Lecture XXXIII

ALGEBRAIC CURVES & RIEMANN SURFACES

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- Fuchsian groups

$G \subset PSL(2,\mathbb{R})$ $\quad G$ discrete

(w.r. to the topology of $PSL(2,\mathbb{R})$)

$z \longmapsto \dfrac{az+b}{cz+d}$, $z \in \mathbb{H}^2$ upper-half plane $\{(x,y)\equiv z \mid y>0\}$

$ad-bc=1$

$a \mapsto -a$, $b \mapsto -b$, $c \mapsto -c$, $d \mapsto -d$

Discontinuous action on $\mathbb{H}^2$:

$\forall\ K \subset \mathbb{H}^2$ compact $\quad G \ni g \quad g(K)\cap K = \emptyset$

but for a finite number of $g \in G$

($G \subset PSL(2,\mathbb{C})$, G preserving Δ disc or half-plane

$A \subset \Delta \qquad \bar{A}$ = closure of A (w.r.to Δ))

- Fundamental domain

G Fuchsian group (1) D domain

(2) $\exists F$: $D \subset F \subset \bar{D}$ and any two distinct points of F do not lie on the same orbit

(3) $\Delta = \bigsqcup_{g\in G} g(F)$

(4) $A(\partial D) = 0$

$\to$ hyperbolic tessellation $\{g(F)\}_{g\in G}$

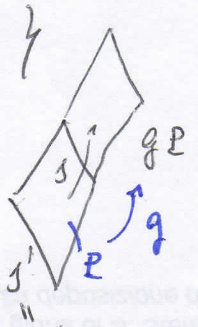
D locally finite: $K \cap g(\bar{D}) = \emptyset$ but for a finite number of g's

$D \rightsquigarrow$ polygon P (convex)

sides of P (s)

$s = \bar{P} \cap g(\bar{P})$ when $\underset{\text{length}}{l(s)} > 0$

$E^0 = \{ g \,/\, \bar{P} \cap g(\bar{P})$ is a side of $P \}$



* Coupling (side pairing) $\phi : E^0 \to S$

$g \longmapsto \phi(g) = s$

is a bijection, E^0 generates E and

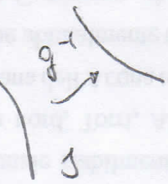
$f : S \to S$

$s \longmapsto s' = f(s) = \phi(g^{-1})$

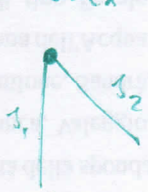
pairs the sides of P, s' conjugate to s

$s' = g^{-1} \cdot s$ indeed

$s' = \phi(g^{-1})$
$s = \phi(g)$



* vertex : (non void) intersection of two sides



Cycles $(z_1 \dots z_n)$: G-orbit (on vertices of P)

4 **Poincaré's theorem**

G given $\rightsquigarrow$ find a tessellation via P

side pairing : G^0 generators for G

cycles : relations $\qquad G^0$: finitely presented group

$$(*) \quad \underbrace{\phi_1 + \dots \phi_n}_{\text{cycle}} = \frac{2\pi}{k} \quad k > 0 \qquad \text{"aliquot part of } 2\pi\text{"}$$

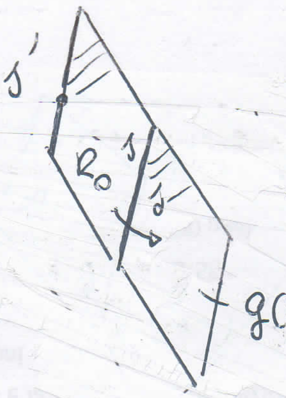
at a vertex

Conversely given a polygonal tessellation of $\mathbb{H}^2$ (or Δ)

one can find G via side pairing

(cond $(*)$)

Poincaré's train of thought



s' conjugate to s : $s' = g^{-1}s$

R_0: generating polygon

three types of cycles

sides: 1st kind $\to 2n$ /n/ conjugate pairs

2nd kind —

Knowing R_0 allows reconstruction of G

necessary conditions
- conjugate sides must be congruent
- Σ(angles) at a cycle $= \frac{2\pi}{n_1}$ or $\frac{2\pi}{n_2}$

Poincaré proves (up to minor corrections) that the above conditions are also sufficient

★ Side/angle conditions

Theorem: If Π, a compact polygon, is a fundamental region for a group Γ of orientation preserving isometries of $S^2, \mathbb{R}^2, \mathbb{H}^2$, Then

(i) $\forall\ s$, side of Π, $\exists!\ s' = gs$, $g \in \Gamma$
(g: side pairing transformations of Π) and

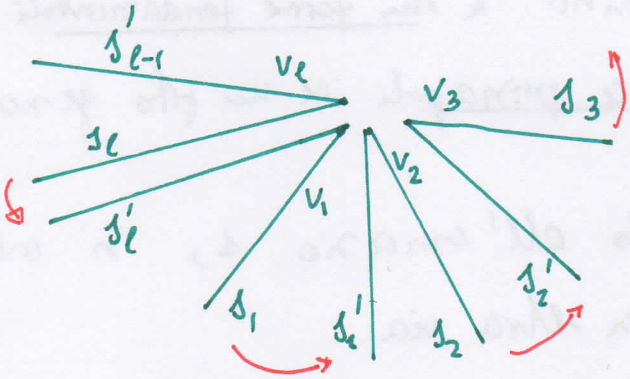
(ii) if each side s is identified with s', then each set of identified vertices corresponds to a set of corners with angle sum $\frac{2\pi}{p}$ $(p \in \mathbb{Z})$

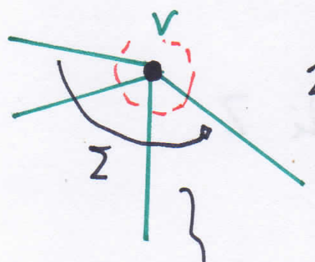
Proof. (i) Let $g_1\Pi \dots g_k\Pi$ tiles sharing sides with Π
Let s be shared by Π and $g_i\Pi$. Then
$s' = g_i^{-1}s$ is a shared side with $g_i^{-1}\Pi$ and
$g_i^{-1}g_i\Pi = \Pi \Rightarrow s'$ is a side of Π
If $s' = s$, $g_i^{-1} = g_i \Rightarrow g_i =$ rotation through π about the mid point of s: this point then becomes a new vertex $\Rightarrow s$ splits into s_1, s_2. Therefore
$g_i s_1 = s_2$ and $g_i^{-1}s_1 = s_2$, i.e. $s_1 = g_i s_2$
$\Rightarrow$ in all cases, $s \in \Pi$ is mapped by $g \in \Gamma$ onto $s' \in \Pi$. s' is unique.

Indeed, if $s_1' = g_1 s$ and $s_2' = g_2 s$ are sides of Π, there would be interior points of Π near s_1' and s_2' in the same Γ-orbit, contrary to the definition of fundamental region.

Ad (ii). Let $\{v_1 .. v_\ell\}$ be a vertex cycle of Π:

$\dots v_\ell\, v_1\, v_2 \dots v_\ell\, v_1 \dots$ (order induced by the side pairing transformations)

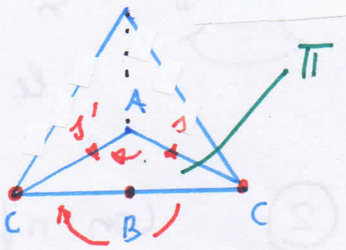




$2\pi = n \cdot \Sigma$, $n \in \mathbb{Z}$

sum of corners at $v_1 .. v_\ell$

Example



Cycles: $A \to \frac{2\pi}{3}$

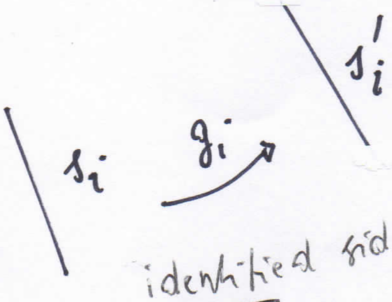
$B \to \pi$

$C \to \frac{\pi}{6} + \frac{\pi}{6} = \frac{2\pi}{6}$

$\left(= \frac{\pi}{3}\right)$

★ Poincaré : A compact polygon Π satisfying the side-angle conditions is a fundamental region for the group Γ generated by the side-pairing transformations of Π

(it states that side/angle condition provide sufficient conditions for Π to be a fundamental polygon)



$s_i' = g_i s_i$

identified sides of Π

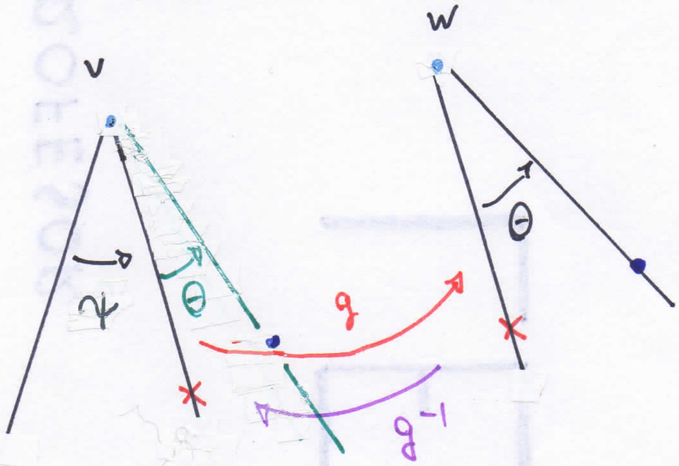
Π polygon satisfying side-angle condition

cycle

$V = (v_1 \dots v_e)$

$$\sum_i \theta_i = \frac{2\pi}{p}$$

"aliquot part of 2π"

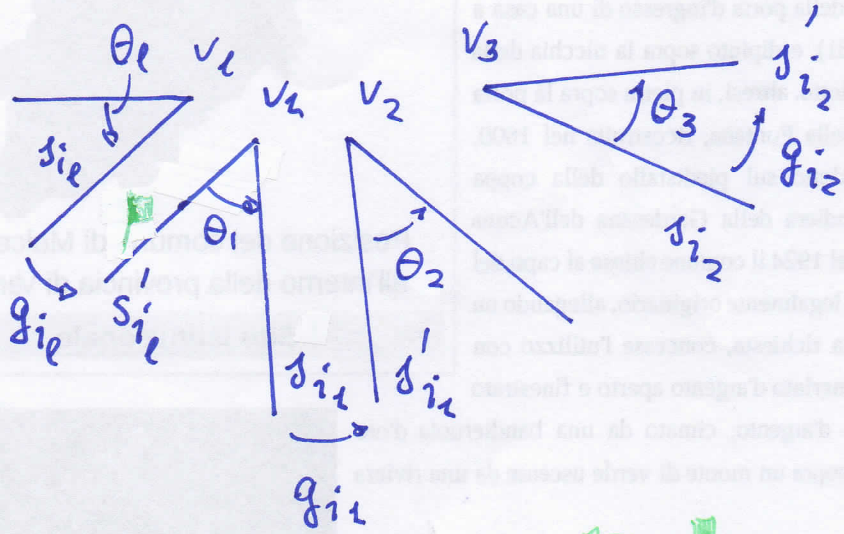


$$\boxed{g R^v_\theta g^{-1} = R^w_\theta} \qquad \boxed{g R^v_\theta = R^w_\theta g}$$

Basic point of the tessellation construction

$v_1 \ldots v_l$ vertex cycle

$$\theta_1 + \ldots + \theta_l = \frac{2\pi}{p} \quad (p > 0)$$



$$g_{i_l} R^{v_l}_{\theta_l} \cdots g_{i_2} R^{v_2}_{\theta_2} g_{i_1} R^{v_1}_{\theta_1} = 1$$

However $R^{v_2}_{\theta_2} = g_{i_1} \cdot R^{v_1}_{\theta_2} \cdot g_{i_1}^{-1}$

That is $R^{v_2}_{\theta_2} \cdot g_{i_1} = g_{i_1} R^{v_1}_{\theta_2}$

In general $R^{v_3}_{\theta_3} g_{i_2} g_{i_1} = g_{i_2} g_{i_1} R^{v_1}_{\theta_3}$ et cetera

Finally

$$g_{i_l} \cdots g_{i_1} \underbrace{R^{v_1}_{\theta_l} \cdots R^{v_1}_{\theta_1}}_{R^{v_1}_{\theta_1 + \cdots + \theta_l}} = 1$$

$\Rightarrow g_{i_l} \cdots g_{i_1}$ = rotation of angle $-\frac{2\pi}{p}$

Then $(g_{i_l} \cdots g_{i_1})^p = 1$ ← least power ensuring this

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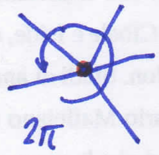
The sequence of corners at any vertex (corr. to $v_1 \dots v_\ell$ (cycle)) closes after p repetitions of the cycle (angle sum $= 2\pi$)

corners at a vertex

$g_{i_\ell},\ g_{i_2} g_{i_1}, \dots, g_{i_\ell} \cdots g_{i_2},\ g_{i_1}(g_{i_\ell} \cdots g_{i_1}) \dots$



$\to$ get a geometric surface

generators, relator $(g_1 \cdots g_{i_\ell})^p = 1$

$\Gamma = \langle g_{i_1}, \dots g_{i_\ell} \mid (g_1 \cdots g_{i_\ell})^p \rangle$

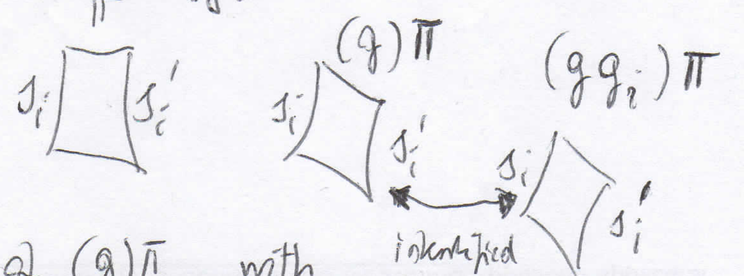
Π : fundamental polygon of $S = \mathbb{H}^2 / \Gamma$ ($\mathbb{R}^2$, S^2)

$\cong \pi_1(S)$

Poincaré's polygon theorem

outline of Stillwell's proof

$\hat{S}_\pi$ constructed by adjoining tiles $(g)\pi$



identify s_i' of $(g)\pi$ with s_i of $(gg_i)\pi$

(if $g=1$) s_i' of π id with s_i of $g_i\pi$

$(gg_i)\pi$ adjacent to $(g)\pi$ as $g_i\pi$ is adjacent to π

$\hat{S}_\pi$ is a complete geometric surface (see details)

$\hat{S}_\pi = \tilde{S}/\pi_1(\tilde{S}_\pi)$

Strategy: First assume $\pi_1(\hat{S}_\pi)=1$

It is then enough to prove that

$(g)\pi = g\pi$

$(1)\pi = \pi$

The proof proceeds by induction on k:

$$g = g_{i_1}^{\varepsilon_1} \cdots g_{i_k}^{\varepsilon_k} \qquad \varepsilon_i = \pm 1$$

for $k=1$ this is clear.

assume then $(g')\pi = \pi$ $\quad g' =$ prod of $\leq k-1$ gen or inverses

$(g' g_i^{\pm 1})\pi$ is to $(g')\pi = g'\pi$ as $g_i^{\pm 1}\pi$ is to π

that is

$$(g' g_i^{\pm 1})\pi = g'(g_i^{\pm 1}\pi) = g' g_i^{\pm}\pi$$

which yields the conclusion

Let us now prove that $\pi_1(\hat{S}_\pi) = 1$

Lift the tessellation of $\hat{S}_\pi$ to $\tilde{S}$

then only one polygon lies over $(g)\pi$, namely $g\pi$

if $g = g_{i_1}^{\varepsilon_1} \cdots g_{i_k}^{\varepsilon_k}$, we reach $(g)\pi$ from $(1)\pi$ by the appropriate sequence of edge crossings

we reach $g\pi$ on $\tilde{S}$ by exactly the same crossings

$\Rightarrow$ any path from $(1)\pi$ to $(g)\pi$ lifts to a path from π to $g\pi$ in $\tilde{S}$