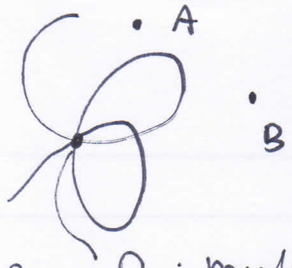
** Resolution of singularities of a plane curve



$\mathcal{C}_n$ O : mult. = s

↑ fundamental pt

quadratic transformation ⟿ $\mathcal{C}'_{2n-s}$ ←

fund. pts: O, A, B

$A, B \notin \mathcal{C}_n$

AB intersecting $\mathcal{C}_n$ in n distinct pts

O': n-ple pt

A', B': $n-s$ ple points

If $\mathcal{C}_n$ has $P_1 \dots P_\nu$ multiple pts with mult $s_1 \dots s_\nu$, not lying on fundamental lines, they become $P'_1 \dots P'_\nu$ with the same multiplicities.

However $O \longrightarrow A'B'$ ✳ blow-up

the h distinct tangents $t_1 \dots t_h$, $h \leq s$

yield $P'_1 \dots P'_h$ on $A'B'$ (distinct)

If t_i has multiplicity s_i, P'_i will contribute s_i among $\mathcal{C}' \cap A'B'$. Thus $\sum_{i=1}^{h} s_i \leq s$

Consequently, if $\mathcal{C}_n$ has at least 2 distinct tangents in O, the quadratic transformation "splits" O into points P' having lower multiplicity, coming from points P "infinitesimally near to O"

The process can be iterated: P'_i can be chosen as a fundamental pt for another quadratic transformation, and will hence become $P''_{i1}, P''_{i2} \dots P''_{ij}$ with multiplicity $s_{i1} \dots s_{ij}$, $\sum_j s_{ij} < s_i$ coming from infinitesimally near pts P_{ij} (in a "second order neighborhood" of O)

If the singularity in O is of type k, i.e. to pts in the $(k+1)$-order neighbourhood of O there correspond simple points of the transformed curve, the singularity will be removed.

Examples

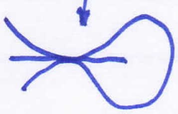
ordinary node   two linear branches

simple pts infinitely nearby

$[2;1,1]$

double pt   simple pts

k-order contact



k blow-ups reduce it to  the last one to 2 simple pts

$[2;\ 2_1;\dots 2_k;\ 1,1]$

$[2;2;1,1]$ tacnode

$[2;2;2;1,1]$ oscnode

superlinear branch

$[2;1]$ cusp of first order

double   simple pt

$[2;2;1]$

in general k double pts are formed before getting a simple pt

$[2_1;\ 2_2;\dots 2_k;\ 1]$

The process of resolution of singularities terminates

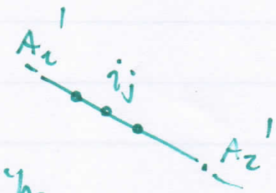
(Bertini) Recall that $\mathscr{C}$, having a single ordinary multiple pt (distinct tangents) in O has genus

$$p = \frac{(n-1)(n-2)}{2} - \frac{r(r-1)}{2}$$

Let us consider a quadratic transformation such that $O \equiv A_3$ is a fundamental point $\mathscr{C}'$, up to the $A_1'A_2'$ line, counted r times, will be of order $2n - r$, and goes through A_3' with multiplicity n and through A_1' and A_2' with multiplicity $n-r$

Then $\underset{\text{genus of } \mathscr{C}'}{p'} = \frac{(2n-r-1)(2n-r-2)}{2} - \frac{2(n-r)(n-r-1)}{2} - \frac{n(n-1)}{2}$

$= \ldots p$ (to be expected...)



Now, if O is not ordinary, one finds multiple pt. on $A_1'A_2'$, with multiplicity $i_1 \ldots$ Then, recomputing the genus yields

$$\bar{p} = p - \sum_j \frac{i_j(i_j-1)}{2} \qquad \text{thus } 0 < \bar{p} < p$$

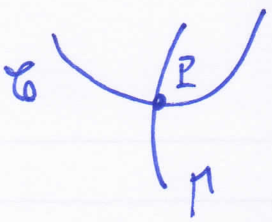
Then one may iterate the procedure, which cannot proceed indefinitely since $p \geq 0$

→ Genus of a curve with non ordinary multiple pts

$$p = \frac{(n-1)(n-2)}{2} - \sum_j \frac{i_j(i_j-1)}{2}$$

⇒ genus invariance persists!

Noether's formula

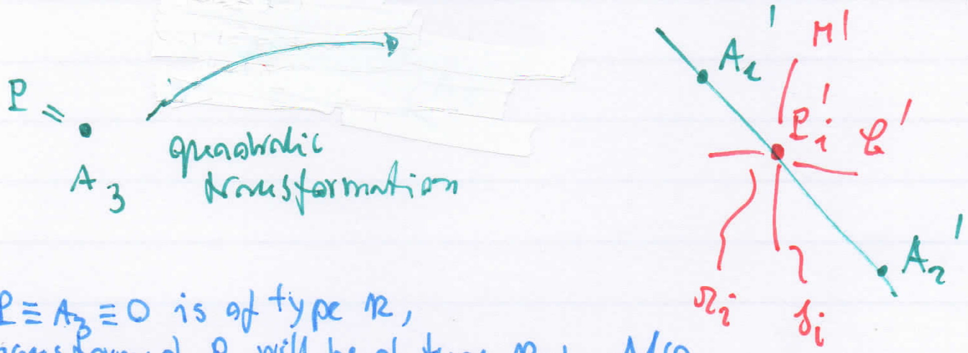


P not ordinary (principal tangents in common)

r-ple for $\mathcal{C}$, s-ple for Γ

N = # intersections absorbed in P

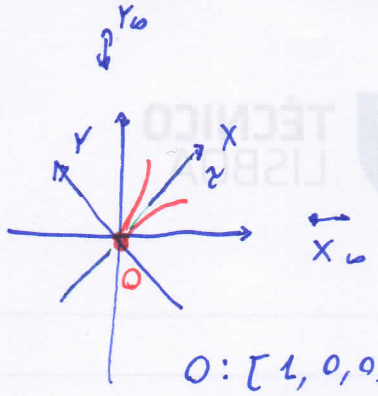
$$\underbrace{N = r\cdot s + \sum r_i s_i + \sum r_{ij} s_{ij} + \dots}_{\text{finite sum}}$$



If $P \equiv A_3 \equiv O$ is of type r, its transformed P will be of type $r-1$. Also, the singularities created in the fundamental pts are generically <u>ordinary</u>. Therefore (Noether), via sequences of quadratic transformations, every plane curve can be transformed in one having <u>only</u> <u>ordinary singularities</u>

Example : The first order cusp $\mathcal{C}_3$

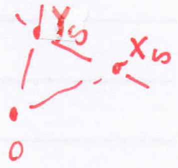
$$Y^2 - X^3 = 0$$



$$\begin{cases} Y = x - y \\ X = x + y \end{cases}$$

$O : [1,0,0]$
$X_\infty : [0,1,0]$
$Y_\infty : [0,0,1]$

$X_\infty \notin \mathcal{C}_3$
$Y_\infty \notin \mathcal{C}_3$



$$(x-y)^2 - (x+y)^3 = 0$$

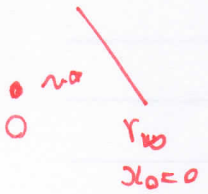
Homogenize: $(x_1 - x_2)^2 x_0 - (x_1 + x_2)^3 = 0$

Blow-up at O:
$$x_0 \longmapsto x_1 x_2$$
$$x_1 \longmapsto x_2 x_0 \qquad \rightsquigarrow$$
$$x_2 \longmapsto x_0 x_1$$

$$(x_2x_0 - x_0x_1)^2 x_1x_2 - (x_2x_0 + x_0x_1)^3 = 0$$

$$x_0^2 (x_2 - x_1) x_1 x_2 - x_0^3 (x_2 + x_1)^3 = 0$$

$$x_0^2 \Big[\underbrace{\overbrace{(x_2 - x_1)x_1x_2 - x_0(x_1+x_2)^3}^{f}}_{C_4} \Big] = 0$$



$\mathcal{C}_4 \cap r_\infty$: $\qquad \begin{cases} (x_2 - x_1)x_1x_2 = 0 \\ x_0 = 0 \end{cases}$

($r_\infty = X_\infty Y_\infty$)

$\Rightarrow P : [0,1,1]$ & X_∞, Y_∞

(P corresp. to z)

$$\frac{\partial f}{\partial x_0}(P) = -(x_1 + x_2)^3 \Big|_P$$

$$= -8 \neq 0 \quad \Rightarrow \quad P \text{ is a simple pt}$$

Singularity: $[2; 1]$

double pt — simple pt

✱4 Cremona's formulae

Cremona transformation

$$\ell \xleftrightarrow[\text{projectivity}]{} \mathcal{C}_\ell^n \quad \text{homaloidal net}$$

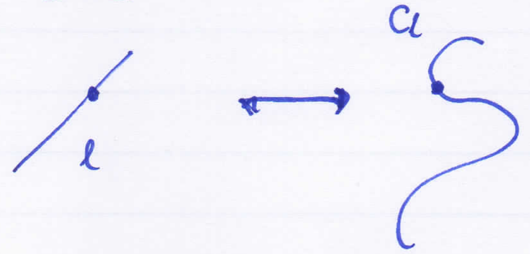
$\cap$

$\mathbb{P}_2$

ν base pts with multiplicities
$A_1 \ldots A_\nu$ $\qquad r_1 \ldots r_\nu$

Then $\left[\, n^2 - \sum_{i=1}^{\nu} r_i^2 = 1 \,\right]$ (homaloidal net condition)
degree $= 1$

Notice that



$\Rightarrow$ $\mathcal{C}_\ell^n$ is rational $\Rightarrow$ it has the maximal number of double points (falling in the base pts)

Hence

$$\frac{\overbrace{(n-1)(n-2)}^{n^2-3n+2}}{2} - \sum_i \frac{r_i(r_i-1)}{2} = 0$$

i.e.

$$n(n-3) - \sum_i r_i(r_i-1) = -2, \quad \text{and}$$

$$\underbrace{n(n-3)}_{n^2-3n} - \underbrace{\sum r_i^2}_{= n^2-1} + \sum r_i = -2$$

$$\Rightarrow \qquad -3n + 3 = -\sum_i r_i \qquad \Rightarrow$$

✱4 Cremona's formulae

$$\boxed{\begin{cases} \sum_{i=1}^{\nu} r_i^2 = n^2 - 1 \\ \sum_{i=1}^{\nu} r_i = 3n - 3 \end{cases}}$$

444 Theorem (Clifford - Noether - Rosanes)

Every Cremona transformation can be expressed as a product of quadratic transformations

Lemma: Let $r_1 \geq r_2 \geq \dots \geq r_\nu$. Then $\sum r_i \geq n+1$

in particular $r_1 + r_2 + r_3 \geq n+1$

[Let us assume that there are no multiple pts infinitely near to the base pts]

Cremona: $r_1^2 + r_2^2 + \dots r_\nu^2 = n^2 - 1$

$r_1 + r_2 + \dots r_\nu = 3n - 3$

From $r_3 \geq r_4 \geq \dots$ we have.

$$r_3 (r_3 + r_4 + \dots r_\nu) = r_3^2 + r_3 r_4 + \dots r_3 r_\nu \geq r_3^2 + r_4^2 + \dots r_\nu^2$$

$$\Rightarrow r_1^2 + r_2^2 + r_3 (\underbrace{r_3 + r_4 + \dots r_\nu}_{\substack{\| \\ 3n-3-r_1-r_2}}) \geq n^2 - 1$$

Then $r_1^2 + r_2^2 + r_3 (3n - 3 - r_1 - r_2) \geq n^2 - 1$

$r_1 (r_1 - r_3) + r_2 (r_2 - r_3) + 3(n-1) r_3 \geq n^2 - 1$

A curve of the net (irreducible) cannot have a multiplicity $\geq n$, so $r_1 \leq n-1$, $r_2 \leq n-1$. Hence $n > 1$

$$(n-1)(r_1 - r_3) + (n-1)(r_2 - r_3) + 3(n-1) r_3 \geq \underset{\substack{\| \\ (n-1)(n+1)}}{n^2 - 1}$$

$$\Downarrow$$

$r_1 - r_3 + r_2 - r_3 + 3 r_3 \geq n+1$ i.e

$$\boxed{r_1 + r_2 + r_3 \geq n+1}$$

The lemma is proved

Let $\lambda_1\varphi_1 + \lambda_2\varphi_2 + \lambda_3\varphi_3 = 0$ a homaloidal net

Perform a quadratic transformation with fundam pts coinciding with the f. pts A_1, A_2, A_3 $r_1 \; r_2 \; r_3$

Then, to φ there will correspond φ' of order

$$\boxed{2n - (r_1 + r_2 + r_3) < n}$$

$r_1 + r_2 + r_3 \geq n+1$ by the preceding lemma

In the same vein, iterating the procedure we can transform our net in a net of conics

From the above theorem, we again find the invariance of the genus p under a general Cremona transformation (since it is true for quadratic transformations) [Recall, however, that p is even a topological invariant...]

Facts:

See Santoro Istituzioni di geometria superiore

irreducible conics, cubics, quartics, quintics with zero genus can be Cremona transformed into lines

This is false for $n > 5$