X compact connected RS

Basic notation

$\mathbb{R}^2 \cong \mathbb{C}$.

$\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C} \cong \mathbb{R}^2$

Tangent space → $T_\circ X \cong T_\circ^{1,0}$

complexification → $T_\circ^{\mathbb{C}} X = T_\circ^{1,0} \oplus T_\circ^{0,1}$

$\alpha = \alpha_1 + i\alpha_2$

Explicitly (local coordinates; nevertheless everything is intrinsic)

$$T_\circ^{1,0} \ni \alpha \frac{\partial}{\partial z} \to \alpha \frac{\partial}{\partial z} + \bar{\alpha} \frac{\partial}{\partial \bar{z}} \in T_\circ X$$

check:

$$(\alpha_1 + i\alpha_2)\underbrace{\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)}_{\frac{\partial}{\partial z}} + (\alpha_1 - i\alpha_2)\cdot\underbrace{\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)}_{\frac{\partial}{\partial \bar{z}}}$$

$$= \overbrace{\frac{1}{2}\left(\alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y}\right)}^{(\spadesuit)} + \frac{1}{2}(\spadesuit)$$

$$\underbrace{+\, i\Big(\underbrace{\tfrac{1}{2}\alpha_2 \frac{\partial}{\partial x} - \alpha_1 \frac{\partial}{\partial y}}_{(\diamondsuit)}\Big) - i(\diamondsuit)}_{=\,0} = \alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y}$$

$$\Rightarrow \boxed{\begin{array}{ccc} TX & \cong & T^{1,0}X \\ \alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} & \longleftrightarrow & (\alpha_1 + i\alpha_2)\frac{\partial}{\partial z} \end{array}}$$

ALGEBRAIC CURVES & RIEMANN SURFACES

Prof. M. Spera

Lecture XXVI

An elementary approach to **Dolbeault cohomology groups**
(following S. Donaldson)

$$H^{0,0}_X = \ker \bar\partial : \Omega^0 \xrightarrow{\bar\partial} \Omega^{0,1} \qquad \text{holomorphic functions}$$
$$f \mapsto \bar\partial f = \frac{\partial f}{\partial \bar z} d\bar z$$

$$H^{1,0}_X = \ker \bar\partial : \Omega^{1,0} \xrightarrow{\bar\partial} \Omega^2 \qquad \text{holomorphic 1-forms}$$
$$\alpha\, dz \qquad \bar\partial\alpha \wedge dz \qquad \bar\partial \Omega^{0,1} = 0$$

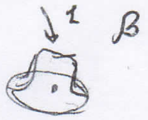
$$H^{0,1}_X = \operatorname{coker} \bar\partial : \Omega^0 \xrightarrow{\bar\partial} \Omega^{0,1} = \Omega^{0,1}/\operatorname{Im}\bar\partial$$
$$f \qquad \bar\partial f$$

$$H^{1,1}_X = \operatorname{coker} \bar\partial : \Omega^{1,0} \xrightarrow{\bar\partial} \Omega^2 = \Omega^2/\operatorname{Im}\bar\partial$$
$$\bar\partial \Omega^2 = 0$$

Let us interpret $\boxed{H^{0,1}_X}$

Fix $p \in X$. Can then we find f meromorphic on X with exactly one simple pole at p? (†) "Mittag-Leffler problem" [special case]

Let z be a local coordinate around p. Then $\frac{1}{z}$ is meromorphic on $U \ni p$ (neighborhood)



introduce a cut-off β

The function $\beta \cdot \frac{1}{z}$ is a smooth function on $X \setminus \{p\}$ (equal to zero outside U). Then finding the above f is tantamount to finding $g : X \to \mathbb{C}$ such that

$$g + \frac{\beta}{z} \text{ is holomorphic on } X \setminus \{p\}$$

(†) Actually, the answer is YES precisely when X is the Riemann sphere

Set $\quad A := \bar\partial \dfrac{\beta}{z} = \bar\partial\beta \cdot \dfrac{1}{z} \in \Omega^{0,1}$

A is compactly supported in $X \setminus \{p\}$

($\beta \equiv 1$ near p , so $\bar\partial\beta = 0$ near p). Then A can be viewed as an element of Ω^{01}, extending to zero over a neighbourhood of p

The condition

$$\bar\partial\left(g + \frac{\beta}{z}\right) = 0$$

becomes

$$\underbrace{\bar\partial g}_{\text{unknown}} = -\overbrace{A}^{\text{given}}$$

Therefore, a solution exists iff $[A] \in \Omega^{0,1}/\mathrm{im}\,\bar\partial$ vanishes. $\left\{ [A] \text{ is intrinsically associated to } p \right\}$:

indeed

$\forall\ \phi$, function on $X \setminus \{p\}$, restricting to a meromorphic function with a pole at p, for some $\lambda \in \mathbb{C}$, the function $\phi - \lambda\dfrac{\beta}{z}$ extends to a smooth function, holomorphic near p

Therefore:

$$[\bar\partial\phi] = \lambda[A]$$

Generalization: Given $p_1 .. p_d$ points in X, a meromorphic function f with simple poles at $p_1 .. p_d$ (and possessing no other poles) exists iff $\exists \lambda_i$, $i = 1..d$ such that the class

$$\boxed{\sum_{i=1}^{d} \lambda_i [A_i] = 0 \text{ in } H_X^{0,1}}$$

obvious notation

In particular, if $H_X^{0,1}$ has dimension h, then given any $h+1$ points $p_1 \dots p_{h+1}$, $\exists f$ meromorphic on X with simple poles at a subset of the p_i's

★ Residue map

f meromorphic, p simple pole : around p

$$f = a_{-1} z^{-1} + a_0 + a_1 z + \dots$$

$\mathrm{Res}_p(f)$. classical definition

Perform a holomorphic coordinate change

$$\tilde{z} = cz + \dots$$
$$z = c^{-1}\tilde{z} + \dots$$

$$\frac{a_{-1}}{z} = \frac{c\,a_{-1}}{\tilde{z}} \equiv \frac{\tilde{a}_{-1}}{\tilde{z}} \qquad \tilde{a}_{-1} = c\,a_{-1}$$

Then

$$a_{-1}\frac{\partial}{\partial z} = \frac{\tilde{a}}{c}\,\frac{\partial}{\partial \tilde{z}}\cdot\frac{\partial \tilde{z}}{\partial z} = \frac{\tilde{a}_{-1}}{\cancel{c}}\cdot\cancel{c}\,\frac{\partial}{\partial \tilde{z}} = \tilde{a}_{-1}\frac{\partial}{\partial \tilde{z}}$$

is a well-defined vector in $T_p^{1,0}X \cong T_pX$

Then $\boxed{R_p : f \longmapsto a_{-1}\frac{\partial}{\partial z}}$

(and obvious extensions) will be called residue map

Linear algebraic digression

$T : V \to W$ linear map $(T \in \mathrm{Hom}(V, W))$
V, W f.d. v. spaces

$T^* : W^* \to V^*$

dual map $(T^* w^*)(v) := w^*(Tv)$, $v \in V$, $w^* \in W^*$, $Tv \in W$

N+R theorem:

$$\dim V = \dim \ker T + \dim \mathrm{Im}\, T$$
$$\dim W = \dim \ker T^* + \dim \mathrm{Im}\, T^*$$

However $\mathrm{Im}\, T = (\ker T^*)^{\perp}$ $\leftarrow$ $\perp$ or 0 : annihilator

$\mathrm{Im}\, T^* = (\ker T)^{\perp}$ (similar...)

Check: introduce inner products $\langle\ \rangle$ $(\Rightarrow V \overset{\langle\ \rangle}{\cong} V^* \ldots)$

$\langle Tv, w\rangle = 0 \quad \forall v \in V$ (i.e. $w \in (\mathrm{Im}\, T)^{\perp}$)

$\Leftrightarrow \langle v, T^* w\rangle = 0 \quad \forall v \in V$

i.e $w \in \ker T^*$

$\Rightarrow (\mathrm{Im}\, T)^{\perp} = \ker T^* \Rightarrow \mathrm{Im}\, T = (\ker T^*)^{\perp}$

Then

$$\dim V = \dim \ker T + \dim (\ker T^*)^{\perp}$$
$$= \dim \ker T + \underbrace{\dim W^*}_{\dim W} - \dim \ker T^*$$

$$\Rightarrow \boxed{\dim \ker T - \dim \ker T^* = \dim V - \dim W}$$

$\mathrm{ind}(T)$ index of T