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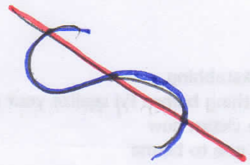
Explicit construction of holomorphic & meromorphic 1-forms on an algebraic curve ("concrete" RS) (Abel, Jacobi)

Lecture XXVII-add

ALGEBRAIC CURVES & RIEMANN SURFACES

T: compact RS $\rightsquigarrow T \subset \mathbb{CP}^2$ as $\mathcal{C}$, algebraic curve ($\deg = d$), whose singular points are double, with distinct tangents (nodes) $F_x = F_y = 0$

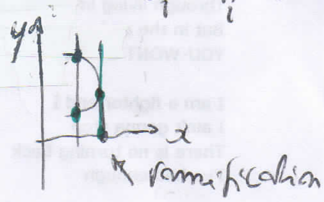
One may assume that $\mathcal{C}$ is transversal to r_∞ and, near a singular pt, $x : \mathcal{C} \to \mathbb{CP}^1$ is a genuine coordinate on each branch



$$E := \{ P = P(x,y) \in \mathbb{C}[x,y] \;/\; \deg P \le d-3 \} \qquad d = \deg \mathcal{C}$$

and vanishing on double pts $\}$

$\frac{\partial F}{\partial y} \neq 0 \quad y = y_i(x)$



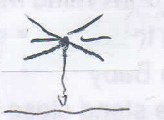
$$\boxed{\omega_P \underset{\text{(pull-back)}}{=} P(x,y)\,\frac{dx}{F'_y}} \qquad F'_y \equiv \frac{\partial F}{\partial y}$$

Proposition
1. $\forall P \in E$, ω_P is holomorphic on T
2. $P \mapsto \omega_P$ is linear and injective
3. $\dim E \ge g$ (genus of T)

Corollary: $\mathcal{C}$ smooth $\Rightarrow r = 0$ $\dim H^{1,0} \ge g$ ($H^{1,0}$: abelian differentials $\shortparallel$ holom. 1-forms)

Proof ω_P is holomorphic on C minus
(a priori)

- pts where x is not a local coord., i.e intersections $C \cap r_\infty$ and branching pts of $x : C \to \mathbb{P}^1$ $F'_y = 0$



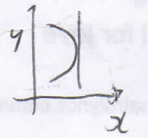
- pts where $F'_y = 0$: double pts & ram pts of x

- pts where $P = \infty$ i.e $C \cap r_\infty$

→ First, notice that the branching points cause no problem: from

$$F'_x dx + F'_y dy = 0 \quad \text{one has} \quad \omega = -\frac{P}{F'_x} dy$$

so this is holom in a neigh. of a ram. pt of x ($F'_x \neq 0$)



In a double pt, F'_y has a zero of order 1
and $R = 0$, $\Rightarrow$ ω is holom at double pts.

At infinity

$$\omega = \frac{R(x,y)\,dx}{F'_y} \qquad x = \frac{1}{X}, \quad dx = -\frac{1}{X^2}\,dX$$

we have:

numerator

$$\deg N \le d-3+2 = d-1$$

$$\deg D = d-1$$

denominator

$a_{IJ}x^I y^J$

$a_{IJ}\frac{1}{X^I}\frac{1}{Y^J}$

$\deg = I+J$

$\Rightarrow$ ω is finite at ∞

Polyn of degree $\le d-3$: v. space of

$$\dim = \frac{(d-2)(d-1)}{2}$$

$\leftarrow$ done... see box below

r linear conditions are needed to insure vanishing at double pts. ($\delta = r$)

$$\dim E \ge \frac{(d-2)(d-1)}{2} - r = \text{genus}$$

0 N

$$1+2+\dots N+1 = \frac{(N+1)(N+2)}{2}$$

$N = d-3$

$$\frac{(d-2)(d-1)}{2}$$

$a_{00} + a_{01}y + a_{02}y^2 + \dots a_{0N}y^N$

$a_{10}x + a_{11}xy + \dots a_{1,N-1}xy^{N-1}$

$\vdots$

$a_{N0}x^N$

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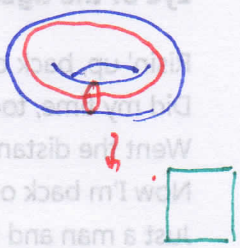
First reciprocity law (Riemann)

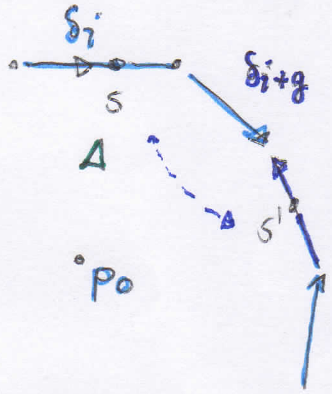
S RS $\quad \delta_i$ cycles issuing from $s_0 \in S$

$$S \setminus \{\delta_i\} = \Delta \quad \text{simply connected}$$



$\partial\Delta \qquad \searrow \delta_i \qquad \delta_i^{-1} \nearrow$

(canonical dissection)
4g-polygon



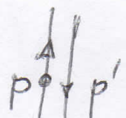
ω holom diff.
η merom 1-form with simple poles at $s_\lambda \in S$
(no poles on δ_i)

Π^i_ω, N^i_η periods along δ_i

Δ simply connected, ω holomorphic

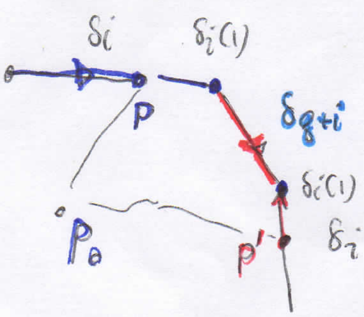
$$\pi(s) = \int_{s_0}^{s} \omega \quad \text{holo on } \bar{\Delta}, \quad d\pi = \omega$$

$p \in \delta_i \mapsto p' \in \delta_i^{-1}$



cancel out

$$\boxed{\pi(p') - \pi(p)} = \int_p^{p'} \omega = \int_p^{\delta_i(1)} \omega + \int_{\delta_{g+i}} \omega + \int_{\delta_i(1)}^{p'} \omega$$



$$= \int_{\delta_{g+i}} \omega = \boxed{\Pi^{g+i}}$$

Similarly, for $p \in \delta_{g+i}$, $p' \in \delta_{g+i}^{-1}$ (identified on S)

$$\boxed{\pi(p') - \pi(p) = -\Pi^i}$$

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Consider $\pi \cdot \eta$ (merom on $\bar{\Delta}$)

Then (residue theorem)

first order poles

$$\int_{\partial\Delta} \pi\cdot\eta = 2\pi i \sum_{\lambda} \operatorname{Res}_{s_\lambda} (\pi\cdot\eta)$$

$$= 2\pi i \sum_{\lambda} \operatorname{Res}_{s_\lambda}(\eta)\cdot \pi(s_\lambda)$$

$$= 2\pi i \sum \operatorname{Res}_{s_\lambda}(\eta) \int_{s_0}^{s_\lambda} \omega$$

But the l.h.s. can also be computed explicitly (suitable pairings made)

$$\int_{\delta_i+\delta_i^{-1}} \pi\cdot\eta = \int_{\delta_i} \pi(s)\,\eta(s) + \int_{\delta_i^{-1}} \underbrace{\pi(s')}_{\pi(s)+\pi^{i+g}}\eta(s')$$

$s \leftrightarrow s'$ $\quad \int_{\delta^{-1}} = -\int_{\delta}$

$$= \int_{\delta_i} \underbrace{(\pi(s) - \pi(s'))}_{-\pi^{i+g} \text{ constant}}\,\eta(s) = -\pi^{i+g} N^{i}$$

$\pi(p') - \pi(p) = \pi^{g+i}$

Similarly $\int_{\delta_{g+i}+\delta_{g+i}^{-1}} \pi\cdot\eta = \pi^{i}\cdot N^{g+i}$

$\Downarrow$

$$\boxed{\sum_{i=1}^{g} \left(\pi^{i} N^{g+i} - \pi^{g+i} N^{i}\right) = 2\pi i \sum_{\lambda} \operatorname{Res}_{s_\lambda}(\eta) \int_{s_0}^{s_\lambda} \omega}$$

taken in the interior

** First reciprocity law

(for first & third kind differentials)

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Corollary: $\underset{\omega'}{\overset{\parallel}{\eta}}$ holom

$$\boxed{\begin{array}{c}\displaystyle\sum_{i=1}^{g} \pi^i \pi'^{\,i+g} - \pi^{g+i}\pi'^{\,i} = 0 \\ \text{(first Riemann bilinear relation)}\end{array}}$$

In the same vein: ω, ω' holo

consider $\pi\overline{\omega'}$ $\quad d(\pi\overline{\omega'}) = d\pi\wedge\overline{\omega'} = \omega\wedge\overline{\omega'}$

$$\Rightarrow \int_{\partial\Delta} \pi\cdot\overline{\omega'} = \int_S \omega\wedge\overline{\omega'}$$

$\Rightarrow$ (proceeding as before)

$$\int_S \omega\wedge\overline{\omega'} = \sum_{i=1}^{g} \pi^i\,\overline{\pi'^{\,i+g}} - \pi^{i+g}\,\overline{\pi'^{\,i}}$$

symplectic basis

closed 1-forms

$$\left[\text{also } \int_S \omega\wedge\omega' = \sum_{i=1}^{g}\left(\int_{a_i}\omega\int_{b_i}\omega' - \int_{b_i}\omega\int_{a_i}\omega'\right)\right]$$

Corollary: if $\omega' = \omega$, then

(+) $$0 < i\int_S \omega\wedge\overline{\omega} = i\sum_{i=1}^{g} \pi^i\,\overline{\pi^{g+i}} - \pi^{g+i}\,\overline{\pi^i}$$

$\Rightarrow$ if $\pi^i = 0\ \forall i$, then $\omega = 0$

(+) check

$$\begin{aligned} dz\wedge d\bar z &= (dx+idy)\wedge(dx-idy) \\ &= i\,dy\wedge dx - i\,dx\wedge dy \\ &= (-2i)\,dx\wedge dy \\ &= \frac{2}{i}\,dx\wedge dy \end{aligned}$$

$$i\cdot\frac{2}{i}\,dx\wedge dy = 2\,dx\wedge dy$$

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25 injective

$$\Rightarrow \dim(\Omega^1_{hol}) \leq g$$

Consequence:

The map $\Omega^1_{hol} \longrightarrow \mathbb{C}^g$

$$\omega \longmapsto (\pi^i) \qquad \pi^i = \int_{a_i} \omega$$

a-periods of ω

is injective

$$\Rightarrow \dim(\Omega^1_{hol}) \leq g$$

However, one has already $\dim(\Omega^1_{hol}) \geq g$

$$\Rightarrow \boxed{\dim(\Omega^1_{hol}) = g} \quad \star\star\star$$

analytical interpretation of g, introduced topologically

Upshot

$$g = \dim H^{1,0} = p = \frac{(n-1)(n-2)}{2} - \delta - r$$

topological genus = $\frac{1}{2} \dim H^1(S)$

h^1

1st singular homology group of S

abelian differentials = holomorphic 1-forms

arithmetic genus

$\star$ Alternative derivation: via de Rham's theorem & Hodge

$$h^1 = h^1_{dR} = h^1_{Hodge}$$

first Betti number $2g$

holomorphy $\updownarrow$ harmonicity

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