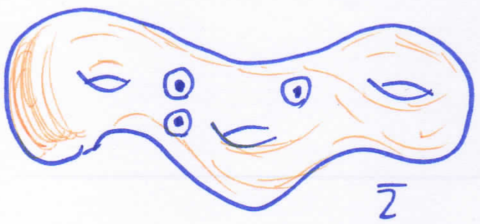
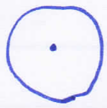
★★★ The Gauss-Bonnet Theorem & the first Chern class



Σ

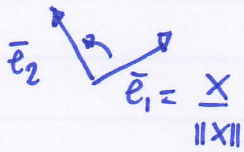
$X \in \mathfrak{X}(\Sigma)$ $\quad$ X with finitely many zeroes

$\boxed{\text{Lecture XXIX}}$

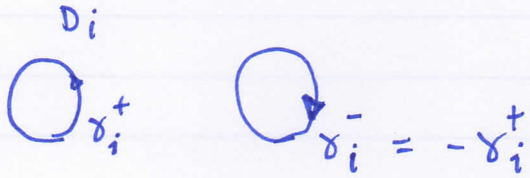


fix a repère mobile

$(\bar e_1, \bar e_2)$ $\quad$ $\bar e_1 = \frac{X}{\|X\|}$

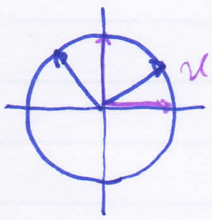


Riemannian metric



in the disc, fix (e_1, e_2)

in general



Levi-Civita connection form

$$\begin{cases} d\bar e_1 = \bar\omega_{12}\, e_2 \\ d\bar e_2 = -\bar\omega_{12}\, e_1 \end{cases}$$

★ Cartan's structure equations

$(K = \bar K$: Theorema egregium$)$

$$\bar\omega_{12} = \omega_{12} + d\nu$$

$$d\omega_{12} = -K\,\omega_1 \wedge \omega_2 \qquad d\bar\omega_{12} = -K\,\bar\omega_1 \wedge \bar\omega_2$$

Compute:

$$\int_{\Sigma \setminus \bigcup_j D_j} K\,\omega_1 \wedge \omega_2 = \int_{\Sigma \setminus \bigcup_j D_j} K\,\bar\omega_1 \wedge \bar\omega_2 = -\int_{\Sigma - \bigcup_j D_j} d\bar\omega_{12} = \text{(Stokes)}$$

$$= \sum_j \int_{\gamma_j^+} \bar\omega_{12} = \sum_j \int_{\gamma_j^+} \omega_{12} + \sum_j \int_{\gamma_j^+} d\nu =$$

$$\overset{\text{Stokes}}{=} \sum_j \int_{D_j} d\omega_{12} + 2\pi \cdot \sum_j I_j$$

indices

$$= -\sum_j K\,\omega_1 \wedge \omega_2 + \underbrace{2\pi \sum_j I_j}_{2\pi\, I(X)}$$

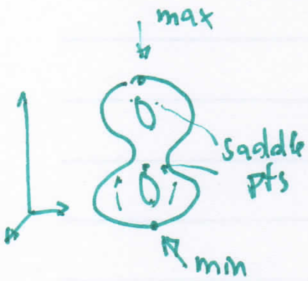
XXIX-1

Regrouping, we get

$$\int_\Sigma K\, \omega_1 \wedge \omega_2 = 2\pi\, i(X)$$

$$i(X) = \frac{1}{2\pi} \int_\Sigma K\, d\sigma$$ (area form) ✶ Poincaré-Hopf

$\Rightarrow$ $i(X)$ is independent of X !



Applying PH to ∇h (riemannian), h height function

we get $\quad i(X) = 2-2g = \dfrac{1}{2\pi}\displaystyle\int_\Sigma K\, d\sigma$

✶✶✶ Gauss-Bonnet

Now (punch line) the above reasoning applies to any (hermitian, holomorphic) line bundle $L \to \Sigma$

I_j = order at j of a generic meromorphic section s (with zeroes/poles at p_j) ; Ω is the curvature of any connection ω on $L \to \Sigma$ (replace $T_\circ\Sigma$ with $L_\circ$) $\quad d\omega = \Omega$ $\quad \cong \mathbb{C}$ $\quad \cong \mathbb{C}$

Then $\quad i(s) = \dfrac{1}{2\pi}\displaystyle\int_\Sigma \Omega\, d\sigma \equiv \deg(L) \equiv c_1(L)[\Sigma]$

$i(s) = \# Z - \# P$ counted with multiplicities

$s \sim z^k$ $\to$ index $= k$

degree of L — first Chern class

If $L \to \Sigma$ has a negative degree $\deg L < 0$, then it does not have non trivial holomorphic sections ($i(s) \geq 0$ for a holomorphic section!)

$T\Sigma \to \Sigma \sim T^{1,0} \to \Sigma$ holomorphic tangent bundle

$(T^{1,0})^* \to \Sigma$ canonical bundle K $\quad \deg K = 2g-2 = -\chi(\Sigma_g)$

* Examples

$g=1$ $\qquad L = K$ $\qquad$ never vanishing hol section

$\omega_1 \equiv$ abelian differential

$\parallel \ \dfrac{dz}{\sqrt{P(z)}}$

* Riemann - Roch

$$h^0(L) - h^0(\overset{K-D}{K \otimes L^{-1}}) = \deg(L) + 1 - g$$

$L = K \qquad h^0(K) - \underbrace{h^0(\mathbb{1})}_{1} = \deg(K) + 1 - g$

$$\left[h^0(K) = \underbrace{\deg K}_{2g-2} + 2 - g = 2g - \cancel{2} + \cancel{2} - g = g \right]$$

dimension of the space of abelian differentials