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ALGEBRAIC GEOMETRY & RIEMANN SURFACES

Lecture XXXVIII

Algebraic miscellanea

k field (k algebraically closed, actually $k = \mathbb{C}$)

$\mathbb{A}^n = \mathbb{A}^n(k)$ affine space $y_1 \dots y_n$ coordinates

$\mathfrak{a}$: ideal in $k[y_1 \dots y_n]$ polynomials

$$V(\mathfrak{a}) := \{ y \in \mathbb{A}^n \mid f(y) = 0 \ \forall f \in \mathfrak{a} \}$$

common zeroes of polynomials in $\mathfrak{a}$

$$= \{ y \in \mathbb{A}^n \mid f_j(y) = 0, \ j = 1 \dots m \}$$

($\mathfrak{a}$ is finitely generated)

$X = V(\mathfrak{a})$: algebraic set (affine alg. set)

X irreducible : $\nexists \ X = X_1 \cup X_2 \qquad X_i \subsetneq X$

Properties $\qquad A = k[Y_1 \dots Y_n]$

(i) $V((0)) = \mathbb{A}^n \qquad V(A) = \emptyset$

(ii) $\mathfrak{a} \subset \mathfrak{b} \Rightarrow V(\mathfrak{b}) \subset V(\mathfrak{a})$

(iii) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$

Proof of $\subset$. Assume $x \in V(\mathfrak{a} \cap \mathfrak{b})$ but $x \notin V(\mathfrak{a}) \cup V(\mathfrak{b})$

$\exists f \in \mathfrak{a}, g \in \mathfrak{b}$ s.t. $f(x) \neq 0$, $g(x) \neq 0$ Then $fg \in \mathfrak{a} \cap \mathfrak{b}$

and $(fg)(x) = f(x) g(x) \neq 0$, contradicting $x \in V(\mathfrak{a} \cap \mathfrak{b})$

($f(x) g(x) = 0$)

iv $\quad V\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcap_{i \in I} V(\mathfrak{a}_i)$

finite sums of elements in the $\mathfrak{a}_i$'s

$\sqrt{\mathfrak{a}} := \{ g \in k[Y_1, Y_2 .. Y_n] \ / \ g^l \in \mathfrak{a} \text{ for some } l \in \mathbb{N} \}$

radical of $\mathfrak{a}$

Then $\bar{V}(\mathfrak{a}) = \bar{V}(\sqrt{\mathfrak{a}})$

Indeed $\mathfrak{a} \subset \sqrt{\mathfrak{a}} \Rightarrow \bar{V}(\sqrt{\mathfrak{a}}) \subset \bar{V}(\mathfrak{a})$. Conversely let $y \in \bar{V}(\mathfrak{a})$, $g \in \sqrt{\mathfrak{a}}$. Then $g^l(y) = g(y)^l = 0$

$\Rightarrow g(y) = 0 \Rightarrow y \in \bar{V}(\sqrt{\mathfrak{a}})$

$X : f = 0 \rightsquigarrow I(X) = (f)$ hypersurface of $\mathbb{A}^n$

principal ideal

(X is irreducible $\Leftrightarrow$ f is irreducible by Hilbert's Nullstellensatz (see below))

* I - correspondence

$I(X) = \{ f \in k[Y_1 .. Y_n] \mid f(x) = 0 \ \forall x \in X \}$

* $I(X)$ is a radical ideal: $I(X) = \sqrt{I(X)} = \{ f \mid f^l \in I(X) \text{ for some } l \in \mathbb{N} \}$

Clearly $I(X) \subset \sqrt{I(X)}$. Then, $f \in \sqrt{I(X)} \Leftrightarrow$

$f^l \in I(X)$ for some $l \in \mathbb{N}$

Therefore $f^l(x) = 0 \quad \forall x \in X$ for $l = 1, 2 ...$

i.e. $f(x) = 0 \quad \forall x \in X$, so $f \in I(X)$

$\Rightarrow \sqrt{I(X)} \subset I(X)$ $\square$

For all $\mathfrak{a}$ ideals in $k[Y_1 .. Y_n]$

strict, in general

$$X \subset V(I(X)) \qquad \mathfrak{a} \subset I(V(\mathfrak{a}))$$

if $X = V(\mathfrak{a})$ algebraic then $X = V(I(X))$

Hilbert's Nullstellensatz

k algebraically closed

(i) $\mathfrak{m}$, maximal ideal of $A = k[Y_1 .. Y_n]$ has the form

$$\mathfrak{m}_P = (Y_1 - a_1, .. Y_n - a_n), \quad P = (a_1 \ldots a_n) \in \mathbb{A}^n(k)$$

i.e. $\mathfrak{m}_P = I(P)$ polynomials vanishing at P

(ii) $\mathfrak{a}$ ideal in A : $\mathfrak{a} \neq A \Rightarrow V(\mathfrak{a}) \neq \phi$

(iii) for any $\mathfrak{a} \subset A$, $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$

Projective variety

$$X \subset \mathbb{P}^n$$

$X = V(\mathfrak{a})$ $\quad$ $\mathfrak{a}$ homogeneous ideal of $k[T_0 \ldots T_N]$

$\mathfrak{a}$ determines X but X does not determine $\mathfrak{a}$

If $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$, then $V(\mathfrak{a}) = V(\mathfrak{b})$

Better, a projective variety is

$(X, \mathfrak{a})$ $\quad$ projective scheme

$X = V(\mathfrak{a})$ $\quad$ cf the (simpler!) classical viewpoint

→ $(X, \mathfrak{a})$ reduced scheme if $\sqrt{\mathfrak{a}} = \mathfrak{a}$

$(X, \mathfrak{a})$ coincides with its support (zero locus) $V(\mathfrak{a})$

Also: projective variety: $(X, k[X])$ ←

$k[X]$ = coordinate ring $k[X_0 \ldots X_n] / I(X)$

rational functions: $k(X)$

quotient field

*** Zariski topology

Affine case $X \subset \mathbb{A}^n$ algebraic sets $X = V(\mathfrak{a})$
$\equiv$ closed sets for the Zariski topology on $\mathbb{A}^n$

base of open sets: $\mathbb{A}^n_f := \mathbb{A}^n \setminus V(f) = \{ y \in \mathbb{A}^n \mid f(y) \neq 0 \}$, $f \in K[Y_1 \ldots Y_n]$

complements of hypersurfaces

principal open sets — principal closed sets

- $\mathbb{A}^n$ is not Hausdorff [Indeed, every U open, $U \neq \emptyset$ is dense]
- $\mathbb{A}^n$ is T_1 (Fréchet): given P, Q, $\exists$ $U_P \ni P$, $U_P \not\ni Q$ and vice versa
- $\mathbb{A}^n$ is compact (from every open covering of $\mathbb{A}^n$ one can extract a finite one)
- If $K = \mathbb{C}$, the Zariski topology is coarser than the usual topology of $\mathbb{C}^n \cong \mathbb{R}^{2n}$

Given X, the Zariski topology on X is the one induced on X from the Z. topology on $\mathbb{A}^n$

$$X_f = X \setminus V(f) = \{ x \in X \mid f(x) \neq 0 \}$$

principal open sets $\longrightarrow$ yield a base for the Z. topology

Correspondences $\mathcal{V}$, $\mathcal{I}$ in the projective case

J : homogeneous ideal of $k[X_0 - X_n]$ $X \subset \mathbb{P}^n$

$\mathcal{V}(J) = \{ x \in \mathbb{P}^n \mid f(x) = 0 \ \forall f \text{ homog. } \in J \}$

$\mathcal{I}(X) = \{ f \in k[X_0 - X_n] \mid f(x) = 0 \ \forall x \in X \}$

$\mathcal{I}(X)$ is homogeneous and $\mathcal{I}(X) = \sqrt{\mathcal{I}(X)}$

(projective) Algebraic set : $X = \mathcal{V}(J)$ J hom.

$\quad\quad = \mathcal{V}(f_1 - f_m)$

projective variety $(X, k[X])$ $k[X] = k[X_0 .. X_n] / \mathcal{I}(X)$

coordinate ring

$X = V(J) = V(\sqrt{J})$

$J \subset \mathcal{I}(V(J)) \ \forall J$ and $V(\mathcal{I}(X)) = X$

X proj algebraic

Irrelevant ideal : $(Y_0 - Y_n)$ $\mathcal{V}(Y_0 - Y_n) = \emptyset$ in $\mathbb{P}^n$!

"standard exception"

Homogeneous Hilbert Nullstellensatz

K algebraically closed. $\forall J$ homogeneous

(i) $\mathcal{V}(J) = \emptyset \iff (Y_0 - Y_n) \subset \sqrt{J}$

(ii) If $\mathcal{V}(J) \neq \emptyset$, then $\mathcal{I}(\mathcal{V}(J)) = \sqrt{J}$

*** Zariski topology (projective case)

Closed sets: $X \subset \mathbb{P}^n$ X algebraic set

base of open sets: $\mathbb{P}^n_f = \mathbb{P}^n \setminus V(f) = \{x \in \mathbb{P}^n / f(x) \neq 0\}$
principal open sets

$\mathbb{P}^n$ can be covered by $n+1$ special open sets

standard affine charts

$$\mathcal{U}_i = \mathbb{P}^n_{x_i} = \{[x_0 \dots x_n] \mid x_i \neq 0\} \quad i = 0, 1, \dots n$$

Zariski on $\mathcal{U}_i$: induced topology

Let $X \subset \mathbb{P}^n$ algebraic, $I(X)$ its associated homogeneous ideal. Let X & $x_i = 0$ $i = 0 \dots n$

$X_{(i)} := X \cap \mathcal{U}_i$ ← affine standard charts of X

* Generic objects
$\{X_p\}_{p \in \mathcal{P}}$ * irreducible algebraic set

"The generic X_p has property P" ≡ the p's such that X_p has property P constitute a nonempty open set

4 Field of fractions of an integral domain

A integral domain

$S := A \setminus \{0\}$ localize:

$\mathrm{Frac}(A) = A \times S / \sim$

$(a, s) \sim (a', s') \iff a s' = a' s$ $\left(" \frac{a}{s} = \frac{a'}{s'} " \right)$

$\mathrm{Frac}(A) = \left\{ \frac{a}{b} , a, b \in A , b \neq 0 ; \frac{a}{b} = \frac{a'}{b'} \iff a b' = a' b \right\}$

Concretely

$X \subset \mathbb{A}^n$ algebraic set (equipped with the Zariski topology)

$K(X) = \left\{ \frac{g}{h} , g, h \in K[X] / h \neq 0 \quad \frac{g}{h} = \frac{g'}{h'} \iff g h' = g' h \right\}$

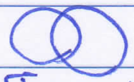
$f \in K(X)$: rational function

$U \subset X$ open ; $P \in U$ $f \in K(X)$ regular at P

if $\exists\, U_P \ni P$ s.t.

$f = \frac{g}{h} , \ h(x) \neq 0 \ \forall x \in U_P$ local representation

f regular on U : f regular at each $x \in U$



U_p $U_{p'}$

$f = \frac{g}{h} = \frac{g'}{h'}$ $\mathrm{dom}(f) = \{ \text{regular pts} \}$

$\mathrm{dom}(f)$ is open dense in X and

$\mathrm{dom}(f) = X$ if and only if $f \in K[X]$ (f is a polynomial)

Field of fractions

quotient field of $K[X]$

X irreducible algebraic set $(\subset \mathbb{P}^n)$
$I(X)$ associated to X

$f = F/_{I(X)}$

homogeneous

f rational function $f : X \to K$

$$f(x) = \frac{g(x)}{h(x)} \quad x \in X$$

g, h homogeneous, having the same degree

$$\boxed{K(X) := \left\{ \frac{g}{h} \;\middle|\; \begin{array}{l} g, h \in K[X_0 \ldots X_n] \text{ hom, same degree} \\ h \notin I(X) \end{array} \right\} / \sim}$$

$$\sim : \quad \frac{g}{h} \sim \frac{g'}{h'} \iff gh' - g'h \in I(X)$$

$\mathrm{dom}(f) = \{ x \in X \mid f \text{ regular at } x \}$

$f = \frac{g}{h} \quad h(x) \neq 0$

$$\mathcal{O}_{X,x} = \{ f \in K(X) \mid f \text{ regular at } x \}$$

local ring of X at x

* **Rational maps** $\varphi : X \to \mathbb{A}^m$
$\mathbb{P}^m$

$$\varphi(x) := (f_1(x) \ldots f_m(x)) \quad x \in X$$

$$f_i \in K(X)$$

well defined in $\bigcap_{j=1}^{m} \mathrm{dom}\, f_i$

$$\varphi(x) := [f_0(x), \ldots f_m(x)] \quad x \in X$$

well defined on the open dense set

$$\bigcap_{i=1}^{m+1} \mathrm{dom}(f_i) - \{x \in X \mid f_0(x) = \ldots f_m(x) = 0\}$$

if $g \neq 0$, then $g f_i$ $i = 0, \ldots m$ yield the same map

* **regular rational map** $\varphi : X \to \mathbb{P}^m$ at $x \in X$

if there exists $\varphi = (f_0 — f_m)$ $f_i \in K(X)$ such that

(i) f_j, $j = 0, \ldots, m$ is regular at x
(ii) $f_j(x) \neq 0$ for some j

The set whereon φ is regular is termed domain of φ ($\mathrm{dom}(\varphi)$)
(it is an open set)

Let $\varphi : X \to W \subset \mathbb{P}^n$ (algebraic set) φ is called dominant (+Zariski)
if $\varphi(\mathrm{dom}(\varphi))$ is dense$^+$ in $\bar{W}$
i.e. $\bar{W} = \overline{\varphi(\mathrm{dom}(\varphi))}$ ← closure in Zariski's topology

* **Morphisms**

$U \subseteq X$ open in X (proj. variety)

$\varphi : U \to W$ morphism

rational map $\varphi : X \to \bar{W}$ with $U \subset dom(\varphi)$
i.e a rational map regular throughout $\bar{U}$.

$\varphi : X \to \bar{W}$ birational isomorphism
(or transformation)

[X and $\bar{W}$ are birationally equivalent]

X, W projective varieties

if $\exists\ \psi : W \to X$ inverting φ :

$$\varphi \circ \psi = id_W \quad , \quad \psi \circ \varphi = id_X$$

Then the following assertions are equivalent :

(i) φ is a birational equivalence

$\varphi : X \to W$

(ii) φ is dominant and

φ^* : $K(W) \longrightarrow K(X)$ is an isomorphism.
pull-back

(iii) $\exists\ X_0 \subset X$, $W_0 \subset W$ (open sets) s.t.

$$\varphi|_{X_0} \xrightarrow{\sim} \bar{W}_0$$

Given a projective (or affine) variety, the following are equivalent

(i) $K(X)$ is a purely transcendental extension of K : $K(X) \cong K(t_1 \dots t_d)$ for some d

(ii) $\exists\ X_0 \subset X$ open dense isomorphic to $U_0 \subset \mathbb{A}^d$ open dense

Such a variety is called ⚡ rational

(ii) tells us that X can be parametrized by d independent variables

★★★ Every projective variety is birationally equivalent to a hypersurface

In particular, every curve (in any $\mathbb{P}^r$) is birationally equivalent to a plane curve

** dimension of $X \subset \mathbb{P}^n$

projective variety (irreducible)

$$\dim(X) = t\text{-}deg_K\, K(X) \leftarrow \text{coordinate ring of } X$$

transcendence degree

field of fractions of (rational functions) $\longrightarrow K[Y_0 \dots Y_n] \,/\, I(X) \equiv K[X]$

$X = V(I(X))$

One can prove that

$\dim X = r \in \mathbb{N}$

such that $S_{n-r} \subset \mathbb{P}^n$, generic, meets X in a finite number of points

Examples • plane curve $\mathcal{C}$ (irreducible) $n = 2$: a generic line ℓ (S_1, $1 = 2-1$) meets $\mathcal{C}$ in a finite number of pts

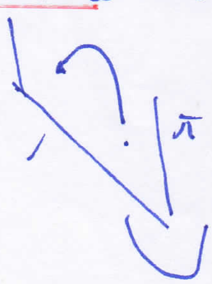
$\dim \mathcal{C} = 1$

• irreducible surface Σ in $\mathbb{P}^3$ $\quad n = 3 \quad \dim \Sigma = 2$

$3 - 2 = 1$ $\quad S_1$ is a line ℓ,

$n \quad r$

• irreducible curve $\mathcal{C}$ in $\mathbb{P}^3$ $\quad \dim \mathcal{C} = 1$



a generic plane π meets $\mathcal{C}$ in a finite # of pts

$n = 3$
$r = 1$
$n - r = 2$

★ degree (order) of a projective r-dimensional variety $X \subset \mathbb{P}^n$

$$\boxed{\deg(X) = \#\text{ points of } X \text{ in common with a generic } S_{n-r}}$$

$\shortparallel$

$\mathbb{P}^{n-r}$

Examples • $n=2$ X plane curve : $r=1$

$\deg X = \#(X \cap \ell)$ ℓ: generic line

• $n=3$ X space curve

$\deg X = \#(X \cap \pi)$ π : generic plane

$\mathcal{C}$ (plane) rational curve $K(\mathcal{C}) = K(\mathbb{P}^1)$

$g=0$

$\dim \mathcal{C} = t\text{-}\deg K(\mathbb{P}^1) = 1$

$\mathcal{C}$ elliptic curve $K(\mathcal{C}) = K(\mathbb{P})(\mathbb{P}')$

$\shortparallel$

transcendental extension of $k=\mathbb{C}$

algebraic extension of $K(\mathbb{P})$

$\dim \mathcal{C} = 1$

★ Tangent space to $X \subset \mathbb{A}^n$ (affine variety) at $x \in X$ (intrinsic formulation)

$x \in X \subset \mathbb{A}^n$ Assume w.l.o.g. $x = 0$

$\mathfrak{m}_x$: ideal of x in $k[X]$

M_x : ideal of x in $k[Y]$

$$\mathfrak{m}_x \cong M_x / I(X)$$

Moreover

$$T_x(X)^{\overset{\text{dual}}{*}} \cong \mathfrak{m}_x / \mathfrak{m}_x^2$$

at least quadratic terms

$\dim X = \dim T_x(X)$

If $f \in k[X]$, with $f(x) \neq 0$, $X_f \subset X$ principal open set, then $T_x(X_f) \xrightarrow[\text{iso}]{\sim} T_x(X)$

Local parameters

X affine variety, $\dim X = n$, x non singular

"algebraic Dini" : $\operatorname{rk}\left(\frac{\partial(f_1 \dots f_m)}{\partial(Y_1, Y_2 \dots Y_n)}\right) = n - \dim X$

$\mathcal{O}_{X,x}$ local ring

$u_1 \dots u_n$ local parameters: basis of $\mathfrak{m}_x / \mathfrak{m}_x^2$

$\mathcal{O}_{X,x} := \{ f \in k(X) \mid f \text{ regular at } x \}$

fractions

local ring (sub ring of $k(X)$)

f regular at x : $f = \frac{g}{h}$ with $h(x) \neq 0$ (deg g = deg h in the proj case)