ALGEBRAIC CURVES & RIEMANN SURFACES

Prof. M. Spera UCSC

Lecture XXV

# ✶ Riemann's Fundamental Theorem (via Hodge Theory)

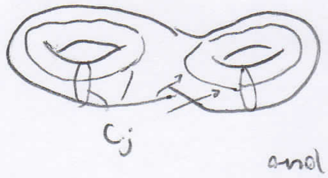
Let $S$ be a closed Riemann surface (compact, without boundary). Given $P_1, \dots, P_m \in S$, and principal parts around each $P_i$:

$$\left[ (A_i z_i^{-1} + B_i z_i^{-2} + \dots) dz_i \right]$$

local parameter around $z_i$

$A_i$ : residue at $P_i$ (principal part). Assume $\boxed{\sum_i A_i = 0}$

Assign $2g$ real numbers $\pi_j$, $j = 1 \dots 2g$ (corresponding to a basis of cycles $C_j$, $j = 1 \dots 2g$



$C_j$

"cuts on T" making T simply connected and not passing through $P_i$

Then there exists a unique meromorphic 1-form $\eta$ with poles exactly at $P_i$, with principal parts $A_i$, $i = 1 \dots m$ and with periods $\mathrm{Re}\left[\int_{C_j} \eta\right] = \pi_j$, $j = 1, 2 \dots 2g$

⇛ In particular: given any $2g$ real numbers $\pi_j$, $j = 1 \dots 2g$, $\exists!$ $\omega$ holomorphic such that $\mathrm{Re}(\int_{C_j} \omega) = \pi_j$

✶ rephrased

$F(P) = \int_{P_0}^{P} \eta$ multivalued potential with appropriate logarithmic singularities (no poles of $\eta$) ($\sum_i A_i =$

and prescribed periods

✶ RR problem: F single-valued : No logarithmic singularities (only poles) & vanishing periods

Riemann-Roch

XXV-1

**Proof.** Let first notice that a holomorphic 1-form $\eta$ is completely determined by its real part and every harmonic 1-form is the real part of a holomorphic 1-form.

Therefore, one has just to show that there exists $\eta$ harmonic with prescribed periods (the $A_i \equiv 0$)

This is clear by Hodge since in each cohomology class in $H^1$ (determined by the periods) one finds exactly one harmonic representative, which then solves the problem.

Let us now treat the general case.

Take $\alpha_0$, smooth on $S \setminus \{P_1 \dots P_m\}$ verifying, in a neighbourhood of $P_i$

$$\alpha_0 = \mathrm{Re}\left[\left(A_i z_i^{-1} + B_i z_i^{-2} + \dots\right) dz_i\right]$$

$\alpha_0$ is then harmonic near $P_i$, hence $d\alpha_0 = 0$

Then $d\alpha_0$ extends to a smooth form on the whole of $S$

One has, furthermore,

$$\boxed{\int_S d\alpha_0 = 0}$$

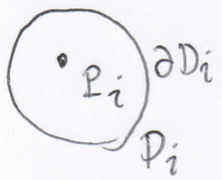
(Stokes)

**Proof.**

$$\int_S d\alpha_0 = \int_{S \setminus \cup_i D_i} d\alpha_0 = -\sum_i \int_{\partial D_i} \alpha_0$$

$$= \mathrm{Re}\left[2\pi i \underbrace{\sum_i A_i}_{=0}\right]$$

$$= 0$$




disc centred at $P_i$, whereupon $d\alpha_0 = 0$

The smooth 1-form $d\alpha_0$ admits a primitive $w$ (since $\int_S d\alpha_0 = 0$). Now consider

$$\alpha_1 = \alpha_0 - \omega \qquad (d\alpha_1 = 0)$$

on $S \setminus \{P_1 \dots P_m\}$. Then, as before, $d(*\alpha_1)$ extends to a smooth 2-form on $S$ with $\int_S d(*\alpha_1) = 0$

$$\begin{aligned} \eta &= f(z)dz \\ &= (u+iv)(dx+idy) \\ &= u\,dx - v\,dy \\ &\quad + i(v\,dx + u\,dy) \\ &\equiv \alpha + i(*\alpha) \end{aligned}$$

$(*dx = dy,\ *dy = -dx$

Hodge $*$ $\quad *^2 = \pm 1$

$\delta = \pm * d *$

Let $\beta$ be a primitive of $d(*\alpha_1)$, so, by Hodge

$$\boxed{\beta = \underset{\substack{\uparrow \\ \text{harmonic part}}}{\beta_h} + dF + * dG}$$

$$\underset{d(*\alpha_1)}{\underset{\parallel}{d\beta}} = \underset{0}{\underset{\parallel}{d\beta_h}} + \underset{0}{\underset{\parallel}{d^2F}} + d(*dG)$$

$$\Rightarrow \boxed{d(*dG) = d(*\alpha_1)} \qquad (*)$$

The 1-form $\alpha_2 = \alpha_1 - dG$ is closed $(d\alpha_2 = d\alpha_1 = 0)$ and coclosed (by $(*)$), hence, outside poles, it is harmonic, and it is the real part of the sought for meromorphic 1-form, this concluding the proof.

$$p \xrightarrow{*} n-p \xrightarrow{d} n-p+1 \xrightarrow{*} n-(n-p+1) = p-1$$

$*dx = dy \quad *dy = -dx$

$*1 = dx\wedge dy \quad *(dx \wedge dy) = 1$