Prof. M. Spera — Hyperbolic case

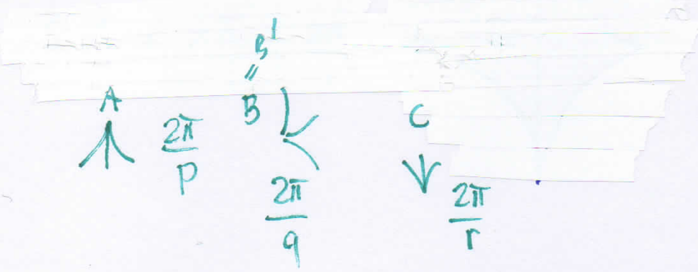
ALGEBRAIC CURVES & RIEMANN SURFACES

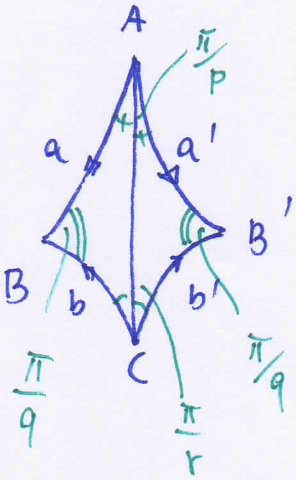
* Triangle tessellations

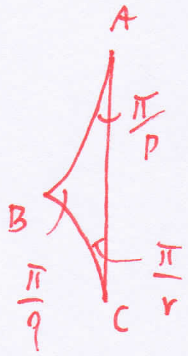
simplest poligons : quadrilaterals, obtained by doubling triangles

Lecture XXXIV

angle/side condition







$$A(\diamond) = \pi(n-2) - \sum \alpha_i \quad \leftarrow \text{recall...}$$

$$= \pi(\overset{n}{4}-2) - 2\pi\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)$$

$$= 2\pi\left[1-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)\right] > 0$$

$$\Rightarrow \boxed{\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1} \quad \leftarrow \text{Hyperbolic tessellations}$$

- spherical case : S^2 $\quad \frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 1$

p	q	r	
2	2	$n \geq 2$	dihedron
2	3	3	tetrahedron
2	3	4	cube / octahedron
2	3	5	icosahedron / dodecahedron

platonic solids

- plane : $\mathbb{R}^2$ $\quad \frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$

p	q	r	
3	3	3	equil. triangle
2	4	4	square
2	3	6	regular exagon (eq. triangle)

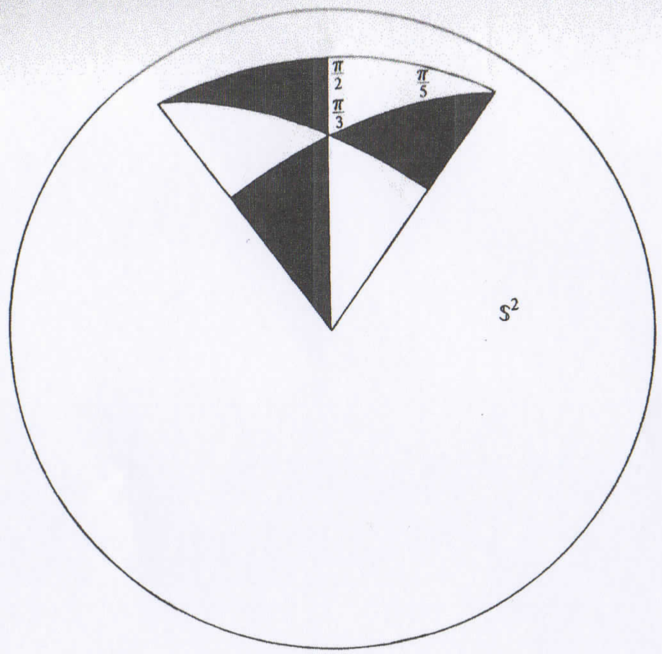


FIGURE 7.16.

All other values of p, q, $r \geq 2$ give $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ and, hence, an angle sum $< \pi$, so the (p, q, r) triangle must be hyperbolic. Proving that all these possibilities can be realized is, in fact, as difficult as proving that an *arbitrary* polygon satisfying the side and angle conditions can serve as a fundamental region; hence, we will postpone the proof until we deal with the general case. Instead we shall look at two examples which help to expose the difficulties involved.

Example 1. The (4,4,4) triangle.

Sixteen $(4, 4, 4)$ triangles can be assembled to form a regular octagon in $\mathbb{H}^2$ with corner angles $\pi/4$ (Figure 7.17, which is from Burnside [1911, p. 395], shows this octagon in the $\mathbb{D}^2$-model). It follows that if we identify sides of the octagon so as to make a genus 2 surface with one vertex, then the angle sum at the vertex will be 2π, and hence S will be a hyperbolic surface by Section 5.5. We can then lift the tessellation of S to its universal cover, which is $\mathbb{H}^2$ by the Killing–Hopf theorem, giving an obviously symmetric tessellation of $\mathbb{H}^2$ by $(4, 4, 4)$ triangles.

From Stillwell "Geometry of Surfaces

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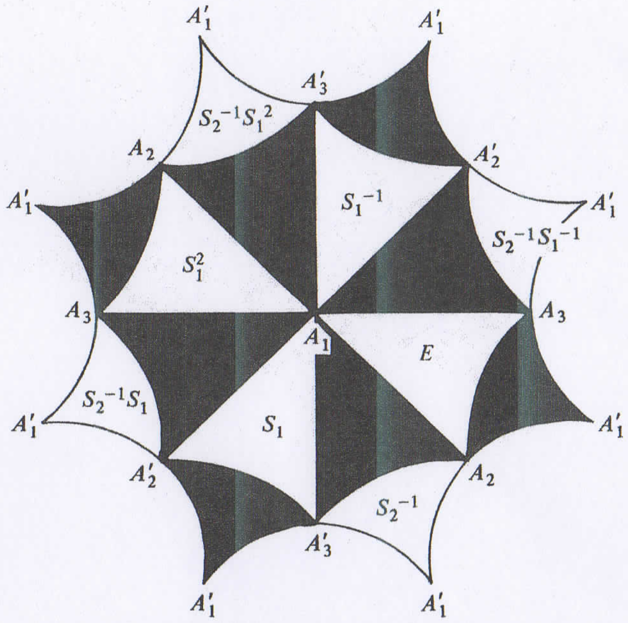

FIGURE 7.17. From Burnside: *The Theory of Groups of Finite Order*, p. 395, with permission from Dover Publications, Inc.

4 Euler characteristic

$$\chi(\Sigma_g) = 2-2g = c - n + 1$$

$\chi(\Sigma_g)$: Π_{2n} polygon

$c = V$, $n = E$, $1 = F$: $V - E + F$

vertices = cycles, edges = $\frac{2n}{2}$, faces

(for a bona fide surface

Klein

check: (2,3,7) triangle

$c = 2 \quad n = 7$

$2-7+1 = -5+1 = -4$

$\Rightarrow g = 3$ ✓

Recall

$$A(\Pi_{2n}) = (2n-2)\pi - \Sigma = 4\pi(g-1)$$

Σ = Σ angles

$$\boxed{\Sigma} = (2n-2)\pi - 4\pi(g-1)$$

$$= (2n - 2 - 4g + 4)\pi$$

$$= (2n + 2 - 4g)\pi$$

$$= \boxed{2(n+1-2g)\pi}$$

$2-2g = c - n + 1$

$2g = 2 - c + n - 1$

$= 1 + n - c$

$$g = \frac{n+1-c}{2}$$

check : 1 reg. polygon $\quad \cancel{2\pi} \stackrel{?}{=} 2n\pi + \cancel{2\pi} - 4g\pi$

$2n = 4g$ ✓

2 Klein (2,3,7) triangle

$$4\pi \stackrel{?}{=} 2(\underbrace{7+1}_{8} - 6)\pi = 4\pi \quad ✓$$

$$\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{7} < 1 \right)$$

$g \geq 2$

$\star \quad A(\Sigma_g) = 4\pi(g-1)$

$(K=-1)$

$g=3$

$\star\star$ Klein tessellation: $(2,3,7)$ How many triangles?

elegantissimum

$$A_\tau = \pi\left(1-\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{7}\right)\right) = \pi\,\frac{\overbrace{42-21-14-6}^{-41}}{42} = \frac{\pi}{42}$$

$$x \cdot \frac{\pi}{42} = 4\pi \cdot 2 = 8\pi \; ; \text{ then}$$

$$A = -2\pi\,\chi(S) = 4\pi(g-1)$$

$$\boxed{x = 42 \cdot 8 = 336}$$

$$\underset{\substack{\| \\ 4\pi(g-1)}}{A(P)} \overset{\sim \Sigma_g}{} = \pi(N-2) - \overset{\text{? angles}}{\overset{\|}{\Sigma}}$$

$\Sigma = \pi\alpha_0 \cdot N$ # sides

Klein: $\Sigma = \frac{2\pi}{7} \cdot 14 = 4\pi \quad N=14 \quad g=3$

$$\underset{\substack{\| \\ 8\pi}}{4\pi \cdot 2} = \pi \cdot 12 - 4\pi = 8\pi \quad \checkmark$$

4 Klein quartic $\mathcal{C}$ hom. coord (x, y, z)

$$F = x^3 y + y^3 z + z^3 x = 0$$

$\mathcal{C}$ is non singular $\left(\Rightarrow g = \frac{\overbrace{(4-1)}^{3}\overbrace{(4-2)}^{2}}{2} = 3 \right)$

Proof.

⚠ We give up giving the full details of the construction

$$\frac{\partial F}{\partial x} = 3x^2 y + z^3 = 0$$

$$\frac{\partial F}{\partial y} = x^3 + 3y^2 z = 0$$

$$\frac{\partial F}{\partial z} = y^3 + 3z^2 x = 0$$

If $z = 0$ then $x = y = 0 \Rightarrow$ non acceptable

Let $z = 1$

$(\Rightarrow x \neq 0, y \neq 0)$

$$\begin{cases} 3x^2 y + 1 = 0 \\ x^3 + 3y^2 = 0 \\ y^3 + 3x = 0 \Rightarrow x = -\frac{y^3}{3} \end{cases}$$

$$3 \cdot \frac{y^6}{9} \cdot y + 1 = 0 \Rightarrow \frac{y^7}{3} + 1 = 0 \qquad y^7 = -3$$

$$\boxed{y^{14} = 9}$$

$$x^3 y^5 + 3\overbrace{y^7}^{-3} = 0 \qquad x^3 y^5 = 9$$

$$\frac{-y^9}{27} y^5 = 9 \qquad -y^{14} = 27 \cdot 9 = 3^6$$

$$\boxed{y^{14} = -3^6}$$

!!

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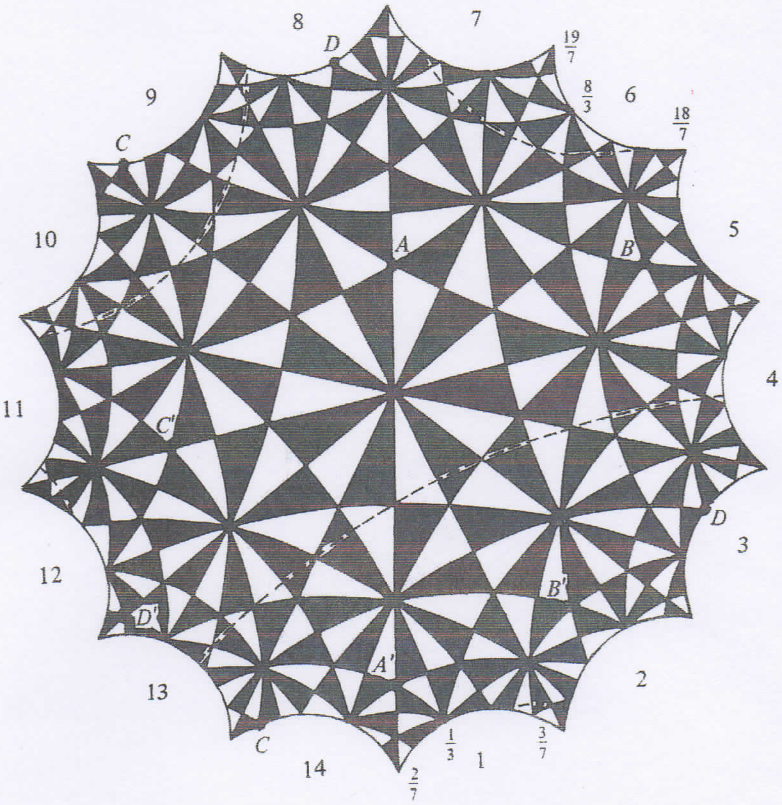

FIGURE 7.18.

Example 2. The $(2, 3, 7)$ triangle.

Klein [1879] made the astonishing discovery that there is a regular 14-gon with corner angles $2\pi/7$ formed by 336 copies of the $(2, 3, 7)$ triangle (Figure 7.18). If side $2i + 1$ of this 14-gon is identified with side $2i + 6$ (mod 14), the result turns out to be a genus 3 surface S with two vertices. The two vertices of S correspond to the sets of even and odd vertices of the 14-gon; hence, they each have angle sum 2π. Thus, we again have a hyperbolic surface, and we obtain the desired tessellation of $\mathbb{H}^2$ by lifting the tessellation of S to its universal cover.

Knowing now that the tessellation of $\mathbb{H}^2$ exists, we can see that it is symmetric because it is generated by reflections in the sides of the $(2, 3, 7)$ triangle. (The mere *appearance* of regularity in a tessellation should not be trusted, incidentally. See Exercise 7.3.3).

In these two examples we have used a compact surface as a shortcut to $\mathbb{H}^2$. Instead of filling $\mathbb{H}^2$ with infinitely many triangles, we have filled a compact surface with finitely many, then used the covering of the com-

from Stillwell "Geometry of surfaces"

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Exercises

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7.3.3. The te
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