\kappa$ branch pts

Now, an ordinary double point does not yield a branch pt, and a cusp is a branch pt of order 1

↑
important!

Hence

$$\underset{\text{\# branch pt}}{b} = d(d-1) - 2\delta - 3\kappa + \kappa$$

$$= d(d-1) - 2\delta - 2\kappa$$

Thus

$$2-2g = 2d \overbrace{- d(d-1) + 2\delta + 2\kappa}^{-b}$$

$$= 2d - d^2 + d + 2\delta + 2\kappa$$

$$= 3d - d^2 + 2\delta + 2\kappa$$

$\Rightarrow$

$$2g = \overbrace{2 - 3d + d^2}^{(d-1)(d-2)} - 2\delta - 2\kappa = (d-1)(d-2) - 2\delta - 2\kappa$$

$$\boxed{g = \frac{(d-1)(d-2)}{2} - \delta - \kappa}$$

bitangents $b = \delta^*$

flexes $f = \kappa^*$

$$d^* = d(d-1) - 2\delta - 3\kappa \qquad g = \frac{(d-1)(d-2)}{2} - \delta - \kappa$$

Dually

$$d = d^*(d^*-1) - 2b - 3f \qquad g^* = g = \frac{(d^*-1)(d^*-2)}{2} - \delta^* - \kappa^*$$

$$= \frac{(d^*-1)(d^*-2)}{2} - b - f$$

★★ **Plücker formulae**

Thus $\boxed{g = p}$

topological genus ↗ (g)

arithmetic genus ↖ (p)

Lüroth's theorem (classical proof)

Let $\mathcal{C}$: $x_i = \dfrac{\phi_i(t)}{\phi_0(t)}$ $i=1..n$ ($\mathcal{C}$ rational)

If every point of $\mathcal{C}$ arises from $k \geq 1$ values of t, then $\exists$ $\tau = \tau(t)$ (rational function) such that the correspondence $\tau \leftrightarrow$ pts of $\mathcal{C}$ is **1-1**.

Proof. Let the ϕ_i without common (non constant) factor and with the same degree. Let t_1 a generic value of t, and $P_1 = \left(\dfrac{\phi_1(t_1)}{\phi_0(t_1)}, \dots, \dfrac{\phi_n(t_1)}{\phi_0(t_1)} \right)$

The values $t_1 .. t_k$ yielding the same P_1 are such that

$$\operatorname{rank}\begin{pmatrix} \phi_1(t) & \cdots & \phi_n(t) \\ \phi_1(t_1) & \cdots & \phi_n(t_1) \end{pmatrix} = 1$$

, namely, the common zeroes of $\Phi_{ij}(t) := \phi_i(t_1)\phi_j(t) - \phi_i(t)\phi_j(t_1)$ $\quad i,j=1..n$, $i \neq j$

i.e. the zeroes of their GCD

$$\Delta(t, t_1) := a_0(t_1)t^k + a_1(t_1)t^{k-1} + .. + a_k(t_1)$$

Upon varying t_1, the coefficients are not constant, otherwise the roots of of Δ would be independent of t_1, and this is absurd since t_1 <u>is</u> a root.

We can also assume that all roots of Δ vary with t_1 (otherwise, we could remove those staying constant).

Therefore, for generic t_1, $\Delta(t, t_1)$ has degree k in t (and $a_0(t_1) \neq 0$). Any other value $t_2 .. t_k$ yielding the same point $P(t_1)$ would have worked equally well

Hence, there exists j ($j=1$, say)

such that $\frac{a_j(t_1)}{a_0(t_1)}$ depends effectively on t_1

and $$\frac{a_1(t_1)}{a_0(t_1)} = \frac{a_2(t_2)}{a_0(t_2)} = \dots \frac{a_k(t_k)}{a_0(t_k)}$$

then $\tau := \frac{a_1(t)}{a_0(t)}$ is the sought new parameter

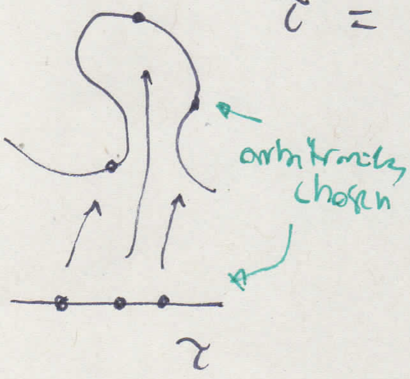
From $$\begin{cases} \tau\, a_0(t) - a_1(t) = 0 \\ x_i\, \phi_0(t) - \phi_i(t) = 0 \end{cases}$$

one has, after eliminating t,
$$g(x_i, \tau) = 0$$ (algebraic)

For every τ, x_i is the i-th coordinate of the point $P(\tau)$

All acceptable τ''s are obtained via

$$\tau' = \frac{a\tau + b}{c\tau + d} \qquad ad - bc \neq 0$$



other definition:

Rational curve in $\mathbb{P}^r$:

$x_i = f_i(\lambda)$ $\quad i = 0 .. r$

* polynomials without a common factor

* A rational curve is algebraic : for simplicity let $r=2$ (plane curve) ; one has $f_1(\lambda)x_0 - f_0(\lambda)x_1 = 0$

$f_2(\lambda)x_0 - f_0(\lambda)x_2 = 0$

Elimination of λ leads to a homogeneous equation $F(x_0 x_1 x_2) = 0$

★ Luröth's theorem

"topological" version

Let $\pi : \mathbb{P}^1 \longrightarrow \overset{\text{compact RS}}{S}$ be a d-sheeted branched covering. Then $S \approx \mathbb{P}^1$.

Proof. From Riemann-Hurwitz we have:

$$\underset{\overset{\shortparallel}{2}}{\chi(\overset{\approx S^2}{\mathbb{P}^1})} = d \cdot \chi(S) - b$$

$$\Rightarrow \quad d \cdot \chi(S) = 2 + b \qquad \Rightarrow \quad \chi(S) > 0$$

$$\Rightarrow \quad g_S = 0$$

Thus S is $\mathbb{P}^1$.