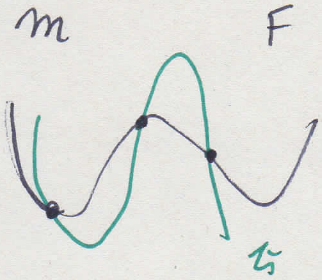
★ linear series : classical approach

F irr. plane curve of order m



C_i order m'

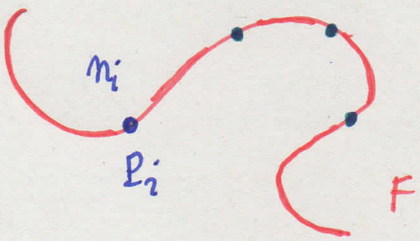
intersections $= m'm$
(Bézout)

◆ Question: given $k \le m'm$ pts on F, how many independent curves of order m' intersect F in these points, plus $mm'-k$ others?

cycle ≡ point group ≡ effective divisor
(Weil) classical terminology arithmetical terminology

$\sum_{i=1}^{k} n_i P_i$, $P_i \in F$ formal sum $n_i \in \mathbb{Z}$, $n_i \geq 0$

★ order of a cycle: $n = \sum_i n_i$



A contains B if $n_P \geq m_P$, i.e. if $A - B$ is a cycle

$A = \sum n_P P$, $B = \sum m_P P$

* virtual cycles : $\sum n_P P$ $n_P \in \mathbb{Z}$

$\updownarrow$

divisors (tout court)

cycle $\equiv$ effective cycle

virtual cycles

$\rightarrow$ cut out by rational functions:

ϕ : rational function interlacing F

in $A - B$

effective cycles without common places

A: zeros of ϕ

B: poles of ϕ

?

ϕ meromorphic, in the previous setting

Recall:

★ Rational functions on a curve $f(x,y)=0$

Algebraic digression

$f(x,y) = 0$ (irreducible)

field

$\Sigma := \mathbb{C}(\xi, \eta)$ ξ transcendental over $\mathbb{C}$

η algebraic over $\mathbb{C}(\xi)$: $f(\xi,\eta)=0$

quotient field of $\mathbb{C}[\xi]$

(exclude $f = f(x)$)

★ More intrinsically: $\Sigma =$ quotient field of $K[x,y]/\mathcal{J}$

$\mathcal{J}$: principal ideal (f)

examples: rational functions $z \mapsto \frac{P(z)}{Q(z)}$ (on $\mathbb{P}^1$)

*** elliptic functions: $\mathbb{C}(\wp(z), \wp'(z))$ ← algebraic over $\mathbb{C}(\wp(z))$

Weierstrass function

cubic equation

$\wp$ and $\wp'$ subject to

$$\frac{P(\wp(z), \wp'(z))}{Q(\wp(z), \wp'(z))}$$

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3$$

$$y^2 = 4x^3 - g_2x - g_3$$

If G intersects F in a cycle B containing a cycle A we say that G cuts out A

$(O_P(G) \geq n_P$

$A = \sum_P n_P P$

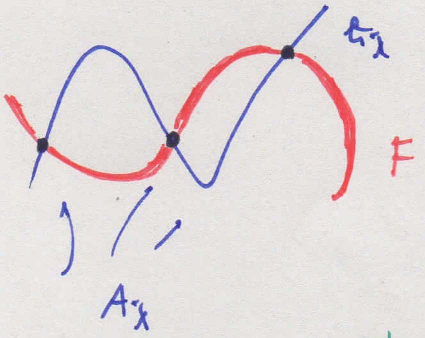
$$\sum_0^r \lambda_i G_i = 0 \qquad G_i \text{ curves of order } m'$$

(linear system)

$$\equiv \quad G_\lambda = 0$$

Assume G_λ does not have F as a component

A_λ : cycle in which G_λ intersects F



curves of the same order m'

order m

* A_λ : linear series on F and G_λ intersects F in this linear series

cycle

(effective)

$\boxed{g_n^r \text{ : linear series of order } n \text{ and dimension } r}$

Every linear series can be cut out by G_λ in a 1-1 manner

a curve G_λ cuts out a cycle of the serie

A cycle is cut out by a unique curve G_λ

Riemann problem : find r

Answer : g_n^r complete (effective, maximal) then

$$r = n - p + i$$

genus — index of speciality

Riemann-Roch

(Riemann : $i = 0$

$r = n - p$

in general $r \geq n - p$)

XXX-3

4 Equivalence of point groups
cycles

(Cf also Enriques - Chisini)

$A \equiv B$ iff $\exists\ \varphi$ rational (meromorphic) on f $(\cdot \equiv f=0)$

Σ

such that A : zeros of φ

B : poles of φ

$B = \sum n_i p_i$

divisor (virtual cycle) $D = A - B$

Riemann - Roch problem

given a cycle A, find all B equivalent to it

$\equiv$ find all rational (meromorphic) functions φ having

poles $B' \leq B$: $m_i \leq n_i$

$B' = \sum m_i p_i$, $B = \sum n_i p_i$

★ Connection rational functions $\leftrightarrow$ linear series

$$\sum_{i=0}^{r} x_i \varphi_i = x_0\varphi_0 + x_1\varphi_1 + \ldots\ x_r\varphi_r = 0$$

Def: linear system cutting a series

$\rightarrow$ Series of level groups of functions

$$x_1 \frac{\varphi_1}{\varphi_0} + x_2 \frac{\varphi_2}{\varphi_0} + \ldots\ x_r \frac{\varphi_r}{\varphi_0} \quad (= -x_0)$$

with poles $\varphi_0 = 0$

Conversely, given r rational functions $t_1 \ldots t_r$ with the same poles, then $t_i = \dfrac{\varphi_i}{\varphi_0}$ (reducing to the same denominator)

$\Rightarrow$ the series of the level sets of the $t_1 \ldots t_r$ will be cut on f by $x_0\varphi_0 + x_1\varphi_1 + \ldots\ x_r\varphi_r = 0$

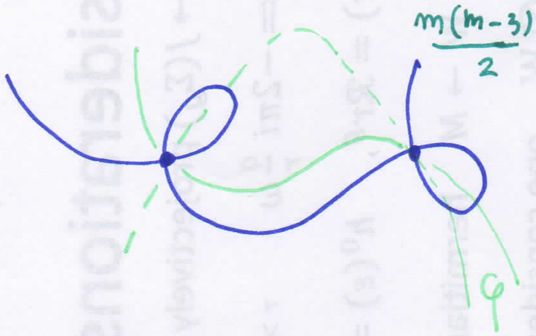
Riemann's formula

(after Enriques–Chisini)

$n \geq r \geq n - \underset{g}{p}$ for a g^r_n

$f = f_m \qquad \delta = \underbrace{\frac{(m-1)(m-2)}{2} - p}_{= \frac{m(m-3)}{2} + 1 - p}$ double points

$f=0$



consider the adjoint curves φ passing simply through the double pts of f

(cf. the treatment of Pascal's limaçon)

take $\varphi = \varphi_{m-3+h} \qquad h \geq 0$

if $h = 0$: $\quad \hat{i}(f, \varphi) = m(m-3) - 2\delta$

intersections outside double pts

$= 2p - 2$ → check

$$m(m-3) - 2\left[\frac{(m-1)(m-2)}{2} - p\right]$$
$$= m(m-3) - (m-1)(m-2) + 2p$$
$$= \cancel{m^2} - 3m - \cancel{m^2} + 3m - 2 + 2p$$
$$= 2p - 2 \quad (= -\chi)$$

for $h > 0$, add $m \cdot h$ intersections

Therefore we have

$$(\ast) \quad \boxed{n = 2p - 2 + m \cdot h}$$

order of the series

ok for $p = 0, 1$ as well...!

Let us estimate the dimension r of the series g^r_n cut out on f by the φ_{m-3+h}

Assume h big enough (at least ≥ 3) such that the double pts of f yield independent conditions

We have

$$r = \underbrace{\frac{(m-3+h)(m+h)}{2}}_{\text{all curves of order } m-3+h} - \underbrace{\delta}_{\text{double points}}$$

$$- \left[\frac{h(h-3)}{2} + 1\right]$$

curves of order $m+h-3$ having f as a component

remove curves $f_m \cdot \psi_{h-3}$

Since every effective cycle belongs to point group

curves of order $\leq N$ (on $\mathbb{P}^2$) $\infty^{\bullet}$

$$\bullet = \frac{(N+1)(N+2)}{2} - 1 = \frac{N^2+3N+2-2}{2} = \frac{N(N+3)}{2}$$

therefore, from $\left[\delta = \frac{m(m-3)}{2} + 1 - p\right]$, see XXX-5, we get

$$\left[r = \frac{\cancel{m^2} - \cancel{3m} + hm + hm - 3h + h^2}{2} - \frac{\cancel{m^2} - \cancel{3m} + 2 - 2p}{2} \right.$$

$$- \frac{h^2 - 3h + 2}{2}$$

$$= \frac{2hm - 2 + 2p - 2}{2} = \boxed{p - 2 + m \cdot h}$$

((♦) page XXX-5)

Finally, from $n = 2p - 2 + mh$ we have

$m \cdot h = n + 2 - 2p$, so eventually

$$\left[r = p - \cancel{2} + n + \cancel{2} - 2p = n - p \right]$$

This dimension can only decrease in general, so

$$\{ r \geq n - p \} \quad (\text{+ Riemann})$$