*** The Riemann-Roch Theorem

analytic version

(à la Donaldson)

ALGEBRAIC CURVES & RIEMANN SURFACES

Lecture XXVII

$p_1 \dots p_d$ distinct points on X (compact RS) genus g

$\downarrow$ D (divisor $D = \sum p_i$)

$H^0(D) = \{ f$ meromorphic having at worst simple poles at $p_i \}$

$H^0(K-D) = \{ \omega$ holomorphic 1-forms vanishing at $p_i \}$

respective dimensions: $h^0(D)$, $h^0(K-D)$

(+) up to $g = h^{1,0} = \dim H^{1,0}_X$ = abelian differential = holom. 1-form

Theorem (Riemann-Roch)$^{(+)}$

genus of X

$$\boxed{h^0(D) - h^0(K-D) = d - g + 1} \qquad (\ast\ast\ast)$$

Proof

The classes $[A_i] \in H^{0,1}$ can be interpreted as linear maps

$A_i : T_{p_i}X \to H^{0,1}$. yielding $\boxed{A : \bigoplus T_{p_i}X \to H^{0,1}}$

We also have a residue map

recall: $\mathbb{C} \ni \lambda \mapsto \lambda [A]$ ← call this map A

$\lambda \frac{\partial}{\partial z} \in T^{1,0}_p \cong T_p$

$$\boxed{\begin{aligned} R : H^0(D) &\longrightarrow \bigoplus_i T_{p_i}X \\ f &\longmapsto \bigoplus_i a_{-1}^{(i)} \frac{\partial}{\partial z^{(i)}} \end{aligned}}$$

Then ** $\boxed{\operatorname{Im} R = \operatorname{Ker} A}$ (cf the above discussion) i.e.

we have an exact sequence$^{(++)}$ $\dots \to A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \to \dots$

$\operatorname{Im} f_{i-1} = \operatorname{Ker} f_i$

$$(\ast) \qquad \boxed{0 \to \mathbb{C} \xrightarrow{i} H^0(D) \xrightarrow{R} \bigoplus_i T_{p_i}X \xrightarrow{A} H^{0,1}}$$

In view of the preceding linear algebraic discussion

$$(\circledast) \quad \boxed{\dim \ker A - \dim \ker \overset{A^*}{A^T} = d - \underbrace{\dim H^{0,1}}_{g}}$$

$\leftarrow$ to be proved!

$$A^T : \underset{\underset{H^{1,0}}{\cong}}{(H^{0,1})^*} \longrightarrow \oplus T_p^* X$$

dual (transpose) of A

<u>Basic</u> <u>claim</u> $(\star)$ $\boxed{A^T = 2\pi i \cdot ev}$ $\quad ev$: evaluation map

$$\boxed{\begin{array}{rcl} ev : H^{1,0} & \rightarrow & \oplus T_{p_i}^* X \\ \omega & \mapsto & \text{values of } \omega \text{ at } p_i \end{array}}$$

Check (dealing with a single p suffices)

$H^{0,1} \ni A$ is represented by $(\bar\partial \beta) \frac{1}{z} \equiv b$

$$\int_{\mathbb{C}} \bar\partial \beta \cdot \frac{\theta}{z} = \int_{\mathbb{C}} \bar\partial \left(\beta \frac{\theta}{z}\right) = \int_\gamma \beta \frac{\theta}{z}$$

compute $(A, \underset{H^{1,0}}{\theta}) = I = \int_X b \wedge \underset{g\,dz}{\theta}$

$$= \int_{\mathbb{C}} (\bar\partial \beta) \frac{1}{z} g(z)\, dz = \text{(Stokes)} \int_\gamma \beta \frac{\theta}{z} = \int_\gamma \frac{\theta}{z}$$

[β has compact support] $\qquad \beta \equiv 1$ on γ

$$= \int_\gamma \frac{g(z)\,dz}{z}$$

$$= 2\pi i \cdot g(0)$$

(<u>Cauchy</u> <u>formula</u>)

$\Rightarrow (\star)$ is <u>established</u>. Thus $\boxed{\ker ev = H^0(K - D)}$

In view of the exactness of (✱)

we have $h^0(D) = \dim\ker R + \dim \operatorname{Im} R$

$= 1 + \dim \ker A$

$(h^{1,0}=1) \Rightarrow \dim\ker A = h^0(D)-1$

Whence (♦) ultimately yields

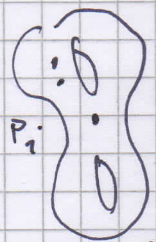
$$\boxed{h^0(D) - h^0(K-D) = d - g + 1}$$

i.e. (✱✱✱)

<u>Comment</u> This is the simplest instance of <u>Serre duality</u> $H^0(K-D) \cong H^1(D)$

RR à la Klein (see Springer)

find F with a single pole at P_i, $i=1\dots m$



Let F_i have a pole at P_i (not single-valued in general). Then

$$F = \sum a_i F_i + \sum_{i=1}^{g} b_j w_j + \overset{\text{constant}}{c}$$

is the most general function

finite complex potential f. ($= \int_{P_0}^{P} \hat{w}_j$ not differential)

when is F single-valued?

The $2g$-periods must vanish $\int_{A_j} dF = \int_{B_j} dF = 0$

⇝ we have $2g$ linear eq. for $m+g$ constants a_i b_i

if the rank of the system is r then (counting c) we have

$x = m+g-r+1$

arbitrary constants making F single valued

But $r \le 2g$ so

$x \geq m+g-2g+1 = m-g+1$

and if $m > g$ then $x \geq 2$

so we have a solution (Riemann's estimate)

Applications of RR

$$\boxed{h^0(D) - h^0(K-D) = d+1-g} \qquad R: h^0(D) \geq d+1-g$$

(Roch's contribution)

① $\boxed{g=0}$: $h^0(K-D) = 0$ (no non trivial holomorphic 1-form)

$$h^0(D) = d+1$$

If $d=1$, then $h^0(D) = 2 \Rightarrow \exists f$ meromorphic on Σ with a single pole, hence with degree 1

∞ is assumed only one, so the same holds in general

From Riemann-Hurwitz $\quad (f: \Sigma \to \bar{\mathbb{C}} \cong S^2 \cong \mathbb{CP}^1)$

$$\chi(\Sigma) = \overset{1}{\overset{\shortparallel}{n}} \cdot \chi(S^2) - R$$

$$2 = 2 - R$$

n: degree of f $= 1$

$$\Rightarrow R = 0$$

$\Rightarrow$ f is a biholomorphism between Σ and $\bar{\mathbb{C}}$

(uniformization theorem for $g=0$)

② $\boxed{g=1}$ Set $D = 0$ (no pts) $d=0$

Then $h^0(D) = 1$ so

$$\underset{\substack{\| \\ 1}}{h^0(D)} - h^0(K) = \underset{\substack{\| \\ 0}}{d} + 1 - \underset{\substack{\| \\ 1}}{g}$$

entails $\boxed{h^0(K) = 1}$ $\Rightarrow$ $\exists\ \omega_1 \in H^{1,0}$, nowhere vanishing

Given a basis (a, b) of 1-cycles of Σ $\quad (H^1(\Sigma) \cong \mathbb{R}^{2g})$
(with intersection number 1, with a suitable orientation)

the map

$$\left\{ P \longmapsto \int_{\underset{\nwarrow \text{ fixed}}{P_0}}^{P} \omega_1 \right\}$$

is well-defined, holomorphic and invertible ($\omega \neq 0$ everywhere)

modulo periods

$$\left\{ \underbrace{\int_a \omega_1}_{\substack{\text{can be set} \\ \| \\ 1}} \qquad \underbrace{\int_b \omega_1}_{\substack{\| \\ \tau}} \right\}$$

generate a lattice Λ (full) in $\mathbb{C} \cong \mathbb{R}^2$

that is

$$\boxed{\begin{array}{ccc} \Sigma & \longmapsto & \mathbb{C}/\Lambda \\ P & \longmapsto & \int_{P_0}^{P} \omega \end{array}}$$

$\equiv J(\Sigma)$ Jacobian of Σ (a torus)

is a biholomorphism (inverse function: Weierstrass $\wp$-function)

$\mathbb{C}$ is then the universal cover of Σ

uniformization for $g=1$

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Variant (more topological)

Let X the field dual to ω : it is nowhere vanishing

Its integration yields an action of $\mathbb{C}$ over Σ, which is transitive since Σ is connected. Therefore

$\Sigma \cong \mathbb{C}/\Lambda$, Λ isotropy group of a point, which is closed. But, given that $\dim \Sigma = \dim \mathbb{C} = 1$, Λ has to be a lattice. [group structure on Σ_1]

Let $f : \Sigma \to \bar{\mathbb{C}}$ meromorphic, with exactly 2 poles
$ \shortparallel$
$ \mathbb{P}^1$

Then

$$\chi(\Sigma) = \overset{\text{degree}}{\overset{\downarrow}{2}}\ \overset{2}{\overset{\shortparallel}{\chi(\mathbb{P}^1)}} - R$$

$$2 - 2g = 4 - R \quad \Rightarrow \quad R = 4 - 2 + 2g = 2 + 2g$$

$\Big\{$ If $g = 1$ $\quad \Sigma_1 \sim$ elliptic curves

$\quad \Sigma_g,\ g \geq 2 \sim$ hyperelliptic curves

RR implies in fact

$$h^0(D) = 2 \geq 2 + 1 - g$$

i.e. $\quad 0 \geq 1 - g \iff g \geq 1$

$g = 3$

ramification points

$2 \cdot 3 + 2 = 8$

Weierstraß $\wp$-function

If $g = 1$, think of $\quad \wp(z) = \dfrac{1}{z^2} + \sum_{\Lambda'} \left(\dfrac{1}{(z-w)^2} - \dfrac{1}{w^2} \right)$

double pole

$\Lambda' = \mathbb{Z}^2 \setminus \{(0,0)\}$