ALGEBRAIC CURVES & RIEMANN SURFACES

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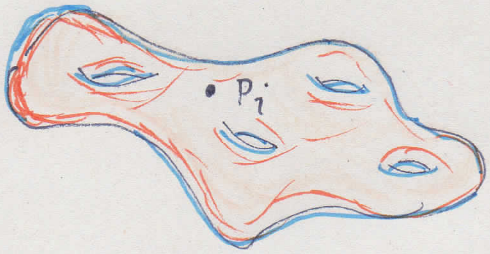
Amplification

* More on divisors (on a RS)

$$D = \sum a_i p_i$$

finite sum

$a_i \in \mathbb{Z}$



p_i ⤳ zero of $f_i \in \mathcal{O}$ (local holomorphic function)

(local coordinate)

trivial Example : p : $z = 0$ in $\mathbb{C}$

* Equivalent description of D

$\mathcal{U} = \{\mathcal{U}_\alpha\}$ open cover

$\{f_\alpha\}$ merom functions not identically zero in $\mathcal{U}_\alpha$

and $f_\alpha / f_\beta \in \mathcal{O}^*(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ $\neq \phi$

$\mathcal{U}_\alpha$ $\mathcal{U}_\beta$

$\forall \; p \quad ord_p(f_\alpha) = ord_p(f_\beta)$

⤳ get $D = \sum_p ord_p(f_\alpha) \cdot p \quad \mathcal{U}_\alpha \ni p$

(if $f_\alpha(p) \neq 0$ then $ord_p(f_\alpha) = 0$)

technically Aside

D : section of the quotient sheaf

$$\mathcal{M}^* / \mathcal{O}^*$$

meromorphic functions not identically zero

non zero hol. functions

see Griffiths-Harris

Vice versa, start from $D = \sum a_i p_i$

Find $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in \Gamma}$ s. that p_i has a local defining function in $\mathcal{U}_\alpha$, $g_{i\alpha} \in \mathcal{O}(U_\alpha)$

Then set:

$$\boxed{f_\alpha = \prod_i g_{i\alpha}^{a_i}} \quad \in \mathcal{M}^*(\mathcal{U}_\alpha)$$

$\{f_\alpha\}$: local defining functions

The functions $\boxed{g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}} \in \mathcal{O}^*(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$

and yield a

holomorphic line bundle $L \sim \{g_{\alpha\beta}\}$

* The tautological and hyperplane line bundles on $\mathbb{P}^1$

$\mathcal{O}(-1) \equiv$ <u>tautological line bundle</u> on $\mathbb{P}^1 (\approx S^2)$

fibre at $[w_0, w_1]$: the line in $\mathbb{C}^2$

given by $\lambda (w_0, w_1) \quad \lambda \in \mathbb{C}$

$$e_0 ([w_0, w_1]) := (1, \underset{\underset{w}{\parallel}}{\tfrac{w_1}{w_0}}) \qquad \text{on } w_0 \neq 0$$

non vanishing meromorphic section, with a pole at ∞ $\quad$ no hol. sections! $\quad \deg(\mathcal{O}(-1)) = -1 \quad D = (-1)\infty$

$$\text{RR}: \quad \underset{0}{\underset{\parallel}{h^0(\mathcal{O}(-1))}} - \underset{0}{\underset{\parallel}{\underbrace{h^0(K \otimes \mathcal{O}(-1))}_{K \setminus D}}} = -1 + 1 - \underset{0}{\underset{\parallel}{g}}$$

$\mathcal{O}(1) \equiv$ <u>hyperplane line bundle</u> (dual to $\mathcal{O}(-1)$)

$$\deg \mathcal{O}(1) = +1$$

$$h^0(\mathcal{O}(1)) = 1 + 1 = 2$$

<u>holomorphic sections</u>: $\quad s = s_{a,b} \equiv a z_0 + b z_1 \qquad a, b \in \mathbb{C}$

$$s_{a,b}([w_0, w_1]) := a z_0 w_0 + b z_1 w_1 \equiv \text{hom. polyn of degree } 1$$

↑ a dual vector... $\qquad$ ↓ possesses a single zero...

In general

$$L = \mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$$

$k \geq 1$

holomorphic sections

$H^0(\mathcal{O}(k))$

$\equiv$ { homogeneous polynomials of degree k in w_0, w_1 }

$$\boxed{h^0(L) = k + 1}$$

Transition functions for $\mathcal{O}(1)$

$$U_0 = \{ [z_0, z_1] \in \mathbb{P}^1 \mid z_0 \neq 0 \}$$

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$$g_{01}^{\mathcal{O}(1)} = \frac{z_0}{z_1} \qquad g_{10}^{\mathcal{O}(1)} = \frac{z_1}{z_0}$$

holomorphic sections: $s = \{ s_k \}$

$$\boxed{s_k = g_{kh} s_h}$$

$$s_0 = g_{01} s_1$$

$$\left(z_0 = \frac{z_0}{z_1} z_1 \right) \qquad s^i = \left\{ s^i_k = \frac{z_i}{z_k} \right\}$$

$$s^0 = \left\{ s^0_0 = 1 , \ s^0_1 = \frac{z_0}{z_1} \right\} \rightsquigarrow z_0$$

$$s^1 = \left\{ s^1_0 = \frac{z_1}{z_0} , \ s^1_1 = 1 \right\} \rightsquigarrow z_1$$

provides global description

$\mathcal{O}(-1)$ does not have non-trivial hol. sections...