Nomenclature (Recap)

* **meromorphic function** $f : U \subset M \to \mathbb{C}$

$$U = \bigcup_i \mathcal{U}_i \qquad f|_{U_i} = \frac{g_i}{h_i} \qquad g_i, h_i \text{ holomorphic}, \; h_i \neq 0$$

in $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$ $\quad \dfrac{g_i}{h_i} = \dfrac{g_j}{h_j}$, $\quad g_i h_j = g_j h_i$

The singularities of f can only be poles.

** **The Weierstrass preparation theorem**

(work in $\mathbb{C}^2$) If f is holomorphic around 0 in $\mathbb{C}^2$, $f(0)=0$

($f = f(z,w)$) and it is not identically zero on the w-axis, then

$$f = g \cdot h \qquad h(0) \neq 0$$

Hartogs: f holomorphic in (z,w) if and only if it is holomorphic in the single variables

g ↓ * **Weierstrass polynomial**

$$w^d + a_1(z)\, w^{d-1} + \ldots a_d(z) \qquad a_i(0) = 0$$

* **Basic observation**

$f = a w^d + \ldots$ $\quad a \neq 0$

$f(z,w) = 0$ roots $b_1 \ldots b_d$ in $|w| < r$

$$\left[\; b_1^q + b_2^q + \ldots b_d^q = \frac{1}{2\pi i} \int_{|w|=r} w^q \, \frac{\frac{\partial f}{\partial w}}{f} \, dw \;\right]$$

$b_i(z)$ analytic for $|z| < \varepsilon$ small

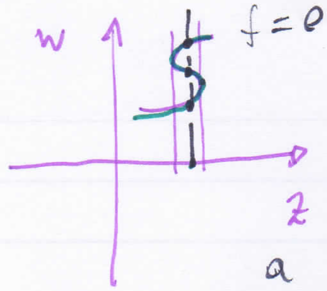
Define
$$g = w^d - \sigma_1(z)\, w^{d-1} + \ldots (-1)^d \sigma_d(z)$$

σ_i: elementary symmetric functions

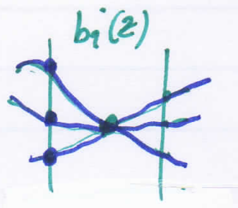
g vanishes precisely on the zero locus of f

$h = \dfrac{f}{g}$ is then holomorphic and $h(0) \neq 0$

Upshot



The zero locus $f=0$ of $f(z,w)$, not identically zero on the w-axis, locally projects on $w=0$ as a finite sheeted cover, branched on the zero locus of an analytic function

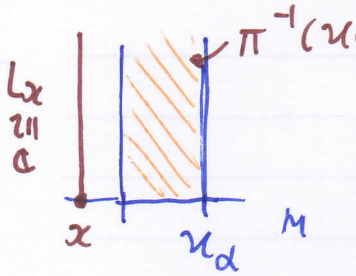


★ Holomorphic line bundles

$L \xrightarrow{\pi} M \qquad M = \bigcup_{\alpha\in\Omega} \mathcal{U}_\alpha$

$\{\mathcal{U}_\alpha\}_{\alpha\in\Omega}$ open cover $\qquad \varphi_\alpha : L_{U_\alpha} \to U_\alpha \times \mathbb{C}$

trivializations $\qquad L_{U_\alpha} = \pi^{-1}(\mathcal{U}_\alpha)$



★ transition functions

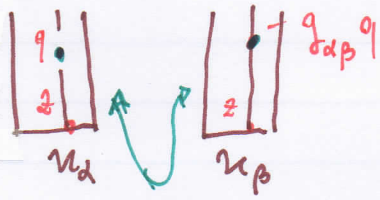
$$g_{\alpha\beta} : \underbrace{\mathcal{U}_\alpha \cap \mathcal{U}_\beta}^{\neq \emptyset} \longrightarrow \mathbb{C}^*$$

holomorphic, non vanishing $\qquad g_{\alpha\beta}(z) = \varphi_\alpha \circ \varphi_\beta^{-1}\Big|_{L_z} \in \mathbb{C}$

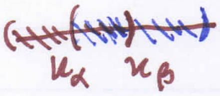
$g_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_\alpha)$

$$\begin{array}{ccccc}
\mathcal{U}_\beta \times \mathbb{C} & \xrightarrow{\varphi_\beta^{-1}} & L_{U_\alpha} & \xrightarrow{\varphi_\alpha} & \mathcal{U}_\alpha \times \mathbb{C} \\
(z, q_\beta) & & \cup & & (z, q_\alpha) \\
 & & L_{U_\alpha} \cap L_{U_\beta} & &
\end{array}$$

$$\{ q_\alpha = g_{\alpha\beta}\, q_\beta \}$$



glue according to



The transition functions satisfy the

★ cocycle conditions

$$\left\{\begin{array}{l} g_{\alpha\beta} \cdot g_{\beta\alpha} = 1 \qquad (★) \\[2ex] g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1 \end{array}\right.$$

on non void triple intersections $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$

conversely, given $\{\mathcal{U}_\alpha\}$ and the $\{g_{\alpha\beta}\}$ fulfilling (★) one manufactures a line bundle $L \to M$ via the above identifications

Given $L \sim \{g_{\alpha\beta}\}$, take $f_\alpha \in \mathcal{O}^*(\mathcal{U}_\alpha)$

and define $\varphi'_\alpha = f_\alpha \cdot \varphi_\alpha$, getting new

transition functions

$$(\star) \quad g'_{\alpha\beta} = \varphi'_\alpha \varphi'^{-1}_\beta = (f_\alpha f_\beta^{-1})\, g_{\alpha\beta}$$

the $\{g_{\alpha\beta}\}$ and the $\{g'_{\alpha\beta}\}$ yield the same bundle

if and only if $(\star)$ holds

$g'_{\alpha\beta} \cdot g_{\alpha\beta}^{-1} = f_\alpha f_\beta^{-1}$

Čech coboundary

$\{g_{\alpha\beta}\}$: Čech 1-cocycle

$\delta(g_{\alpha\beta}) = 0$

group structure:

Line bundles

$H^1(M, \mathcal{O}^*) \equiv Pic(M)$

Picard group

$L \sim \{g_{\alpha\beta}\}$

$L' \sim \{g'_{\alpha\beta}\}$

$L \otimes L' \sim \{g_{\alpha\beta}\, g'_{\alpha\beta}\}$ tensor product

$L^* \sim \{g_{\alpha\beta}^{-1}\}$ dual bundle

* Divisors & line bundles (M : Riemann surface)

D : divisor: $\sum_i k_i a_i$ finite formal sum $k_i \in \mathbb{Z}$

D effective: $k_i \geq 0$

Local defining functions (local coordinates..) $\{f_\alpha\}$

ex: f meromorphic

$(f) = (f)_0 - (f)_\infty$

divisor zeros poles

$(f) = \sum_p \mathrm{ord}_p(f) \cdot p$

$f \underset{loc}{=} \dfrac{g}{h}$ $\mathrm{ord}_p(f) = \mathrm{ord}_p(g) - \mathrm{ord}_p(h)$

g, h holomorphic

Take $g_{\alpha\beta} = \dfrac{f_\alpha}{f_\beta}$; the $g_{\alpha\beta}$

are non zero in $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$

and in a triple intersection satisfy

$g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1$

$\Rightarrow$ get $L \sim \{ g_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \}$

$\boxed{L \equiv [D] \quad \text{line bundle associated to } D}$

(this correspondence is independent of the local data)

$$[\underbrace{D + D'}_{f_\alpha \cdot f'_\alpha}] = [D] \otimes [D']$$

$[\ ] : \mathrm{Div}(M) \to \mathrm{Pic}(M)$ is a homeomorphism

If $D = (f)$ (f meromorphic) $f_\alpha = f|_{\mathcal{U}_\alpha} \Rightarrow$

$g_{\alpha\beta} \equiv 1$

Also, if D, given by $\{f_\alpha\}$ induces a trivial $[D]$,

then $\frac{f_\alpha}{f_\beta} = g_{\alpha\beta} = \frac{h_\alpha}{h_\beta}$ $h_\cdot \in \mathcal{O}^*$

$\Rightarrow f = f_\alpha h_\alpha^{-1} = f_\beta \cdot h_\beta^{-1}$ is a global merom. f. on M, with divisor D

Conclusion: $[D]$ is trivial precisely when

$D = (f)$ (divisor of a meromorphic function)

crucial notion

$D \sim D'$: $D = D' + f$ $D - D' = (f)$

linearly equivalent

$[D] = [D']$

$[\ ]$ is functorial

linear equivalence of divisors

XXVIII-5

Holomorphic and meromorphic sections

$L \to M$ hol. line bundle, trivializations

$\varphi_\alpha : L_{U_\alpha} \to \mathcal{U}_\alpha \times \mathbb{C}$, $L \sim \{g_{\alpha\beta}\}$

$$\varphi_\alpha^* : \mathcal{O}(L)(\mathcal{U}_\alpha) \to \mathcal{O}(\mathcal{U}_\alpha)$$

holomorphic

$$s \to \{ s_\alpha = \varphi_\alpha^* s \in \mathcal{O}(U \cap \mathcal{U}_\alpha) \}$$

$$\Rightarrow \boxed{ s \to \{s_\alpha\} \qquad s_\alpha = g_{\alpha\beta}\, s_\beta }$$

similar definition for s meromorphic

if $s' \neq 0$, then s/s' is a meromorphic function

$$(s) = \sum_p \mathrm{ord}(s)\cdot p \qquad \text{divisor of } s$$

caveat for non effective divisors

$D \in \mathrm{Div}(M)$ given by $\{f_\alpha\}$. Then get

a meromorphic section $s_f \in \mathcal{M}([D])$ $\quad s_f = D$

Conversely let L be given by $\{\varphi_\alpha\}$, $\{g_{\alpha\beta}\}$.

Let s be a global meromorphic section

$$\frac{s_\alpha}{s_\beta} = g_{\alpha\beta} \qquad \text{i.e.} \quad L = [(s)]$$

Therefore if $[D] = L$, $\exists\, s$ with $(s) = D$ and for any s', mer. section of L, $L = [(s')]$

Amplification

$$D = \sum a_i p_i$$

$$\mathcal{L}(D) = \{ f : M \to \mathbb{C} \mid f \text{ meromorphic}, \ D + (f) \geq 0 \}$$

(i.e, they are holomorphic on $M - \cup p_i$ and $\mathrm{ord}_{p_i}(f) \geq -a_i$

$|D|$ = effective divisors linearly eq. to D

if $L = [D]$, write $|L| \equiv |D|$

Let s_0 a global meromorphic section of $[D]$, $(s_0) = D$. Then if s is a global holomorphic section of $[D]$ ($[D]$ must then be effective)

$$f_s = \frac{s}{s_0} \quad \text{is a meromorphic}$$

function on M with

$$(f_s) = (s) - (s_0) \geq -D$$

i.e $f_s \in \mathcal{L}(D)$

We have: $(s) = D + (f_s) \in |D|$

Moreover, if $f \in \mathcal{L}(D)$, then $s = f \cdot s_0$ is holomorphic $\Rightarrow$

holomorphic sections

$$\mathcal{L}(D) \xrightarrow[\simeq]{\otimes s_0} H^0(M, \mathcal{O}([D]))$$

If M is compact, $\forall D' \in |D|$

$\exists f \in \mathcal{L}(D)$ s.t. $D' = D + (f)$

and two such functions differ by a non-zero constant. Thus

$$|D| = \mathbb{P}(\mathcal{L}(D)) \cong \mathbb{P}(H^0(M, \mathcal{O}([D])))$$

effective divisors corresponding to a linear subspace of $\mathbb{P}(H^0(M, \mathcal{O}(L)))$:

** linear system

* complete linear system : $|D|$

that is, it contains every effective divisor linearly equivalent to any of its members

$$\dim |D| = \underbrace{h^0(M, \mathcal{O}(D))}_{\dim H^0} - 1$$

dim 1 : pencil
dim 2 : net
dim 3 : web

XXVIII-8