☆ Geodesic polygons on the hyperbolic disc

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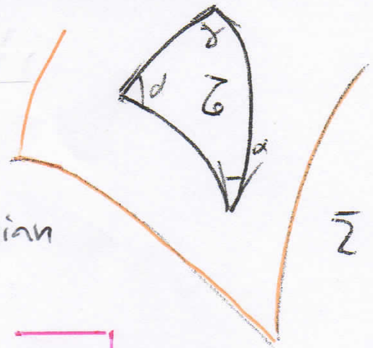
Remember the "Theorema elegantissimum" (Gauss, 1827)

For a geodesic triangle

τ on a regular surface

Σ (or on an abstract Riemannian surface)



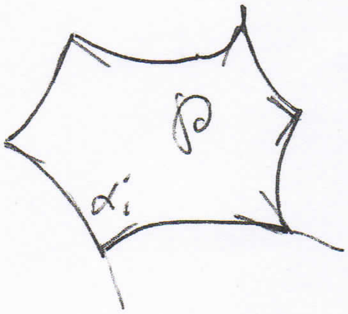
$$\boxed{\alpha+\beta+\gamma-\pi = \iint_{\tau} K\, d\sigma}$$

angular defect / generalised spherical excess

If Σ is a pseudosphere or, abstractly, a hyperbolic plane one has $K \equiv -1$ (upon normalization)

$$\boxed{A(\tau) = \pi - (\alpha+\beta+\gamma)}$$

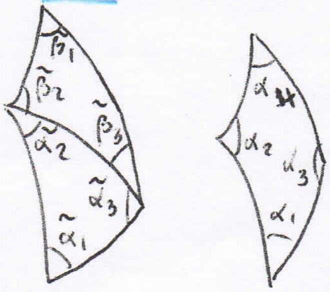
☆ The above formula immediately generalises to a convex geodesic polygon

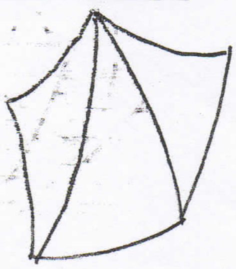


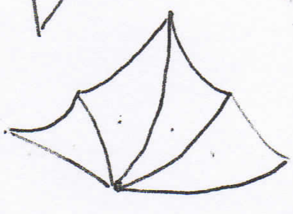
$$\boxed{A(P) = \pi(n-2) - \sum \alpha_i}$$

triangles

angular defect







Example: The area of a regular n-polygon with vertices on the boundary of the disc (thus $\alpha_i = 0 \ \forall i$)

vale $\pi(n-2)$.

arcs of circles intersecting the boundary circle orthogonally

If $\alpha_i = \dfrac{2\pi}{n}$, then $\boxed{A(P) =} \ \pi(n-2) - 2\pi$

So $\boxed{\sum_i \alpha_i = 2\pi}$

$= \pi n - 4\pi = \boxed{\pi(n-4)}$ Standard regular n-igon

Does there exist such a polygon? Yes. Let us check this

If r (Euclidean radius of the circumscribed circle) goes to zero ($r \to 0$) one finds:

$$0 = \pi(n-2) - \sum \alpha_i \Rightarrow \sum_{i=1}^{n} \alpha_i = \pi(n-2)$$

(the Euclidean value)

Since the function $r \longmapsto A(P)(r)$ is continuous an strictly increasing, $\exists!$ $0<R<1$ such that

$$\alpha = \alpha_i = \frac{2\pi}{n} \quad \text{(Standard regular } n\text{-gon)}$$

(For such a value $\boxed{\sum \alpha_i = 2\pi}$,

$A(P) = \pi(n-2) - 2\pi = \pi(n-4)$)

since $0 < \pi(n-4) < \pi(n-2)$ if $n \geq 5$

The interesting case for hyperbolic geometry is $n = 4g$, $g \geq 2$. The area of a standard regular $4g$-gon is

$$\pi \cdot (4g-4) = 4\pi(g-1)$$

$$\boxed{A(P) = 4\pi(g-1) = -2\pi(2-2g) = -2\pi\,\chi(\Sigma_g)}$$

see figure

standard, regular $4g$-gon

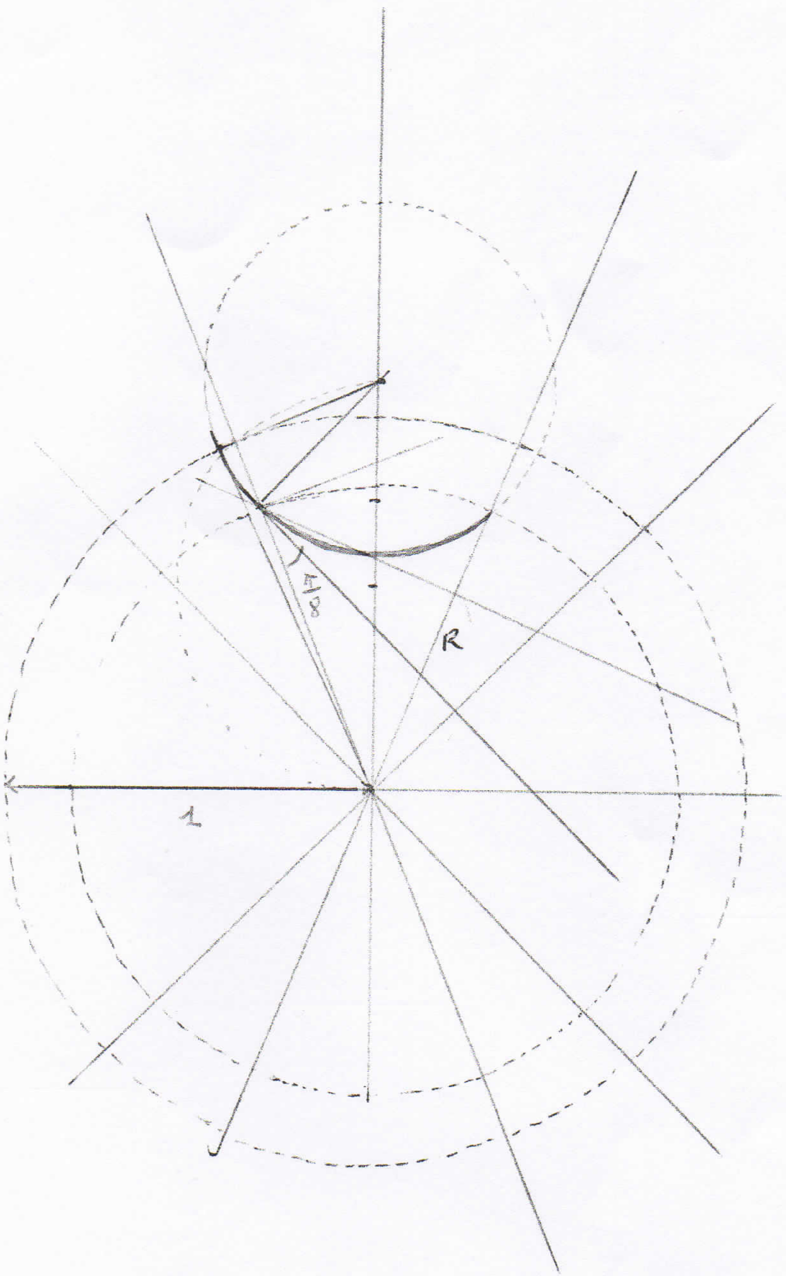
$g \geq 2$

closed orientable surface of genus $g \geq 2$

This suggests an analogy between hyperbolic geometry and such surfaces. [The area of the standard $4g$-gon would then be a topological invariant]

Indeed, this analogy is quite stringent

Σ_g can be obtained by suitably identifying the sides of a $4g$-gon (whose sides are geodesics) in pairs



construction of standard regular 4g-gon with $g = 2$ (octogon) $\measuredangle$ $\frac{\pi}{4}$ $\frac{2\pi}{8} = \frac{\pi}{4}$ $A(P) = 4\pi$

[⚠ Classically what we depicted is a kind of <u>analysis</u> and not a <u>synthesis</u>, in the sense that first we constructed the polygon and subsequently the unit circle]

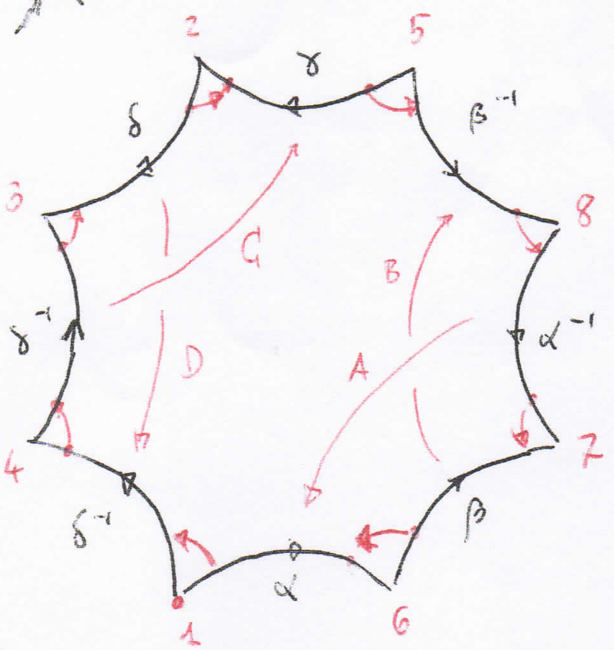
$g = 2$

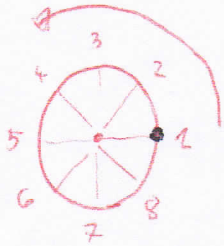
Construction of Σ_g via an identification
4g-gon with geodesic sides

$\mathbb{H}$ is 2-point homogeneous:
Given p, q, p', q' with
$d(p,q) = d(p', q')$, $\exists (!)$
isometry $p \mapsto p'$
$q \mapsto q'$

$\frac{\pi}{4} \cdot 8 = 2\pi$

$\pi/4$





first

last

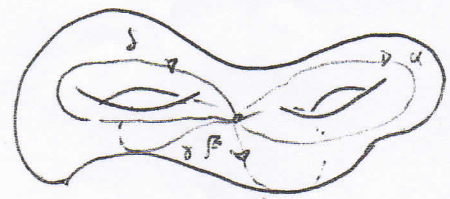
$$A\, B\, A^{-1} B^{-1} C\, D\, C^{-1} D^{-1} = I$$

$$\alpha \beta \alpha^{-1} \beta^{-1} \gamma \delta \gamma^{-1} \delta^{-1} = 1$$

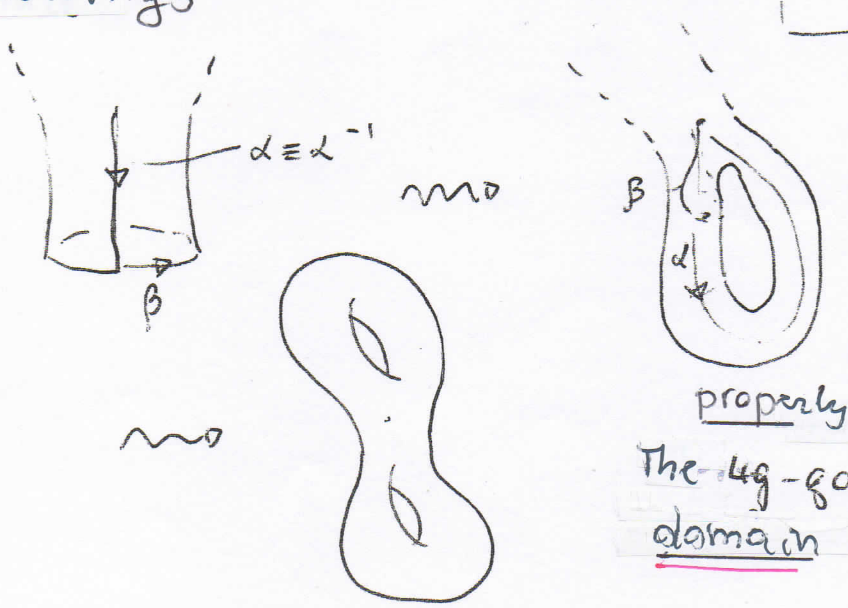
hyperbolic plane
upper-half plane
disc

$$\boxed{\Sigma_g = \mathbb{H}/\Gamma}$$

$\Gamma \cong \pi_1(\Sigma_g)$



"Sewings"



Γ acts on $\mathbb{H}$ in a properly discontinuous manner

The 4g-gon is a fundamental domain

The metric of $\mathbb{H}$ $(K=-1)$

goes down to the <u>quotient</u>

$\Rightarrow$ Every Σ_g can be equipped with a metric with curvature $K=-1$

Aside

7. Sphere with g handles Σ_g

(connected sum of g tori)

$g=3$

$g=2$

generators — relators

$$\pi_1(\Sigma_g) = \langle a_1 b_1, a_2 b_2 \ldots a_g b_g \;/\; a_1 b_1 a_1^{-1} b_1^{-1} \cdot a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle$$

$$\prod_{i=1}^{g} [a_i; b_i] = 1$$

$H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$

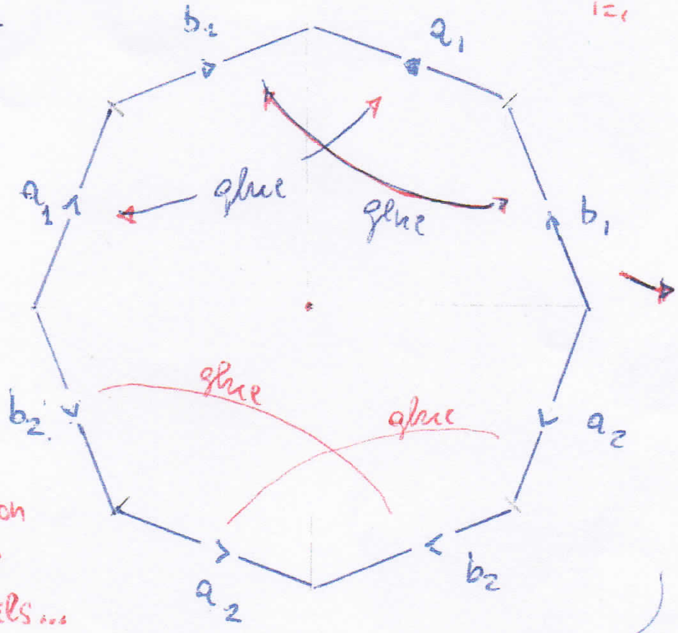
(abelianization)

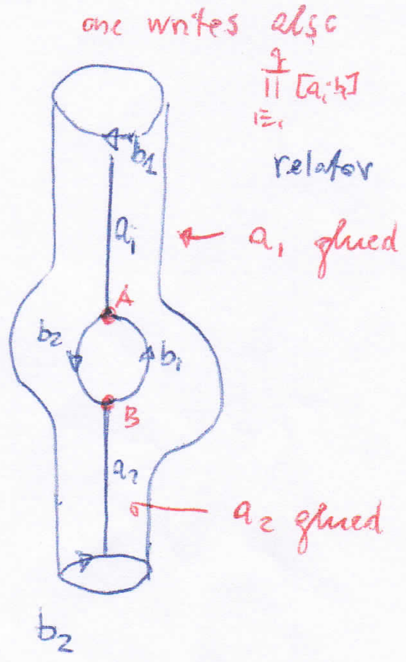
$H_0(\Sigma_g) \cong \mathbb{Z}$

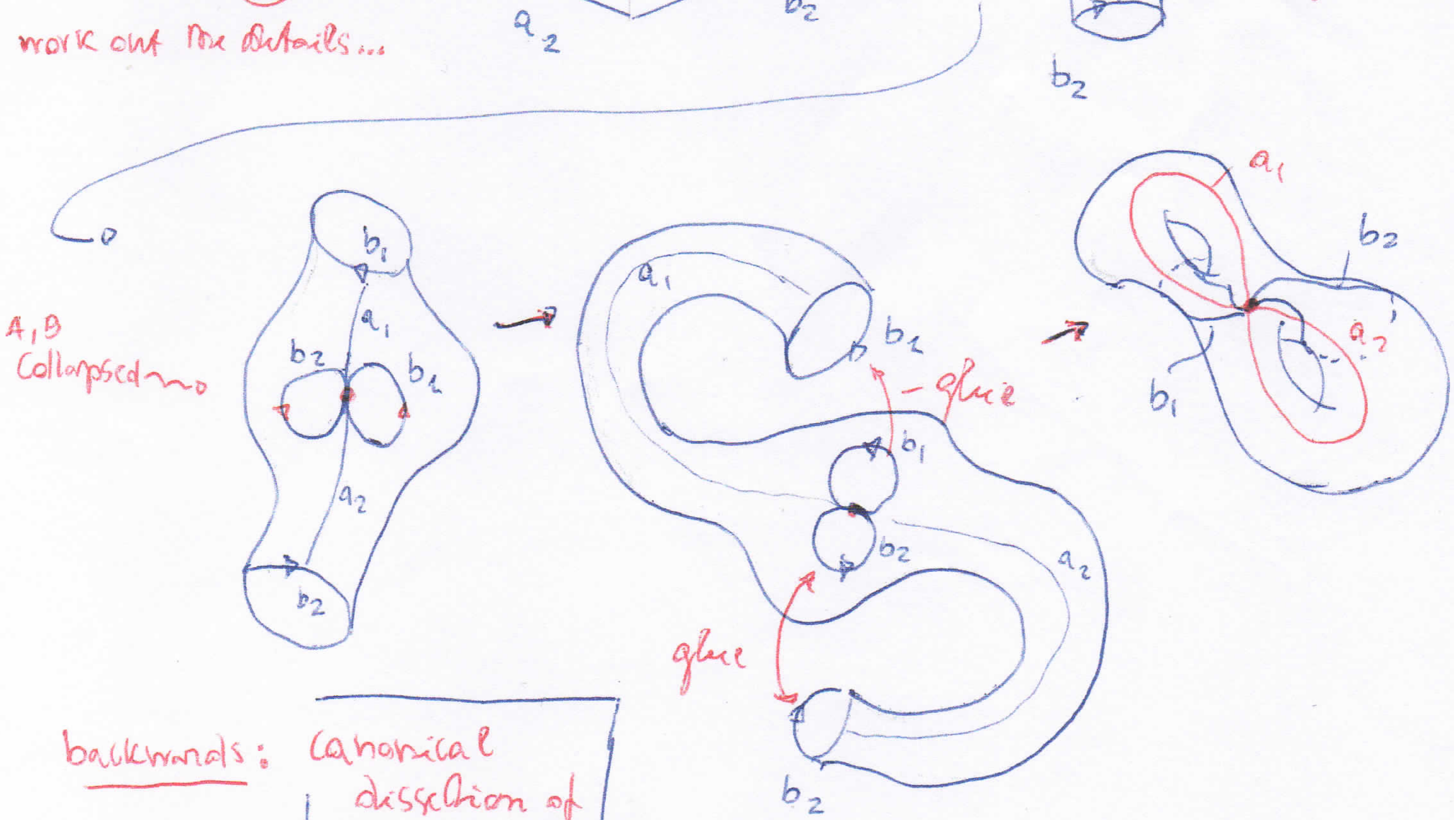
$H_2(\Sigma_g) \cong \mathbb{Z}$

also directly from a triangulation ~>

work out the details...







backwards: canonical dissection of Σ_g