** Puiseux series

$f(x,y)=0$ polynomial equation

try to solve $y=y(x)$ $f(x,y(x))=0$

(cf. Dini)

Ansatz: $y = t x^{\mu}$, $t \in \mathbb{C}$, $\mu = \frac{p}{q}$

$$f(x,y) = \sum_{\alpha+\mu\beta=\nu} a_{\alpha\beta}\, x^{\alpha} y^{\beta}$$

quasi-homogeneous polynomial

$$f(x, t x^{\mu}) = \sum a_{\alpha\beta}\, x^{\overbrace{\alpha+\mu\beta}^{\nu}} \cdot t^{\beta} = x^{\nu} \underbrace{\sum a_{\alpha\beta} t^{\beta}}_{g(t)}$$

$$f(t^{w_1}x_1, \dots t^{w_k}x_k) = t^{n} f(x_1 \dots x_k)$$

$$f = \sum a_{i_1\dots i_k}\, x_1^{i_1} \dots x_k^{i_k}, \qquad \sum_{j=1}^{k} w_j i_j = n$$

If t_0 is a root of g, $g(t_0)=0$, then $y = t_0 x^{\mu}$ solves $f(x,y)=0$

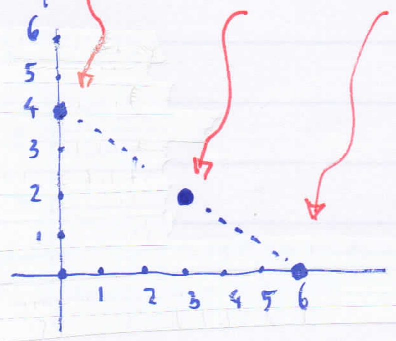
$t_0 \neq 0$ exists only when f contains at least two distinct monomials

** Geometric interpretation (Newton) of $\alpha + \mu\beta = \gamma$

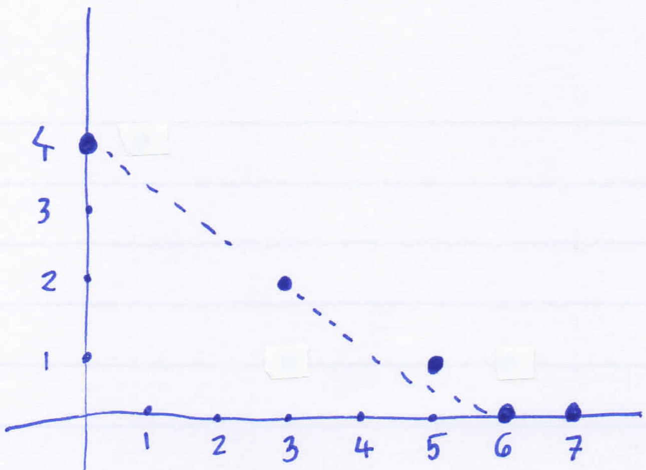
$\rightsquigarrow (\alpha,\beta) \in \mathbb{R}^2$ $f(x,y) = \sum a_{\alpha\beta} x^{\alpha} y^{\beta}$

Carrier of f : $\{(\alpha,\beta) \,/\, a_{\alpha\beta} \neq 0\} =: \Delta(f)$

ex: $f(x,y) = y^4 - 2x^3y^2 + x^6$



$$f(x,y) = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$$



Existence of μ and ν such that

$\alpha + \mu\beta = \nu$ for all $(\alpha, \beta) \in \Delta(f)$

means: (α, β) all lie on a straight line

slope: $-\frac{1}{\mu}$ $\quad$ $\beta = 0$ for $\alpha = \nu$

Example 1: $f = y^4 - 2x^3y^2 + x^6$: $\mu = \frac{3}{2}$ $\quad \nu = 6$

$$\alpha + \frac{3}{2}\beta = 6$$

$(0,4)$ ✓
$(3,2)$ ✓
$(6,0)$ ✓

so $y = t x^{3/2}$

Substituting into f gives

$$f(x, t x^{3/2}) = x^6 (\underbrace{t^4 - 2t^2 + 1}_{(t^2-1)^2}) \quad \text{so } t = \pm 1$$

$$\rightsquigarrow \quad y = \pm x^{3/2}$$

Example 2 $\quad$ The points in $\Delta(f)$ do not all lie on a single line.

write $f = \hat{f} + h$ $\sim$ $h = -4x^5y - x^7$

$\hat{f} = y^4 - 2x^3y^2 + x^6$

h of "higher order"

⚠ order: $\alpha + \mu\beta$

x^5y : $5 + \frac{3}{2} = \frac{13}{2}$ $\qquad$ x^7 : $7 + \frac{3}{2}\cdot 0 = 7$

Write then

$$y = x^{3/2} + \text{higher order terms}$$

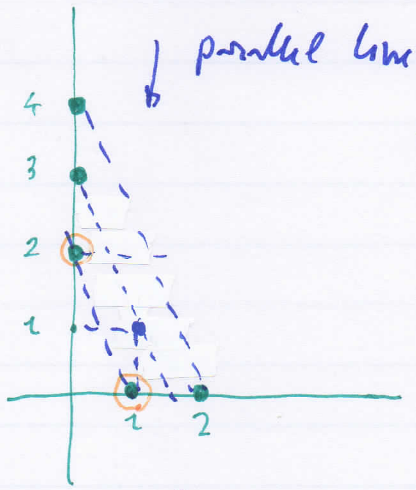
$$y = x^{3/2}(1+y_1) \qquad x^{1/2} \equiv x_1$$

$$\begin{cases} y = x_1^3(1+y_1) \\ x = x_1^2 \end{cases}$$

$$f(x,\ ) = x_1^{12} f_1(x_1, y_1)$$

$$= y_1^4 + 4y_1^3 + 4y_1^2 - 4x_1 y_1 - 4x_1 - x_1^2$$

Compute $\Delta(f_1)$:



parallel lines with slope $-2 = -\frac{1}{\mu}$ $\quad \mu = \frac{1}{2}$

try then $y_1 = t_1 x_1^{\frac{1}{2}}$

substitute this into the lower order terms

$$y_1^4 + 4y_1^3 + 4y_1^2 - 4x_1 y_1 - 4x_1 - x_1^2$$

$$4y_1^2 - 4x_1 = 4t_1^2 x_1 - 4x_1 = 4(t^2-1)x_1 = 0$$

$\Rightarrow t = \pm 1$. If $t = +1$, then

$$f_1(x_1, x_1^{\frac{1}{2}}) = \dots = 0$$

$y_1 = x_1^{\frac{1}{2}}$ solves $f_1(x_1, y_1) = 0$

$$\Rightarrow y = x^{3/2}(1+y_1) = x^{3/2}(1+x^{\frac{1}{4}}) = x^{3/2} + x^{7/4}$$

solves $f(x,y) = 0$

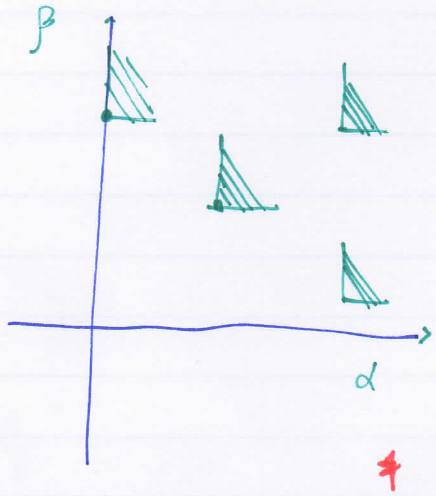
★ The Newton process in general

$$f(x,y) = \sum a_{\alpha\beta}\, x^{\alpha} y^{\beta}$$ convergent power series

y-general (of order $m > 0$)

$$a_{0m} \neq 0 \quad a_{0i} = 0 \quad i < m$$

$$\Delta(f) = \{ (\alpha, \beta) \in \mathbb{N}^2 \mid a_{\alpha\beta} \neq 0 \}$$ carrier of f

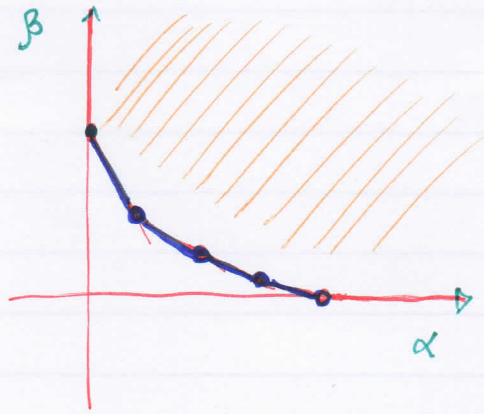


Take the convex hull of the displaced quadrants

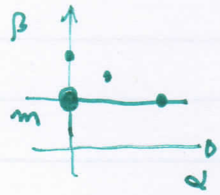
$$\operatorname{conv}\Big(\bigcup_{p \in \Delta(f)} \big(p + (\mathbb{R}^+)^2\big) \Big)$$

$\parallel\!\!\mid$

★ Newton polygon



If



$$f = y^m \tilde{g}(x,y) \qquad \tilde{g}(0,0) \neq 0$$

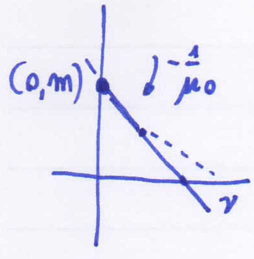
$$\Rightarrow \quad y = 0 \ \text{ solves } \ f = 0$$

Otherwise start with the line having steepest slope μ_0

and partition f in the following manner

(f y-general of order m $\quad a_{0i} = 0 \quad i < m, \ a_{0m} \neq 0$)

$$f = \overbrace{\sum_{\underset{\text{at least two terms}}{\alpha+\mu_0\beta=\nu}} a_{\alpha\beta}x^\alpha y^\beta}^{\tilde f} + \sum_{\alpha+\mu_0\beta>\nu} a_{\alpha\beta}x^\alpha y^\beta$$



$\nu = m\mu_0$

Solve $\tilde f = \sum_{\alpha+\mu_0\beta=\nu} a_{\alpha\beta}x^\alpha y^\beta$, $\nu = m\mu_0$

Ansatz: $y = t x^{\mu_0}$

$$\tilde f(x,y) = x^\nu \Big(\underbrace{\sum_{\alpha+\mu_0\beta=\underbrace{\mu_0 m}_{\nu}} a_{\alpha\beta}t^\beta}_{g(t)}\Big) = x^\nu g(t)$$

At least two coefficients in g are non-zero, hence g has a nonzero root t_0 $\longrightarrow$ $y_0 = t_0 x^{\mu_0}$ first approximate solution

$\mu_0 = \frac{p_0}{q_0}$, $q_0 \neq 0$ $\quad (p_0, q_0) = 1$ (relatively prime)

Set $x_1 = x^{\frac{1}{q_0}}$ $\qquad \boxed{y_0 = t_0 x_1^{p_0}}$

Improve this solution.

$y = x_1^{p_0}(t_0 + y_1) \longrightarrow$ substitute, getting

$f(x_1^{q_0}, x_1^{p_0}(t_0+y_1))$ a power series in x_1, y_1,

having $x_1^{\nu q_0}$ as a factor

$f(\ ,\ ) = x_1^{\nu q_0}\cdot f_1(x_1,y_1)$ $\qquad f_1$ y-general of order $m_1 \le m$

$x_1^{\nu q_0} = x^{\nu\frac{1}{q_0}q_0} = x^\nu = x^{m\mu_0}$

Then construct the Newton polygon for f_1,

with $-\frac{1}{\mu_1}$ its steepest negative slope $\left(\mu_1 = \frac{p_1}{q_1}\right)$,

find $y_1 = t_1 x_1^{\mu_1}$. Then put $x_2 := x_1^{\frac{1}{q_1}}$

and $y_1 = x_2^{p_1}(t_1 + y_2)$, substituted into $f_1(x_1, y_1) = 0$

Pull out all powers of x_2

$$f_1(\ ,\ ) = x_2^{\nu_1 q_1} f_2(x_2, y_2) \qquad f_2 \quad y\text{-general of order } m_2 \leq m_1$$

Upshot: a sequence of convergent

power series $f_i(x_i, y_i)$ $x_{i+1} = x^{1/q_i}$.

f_i y-general of order m_i, and $m = m_0 \geq m_1 \geq m_2 \ldots$

and approximate solutions

$$y = x^{\mu_0}(t_0 + y_1)$$
$$y_1 = x_1^{\mu_1}(t_1 + y_2)$$
$$y_2 = x_2^{\mu_2}(t_2 + y_3)$$
$$\vdots$$

Therefore:

$$y = x^{\mu_0}(t_0 + x_1^{\mu_1}(t_1 + x_2^{\mu_2}(t_2 + \ldots)) =$$

$$t_0 x^{\mu_0} + t_1 x^{\mu_0} x_1^{\mu_1} + t_2 x^{\mu_0} x_1^{\mu_1} x_2^{\mu_2} + \ldots$$

$$= t_0 x^{\mu_0} + t_1 x^{\mu_0 + \mu_1/q_0} + t_2 x^{\mu_0 + \mu_1/q_0 + \mu_2/q_0 q_1} + \ldots$$

ascending fractional powers of x

Now, the denominators do not increase indefinitely (and the series converges)

If the process breaks off giving $y_i = 0$

(Newton polygon reduced to a single pt)

The series is a polynomial in $x^{\frac{1}{n}}$ for some n, satisfying $f(x,y)=0$ (★)

general case $\boxed{\exists\ i_0 \text{ such that } \mu_i \in \mathbb{Z} \text{ for } i \geq i_0}$.

Thus $q_i = 1$ and $x_{i+1} = x_i$ for $i \geq i_0$. Then

$\boxed{n := q_0 q_1 \cdots q_{i_0}}$ yields a common denominator for all exponents, namely y is a power series in $x^{\frac{1}{n}}$

$y = \sum a_i x^{i/n}$ and $f(x, y(x)) \equiv 0$ Puiseux series ★★

(a formal solution)

Let us check that if $m_i = m_{i+1}$, then $\mu_i \in \mathbb{N}$

(Thus, if $\mu_i \notin \mathbb{N}$, $m_i > m_{i+1}$; so $\mu_i \in \mathbb{N}$ starting from some i_0)

Take $i=0$ for simplicity and w.l.o.g.

$x = x_1^{q_0} \quad y = x_1^{p_0}(t_0 + y_1) \longrightarrow$

$$x_1^{\nu q_0} f_1(x_1, y_1) = f(x_1^{q_0}, x_1^{p_0}(t_0+y_1))$$

$$= x_1^{\nu q_0} \Big(\underbrace{\sum_{\alpha+\mu_0\beta = \mu_0 m} a_{\alpha\beta} (t_0+y_1)^\beta}_{(*)} + x_1(\ ) \Big)$$

$\Rightarrow f_1(0, y_1) = (*) = g(t_0 + y_1)$

to solve $g(t)=0$ g is a polynomial of degree $m = m_0$

m_1 = order of t_0 (= order of the zero y_1 in $f(0,y)=0$)

But $m_1 = m_0 \Rightarrow g(t) = c(t-t_0)^m$

$\Rightarrow$ the coefficient $\underset{\wr}{a_{\alpha, m-1}} \neq 0$ for some $\alpha \in \mathbb{N}$ s.t.

$$g(t) = \sum_{\alpha + \mu_0 \beta = \mu_0 m} a_{\alpha\beta} t^{\beta}$$

$\alpha + \mu_0 (m-1) = \mu_0 m$

hence $\mu_0 = \alpha \in \mathbb{N}$

set $z = x^{\frac{1}{n}}$

$\rightsquigarrow \quad y = y(z) = \sum_i a_i z^i \qquad f(z^n, y(z)) = 0$

Theorem. Let $f(x,y) \in \mathbb{C}\{x,y\}$ y-general of order $m > 0$ and irreducible. Then $\exists\, \varepsilon_0 > 0$ such that for each $0 < \varepsilon < \varepsilon_0$, $\exists\, \delta_\varepsilon > 0$ such that, setting

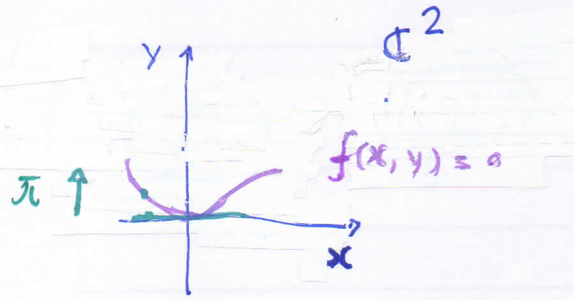
$$X = \{ (x,y) \in \underbrace{\overline{U}_{\varepsilon,\delta}}_{\text{polydisc}} \mid f(x,y) = 0 \}$$

there exists $y(z) \in \mathbb{C}\{z\}$ for which

$\pi : B \to \mathbb{C}^2 \qquad B = \{ z \in \mathbb{C} \mid |z| < \delta^{\frac{1}{m}} \}$

$\pi(z) = (z^m, y(z))$ is holomorphic and onto X — uniformizing parameter

$\pi : B \setminus \{0\} \to X \setminus \{0\}$ biholomorphic and $\pi^{-1}(0) = 0$



Actually

X = the set of

$$f(x,y) = \prod_{\varepsilon^m = 1} (y - \underbrace{y(\varepsilon x^{\frac{1}{m}})}_{\tilde{y}_i})$$

$$\begin{array}{ccc} z & & x^{\frac{1}{m}} \\ \downarrow & & \updownarrow \\ z^m & & x \end{array}$$

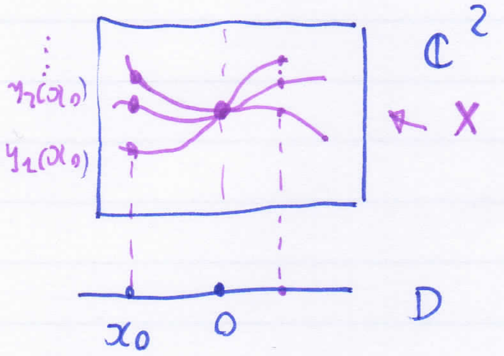
$f = y^m + c_1(x) y^{m-1} + \dots$ in a neighbourhood of 0 Weierstrass $c_i(0) = 0$

remember... $f(x_0, y) = 0$ in $|x_0| < \delta$ has exactly m zeroes in $\overline{U}_{\varepsilon,\delta}$ (by Weierstrass preparation theorem)

The y_i are the branches of the genuine holomorphic function y of the uniformizing parameter z

Amplification

branched covering



fixed choice of the m-root

$$y_i(x_0) = y(\varepsilon^i x_0^{\frac{1}{m}})$$

$$\varepsilon^m = 1$$

$$(x_0 \cdot e^{2\pi i})^{\frac{1}{m}} = \underset{\text{fixed}}{x_0^{\frac{1}{m}}} \cdot \underbrace{e^{\frac{2\pi i}{m}}}_{\varepsilon}$$

Let $x = x(t) = e^{2\pi i t}$

$$x^{\frac{1}{m}} = e^{\frac{2\pi i t}{m}} \qquad t = 1$$

$$e^{\frac{2\pi i}{m}} \equiv \varepsilon$$

<u>Example</u> $X: \; y^2 - x^3 = 0$

(typical) $\pi(z) = (z^2, z^3)$

$$x = z^2 \qquad x^3 = z^6$$
$$y = z^3 \qquad y^2 = z^6$$

$f(x(t), y) = 0$

$$y_1(t) = e^{2\pi i \frac{3}{2} t} \qquad \begin{matrix} t=0 \\ +1 \end{matrix} \qquad t = 1 \qquad e^{\pi i} = -1$$

$$y_2(t) = -e^{2\pi i \frac{3}{2} t} \qquad \begin{matrix} t=0 \\ -1 \end{matrix} \qquad t=1 \qquad -(-1) = +1$$

The solutions are exchanged

In general get a <u>braid</u>

Artin braid group

★ Puiseux series, Puiseux pairs, toral knots
(outline)

Consider the Puiseux series $\qquad f(x,y)=0$

$$y = \underset{\neq 0}{c}\, x^{\alpha} + \text{higher} \qquad \alpha \geq 1$$
$$= \sum a_x x^{x} \qquad a_x \neq 0 \quad x \in \mathbb{Q} \quad x \geq 1$$

One can find a standard form

$$\boxed{y = x^{\kappa_1} + x^{\kappa_2} + \dots x^{\kappa_g}} \qquad \kappa_j > 1$$

$\kappa_1 = \dfrac{n_1}{m_1}$ $\qquad j=1\dots g \qquad m_1 < n_1$

$\kappa_2 = \dfrac{n_2}{m_1 m_2}$ $\qquad n_{j-1} m_j < n_j \quad j \geq 2$

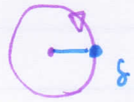
$\vdots$

$\kappa_g = \dfrac{n_g}{m_1 \cdots m_g}$ $\qquad \gcd(n_j, m_j) = 1$

$(m_1, n_1), \dots (m_g, n_g)$ $\qquad j = 1, 2, \dots, g$

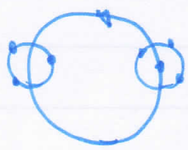
★ Puiseux pairs of f

$$x(t) = \underset{\text{small}}{\delta}\, e^{2\pi i t}$$



$y_k = \delta^{\kappa_1} e^{2\pi i \kappa_1 \cdot k} \qquad k = 1 \dots m_1 \qquad$ initial points

$x = \delta \qquad \rightsquigarrow$ braid



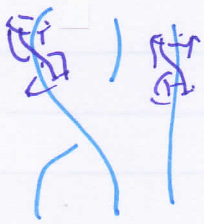
m_2 pts on circles of radii $\delta^{\kappa_2} < \delta^{\kappa_1}$

et cetera (Hipparcus & Ptolemy epicycles ★★★

get

Cabled braids

closing the braids leads to sequences of torus knots



★ **Why knots?** (The closure of a braid is a link, in general)

In $\mathbb{C}^2 \cong \mathbb{R}^4 \supset S^3$ $\qquad f(x,y)=0$

$X := \{f=0\}$

polydisc $\bar{U}_{\varepsilon,\delta} = \{(z,w) \mid |z|<\varepsilon,\ |w|<\delta\}$

its boundary $\partial\bar{U}$: union of two solid tori, meeting on a torus $\{|z|=\varepsilon\}\times\{|w|=\delta\} = \Pi$

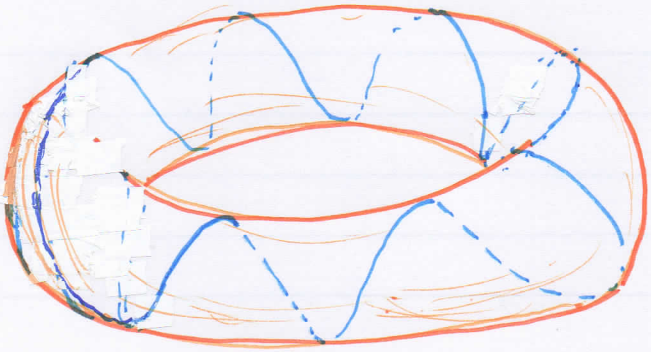
★★ $\partial\bar{U}$ is homeomorphic to a sphere S^3 (†) $\quad |z|^2+|w|^2=\varepsilon^2+\delta^2$

$\dim = \dim_{\mathbb{R}}$ $\qquad \underbrace{\dim(X\cap\Pi)}_{x} + \underset{\dim S^3}{\overset{\shortparallel}{3}} = \underset{2}{\overset{\shortparallel}{\dim X}} + \underset{2}{\overset{\shortparallel}{\dim \Pi}}$

$\Rightarrow x = \dim(X\cap\Pi) = 1$

$\Rightarrow X\cap\Pi$ a collection of circles (a link)

If the singularity is irreducible, $X\cap\Pi$ is a knot (toral knot)



Any line goes to another line' after some circling around 0 $\Rightarrow$ get a knot

Also:

$\underbrace{\dim(X\cap \overset{\partial\bar{U}}{\overset{\shortparallel}{S^3}})}_{y} + \underbrace{\dim \mathbb{C}^2}_{4} = \underbrace{\dim X}_{2} + \underbrace{\dim \overset{\partial\bar{U}}{\overset{\shortparallel}{S^3}}}_{3}$

$y=1 \qquad X\cap S^3 = X\cap\Pi$

(†) $S^3 = \Pi^+ \cup \Pi^- /\sim$ $\quad$ glueing on the common boundary torus Π

solid tori

- **Example** $\quad x^n - y^m = 0 \qquad (m,n)=1, \quad m<n$

★ Puiseux: $\quad y = x^{\frac{n}{m}}$

★ braid: $\quad y_k(t) = \eta \cdot e^{\frac{2\pi i}{m}(k+t)n} \qquad 0 \leq t \leq 1 \qquad \eta = \delta^{n/m}$, $\delta > 0$ small

★ knot: $\quad t \longmapsto (\delta t^m, \eta \cdot t^n) \in S^1 \times S^1 \cong$ $\{|x| = \delta, \ |y| = \eta\}$, $\quad t \in S^1$, $|t| = 1$

closing the above braid

$(\delta t^m)^n = \delta^n t^{mn}$

$(\eta t^n)^m = \eta^m t^{nm}$

In general one forms iterated torus knots ("cable knots")