Lecture XXXV

** Birational transformations (on $\mathbb{P}^2$)

$\mathbb{P}_2$ $\mathbb{P}_2'$ copies of $\mathbb{P}^2$

$P : [x_1 x_2 x_3]$
$P' : [x_1' x_2' x_3']$

$$(*)\quad \begin{cases} x_1' = \varphi_1(x_1, x_2, x_3) \\ x_2' = \varphi_2(x_1, x_2, x_3) \\ x_3' = \varphi_3(x_1, x_2, x_3) \end{cases}$$

φ_i : homogeneous polynomials of the same degree n without a common factor

$$\det \begin{bmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \frac{\partial \varphi_3}{\partial x_3} \\ \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} & \frac{\partial \varphi_2}{\partial x_3} \\ \frac{\partial \varphi_3}{\partial x_1} & \frac{\partial \varphi_3}{\partial x_2} & \frac{\partial \varphi_3}{\partial x_3} \end{bmatrix} \neq 0$$

$$\equiv \quad \frac{\partial(\varphi_1 \varphi_2 \varphi_3)}{\partial(x_1 x_2 x_3)} \neq 0$$

Jacobian determinant

$$\frac{\partial(\varphi_1, \varphi_2, \varphi_3)}{\partial(x_1 x_2 x_3)} = 0$$

* Jacobian curve
order : $3(n-1)$

$(*)$ maps a line

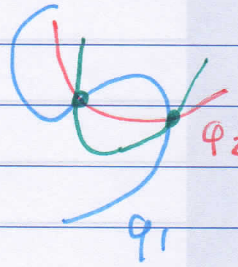
$$\ell : \quad a x_1 + b x_2 + c x_3 = 0 \qquad (a, b, c) \neq (0,0,0)$$

to

$$\mathcal{C}_\ell : \quad a\varphi_1 + b\varphi_2 + c\varphi_3 = 0$$

$P \longmapsto P'$
generally well-defined
problems arise on
P' not defined

$$\begin{cases} \varphi_1 = 0 \\ \varphi_2 = 0 \\ \varphi_3 = 0 \end{cases}$$



Conversely, from P' one gets n^2 points P as preimages: indeed

$$\begin{cases} x' = \dfrac{x_1'}{x_3'} = \dfrac{\varphi_1(x_1,x_2,x_3)}{\varphi_3(x_1,x_2,x_3)} = \dfrac{\varphi_1(x\ y)}{\varphi_2(x,y)} \\[2ex] y' = \dfrac{x_2'}{x_3'} = \dfrac{\varphi_2(x,y|}{\varphi_3(x,y} \end{cases}$$

$$\Rightarrow \quad \begin{cases} x'\varphi_3(x,y) - \varphi_1(x,y) = 0 \\ y'\varphi_3(x,y) - \varphi_2(x,y) = 0 \end{cases} \quad \leftarrow \text{degree } n$$

fixed

$\Downarrow$

get n^2 solutions (Bézout)

Thus $(*)$ is not 1-1 in general

<u>homaloidal</u> <u>systems</u>

degree of a ∞^n system of plane algebraic curves = # intersections of any two curves of the system outside the base points

r base points ; curves $\mathcal{C}^n$; degree $= n^2 - r$

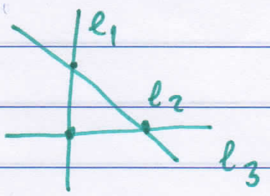
homaloidic system: n^2-1 base points

$\Rightarrow$ degree $= n^2 - (n^2-1) = 1$

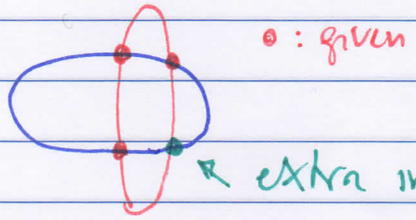
$n = 2$ homaloidic nets

single intersections outside the n^2-1 base points

Examples : • lines on a plane $n = 1$ $r = 2$



• conics passing through 3 given pts



✱✱ plane birational transformations

one wishes to "invert" (✱)

Net
$$(\blacklozenge)\ \lambda_1\varphi_1 + \lambda_2\varphi_2 + \lambda_3\varphi_3 = 0$$

Assume (◆) has γ base points
Then γ among the n^2 intersections would fall in these base points, so the effective intersections would be just $n^2 - \gamma$
hence can be "neglected"

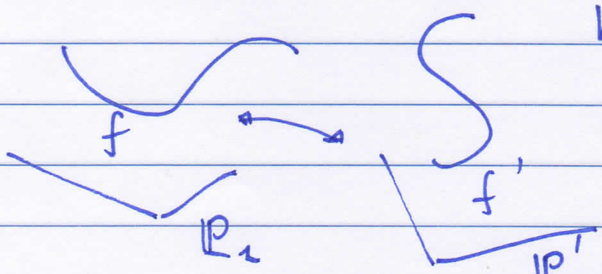
Moreover, if $\nu = n^2 - 1$, then there would be just one effective intersection, and the transformation (generically varying) would be invertible : a birational transformation of the plane

homaloidal system!

$$\begin{cases} x_1 = \bar{\varphi}_1(x_1', x_2', x_3') \\ x_2 = \bar{\varphi}_2(x_1', x_2', x_3') \\ x_3 = \bar{\varphi}_3(x_1', x_2', x_3') \end{cases}$$

✱ Cremona transformation (they form a group)

The birational transformation may hold between two curves f and f' but may not be extendable to the whole ambient planes : f and f' can be birationally equivalent without being Cremona equivalent



Let us elaborate on this point

Consider a net of degree $d > 1$

It determines an "involution of order d" in the following sense. All curves passing through P (different from the base pts) form a pencil, which then has n^2 base pts.

Being the net of degree d, one has

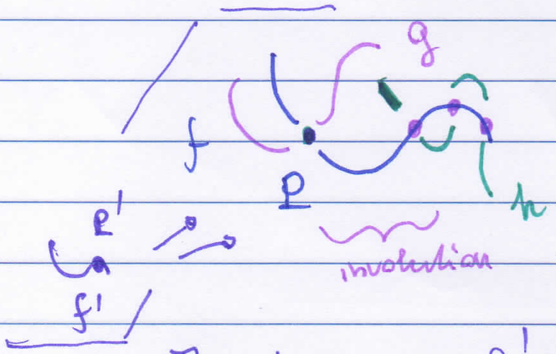
$$n^2 = \underset{\substack{\uparrow \\ \#\,\text{base pts}}}{\nu} + d$$

Therefore, every curve passing through P (non base) must go through the extra $d-1$ pts varying with P (plus the ν base points of the net)

Hence we get a correspondence $1-d$ between $\mathbb{P}_2'$ in $\mathbb{P}_2$

Given f' in $\mathbb{P}_2'$ let its transformed curve be f

If f passes through some P, it must pass through the full group of points conjugate to P



Now $f = \underset{\substack{\uparrow \\ \text{not passing through the extra pts}}}{g} \cdot \underset{\substack{\uparrow \\ \text{passing through the } d-1 \text{ pts}}}{h}$

Therefore $f' \xleftrightarrow{\text{birationally}} g$ but this correspondence cannot be extended to the whole planes, i.e. it cannot be promoted to a Cremona transformation

4 Cremona transformations

$$\begin{cases} x_i' = \varphi_i(x) \\ x_i = \bar{\varphi}_i(x') \end{cases}$$

One has a projectivity between the net of lines in $\mathbb{P}_2'$ and the curves of the corresponding homaloidal system (and vice versa)

Fundamental points: base points of the homaloidal systems

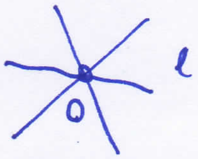
bijectivity ceases. Task: eliminate the ambiguity

Let $O:[0,0,1]$ be an r-ple base point of the net then:

homogeneous polynomials $h_r^{(i)}$ has degree r

$$(4)\begin{cases} \varphi_1(x_1,x_2,x_3) = h_r^{(1)} x_3^{n-r} + h_{r+1}^{(1)} x_3^{n-r-1} + \dots h_{n-1}^{(1)} x_3 + h_n^{(1)} \\ \varphi_2(x_1,x_2,x_3) = h_r^{(2)} x_3^{n-r} + h_{r+1}^{(2)} x_2^{n-r-1} + \dots + b_{n-1}^{(2)} x_3 + h_n^{(2)} \\ \varphi_3(x_1 x_2 x_3) = h_r^{(3)} x_3^{n-r} + h_{r+1}^{(3)} x_3^{n-r-1} + \dots h_{n-1}^{(3)} x_3 + h_n^{(3)} \end{cases}$$

l: line through O



$$l: \begin{cases} x_1 = \lambda \\ x_2 = \lambda m \\ x_3 = 1 \end{cases} \xrightarrow[\text{substitute into}]{} (4)$$

$$x_j' = \lambda^r h_r^{(j)}(m) + \lambda^{r+1} h_{r+1}^{(j)}(m) + \dots \lambda^n h_n^{(j)}(m)$$

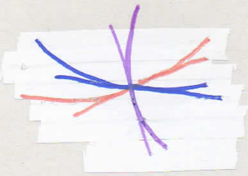
removing λ^r:

$$x_j' = h_r^{(j)}(m) + \lambda\, h_{r+1}^{(j)}(m) + \dots$$

if $\lambda \to 0$ one has $(x_1', x_2', x_3') = (h_r^{(1)}(m), h_r^{(2)}(m)\ h_r^{(3)}(m))$

Upshot: upon varying X in the "first order neighborhood of 0", one gets a rational algebraic curve $x' = h_r(m)$

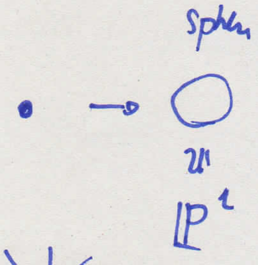
If the tangents are distinct, one has distinct pts on the curve

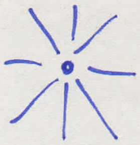


If the tangents are partially coinciding, the indeterminacy can be still removed. If they coincide, the indeterminacy persists.

★★★ **blow-up**

a point is replaced by lines through it (a copy of $\mathbb{P}^1$)





★ Invariance of the genus under a quadratic transformation

This follows from birational (or topological) invariance. However, it can be proven directly.

Let $\mathcal{C}_n$ (irreducible) not passing through the fundamental points, and possessing ν ordinary multiple pts (distinct tangents) $P_1, P_2, \ldots P_\nu$ with mult r_i $i = 1, \ldots \nu$

$$p = \frac{(n-1)(n-2)}{2} - \underbrace{\sum \frac{r_i(r_i-1)}{2}}_{A}$$

$$\binom{r_i}{2} = \frac{r_i(r_i-1}{2}$$

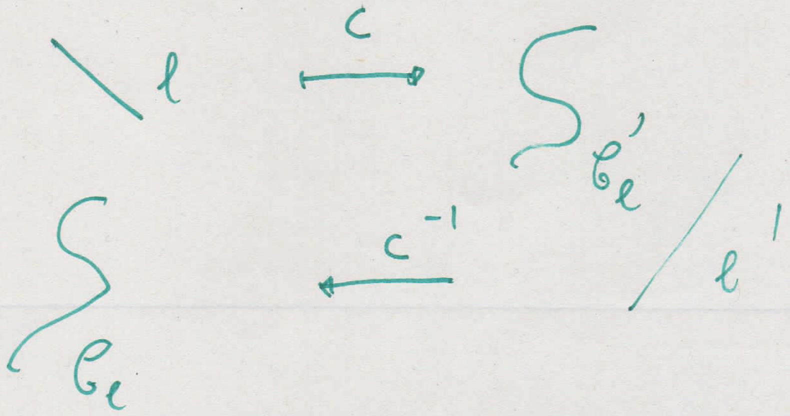
$\mathcal{C}'_{2n}$ passes through the fund points, with multiplicity n, and $P'_1 \ldots P'_\nu$ have the <u>same</u> multiplicities

Therefore $$p' = \frac{(2n-1)(2n-2)}{2} - 3\,\frac{n(n-1)}{2} - A$$

$$= \frac{4n^2 - 2n - 4n + 2 - 3n^2 + 3n}{2} - A$$

$$= \frac{n^2 - 3n + 2}{2} - A = \frac{(n-1)(n-2)}{2} - A = p$$

* Order of a Cremona transformation



in general

$n^2 - \nu = d$

$\uparrow$

base points

$\nu = n^2 - d$

$$\underset{n}{\underset{\parallel}{m(l, \mathcal{C}_l)}} = \underset{n'}{\underset{\parallel}{m(l', \mathcal{C}'_l)}}$$

n: order of c

first order: collineations

$n^2 - \nu = 1$ homaloidal net

If $\nu = 0$, $n^2 = 1 \Rightarrow n = 1$

* quadratic transformations (order 2)

$$\mathbb{P}_2 \longrightarrow \mathbb{P}_2'$$

$$l \longmapsto \mathcal{C}_l^2$$ ← homaloidal net $(d=1)$

$\nu = n^2 - 1 = 4 - 1 = 3$

base points

if these three pts are distinct, one speaks of a general quadratic transformation

(otherwise, one has a special quadratic transformation

★ general quadratic transformations

A_1, A_2, A_3 base points (distinct, not collinear) of a homaloidal net of conics

w.l.o.g A_i = fundamental pts

Take all conics: $\lambda_1 x_2 x_3 + \lambda_2 x_3 x_1 + \lambda_3 x_1 x_2 + \lambda_4 x_1^2 + \lambda_5 x_2^2 + \lambda_6 x_3^2$

Impose passage through A_i : get $\lambda_4 = \lambda_5 = \lambda_6 = 0$

$$\Rightarrow \left[\lambda_1 \underbrace{x_2 x_3}_{\varphi_1} + \lambda_2 \underbrace{x_3 x_1}_{\varphi_2} + \lambda_3 \underbrace{x_1 x_2}_{\varphi_3} = 0 \right]$$

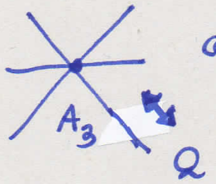
$$\begin{cases} x_1' = x_2 x_3 \\ x_2' = x_3 x_1 \\ x_3' = x_1 x_2 \end{cases} \qquad \begin{cases} x' = \frac{1}{x} \\ y' = \frac{1}{y} \end{cases} \qquad \begin{cases} x = \frac{1}{x'} \\ y = \frac{1}{y'} \end{cases}$$

$$x_1' : x_2' : x_3' = \frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3} \qquad \begin{cases} x_1 = x_2' x_3' \\ x_2 = x_3' x_1' \\ x_3 = x_1' x_2' \end{cases}$$

→ the conics of the homaloidal net in $\mathbb{P}_2'$ have the fundamental pts of that plane as base pts.

→ Take $A_3 : [0,0,1]$ then $x_i' = 0 \quad \forall i$

not defined!



consider the pencil through A_3

$$\begin{cases} x_1' = \lambda m \\ x_2' = \lambda \\ x_3' = \lambda^2 m \end{cases}$$

$A_3 : [0,0,1]$

$Q : [1, m, 0]$

→ at ∞

$$\begin{cases} x_1 = \lambda \\ x_2 = \lambda m \\ x_3 = 1 \end{cases}$$

$$\Big\downarrow$$

$$\begin{cases} x_1' = m \\ x_2' = 1 \\ x_3' = \lambda m \end{cases}$$

If $\lambda \to 0$ we get $[m, 1, 0] \in A_1'A_2'$

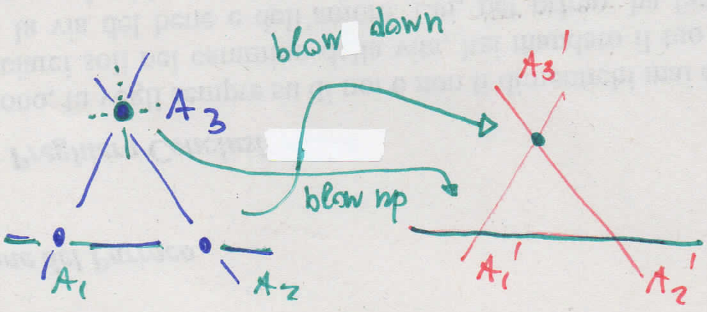
pts of the "first order neighbourhood" of A_3

$$\Updownarrow$$

pts of $A_1'A_2'$

projectivity

$$\begin{cases} x_1' = x_2 \\ x_2' = x_1 \end{cases}$$



$(a,b,c) \neq (0,0,0)$

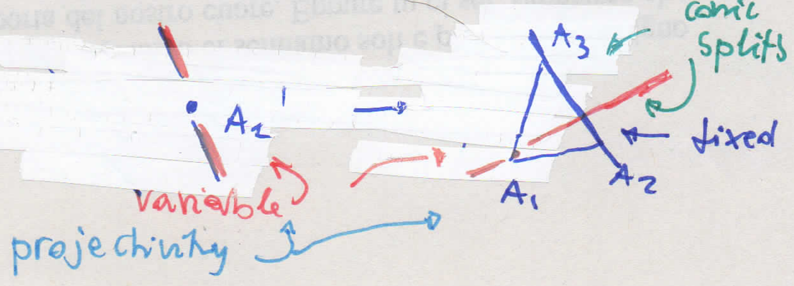
$r : a x_1' + b x_2' + c x_3' = 0 \longmapsto a x_2 x_3 + b x_3 x_1 + c x_1 x_2 = 0$ (passes through A_i)

if $r \ni A_1$

then $a = 0$

$r : b x_2' + c x_3' \longmapsto b x_3 x_1 + c x_1 x_2 = x_1 (b x_3 + c x_2) = 0$

$(b,c) \neq (0,0)$



- given a conic in $\mathbb{P}_2$ not passing through the fundamental pts

otherwise the transformed curve splits

$$C: \; a x_2x_3 + b x_3x_1 + d x_1^2 + e x_2^2 + f x_3^2 = 0$$

$$\downarrow$$

$$Q \qquad a x_3' x_1'^2 x_2' + b x_1' x_2'^2 x_3' + c x_2' x_3'^2 x_1^2 + \\ + d\, x_1^2 x_3'^2 + e\, x_3'^2 x_1'^2 + f\, x_1'^2 x_2'^2 = 0$$

Q is a quartic having 3 double pts in A_1', A_2', A_3'

$\Rightarrow$ Q is rational (this was to be expected....)

- in general $\mathcal{C}^n$: $f_n = 0$

$$f_n = \sum_{i+j+k=n} a_{ijk}\, x_1^i x_2^j x_3^k = 0 \longmapsto$$

$$f_{2n}' = \sum_{i+j+k=n} a_{ijk}\, x_1^{j+k} x_2^{i+k} x^{i+j} = 0 \qquad \mathcal{C}'^{2n}$$

indeed: $j+k + i+k + i+j = 2(i+j+k) = 2n$

if one index is zero, the sum of the remaining ones is n $\Rightarrow$ $\mathcal{C}'$ goes through A_1' with multiplicity

$2n - (j+k) = 2n - n = n$, and the same occurs for the other fundamental pts.

More generally if $\mathcal{C}n$ goes through the fundamental pts with multiplicities $r_1, r_2\, r_3$, then $\mathcal{C}'$ has $A_2'A_3'$, $A_3'A_2'$, $A_1'A_2$ as fixed components $\underset{\text{with multiple } r_1, r_2, r_3}{}$: up to those components the order will be $2n-(r_1+r_2+r_3)$.

$\mathcal{C}'$ passes through A_1 with multiplicity $n-(r_1+r_2)$ et cetera