GEOMETRIC METHODS IN QUANTUM MECHANICS

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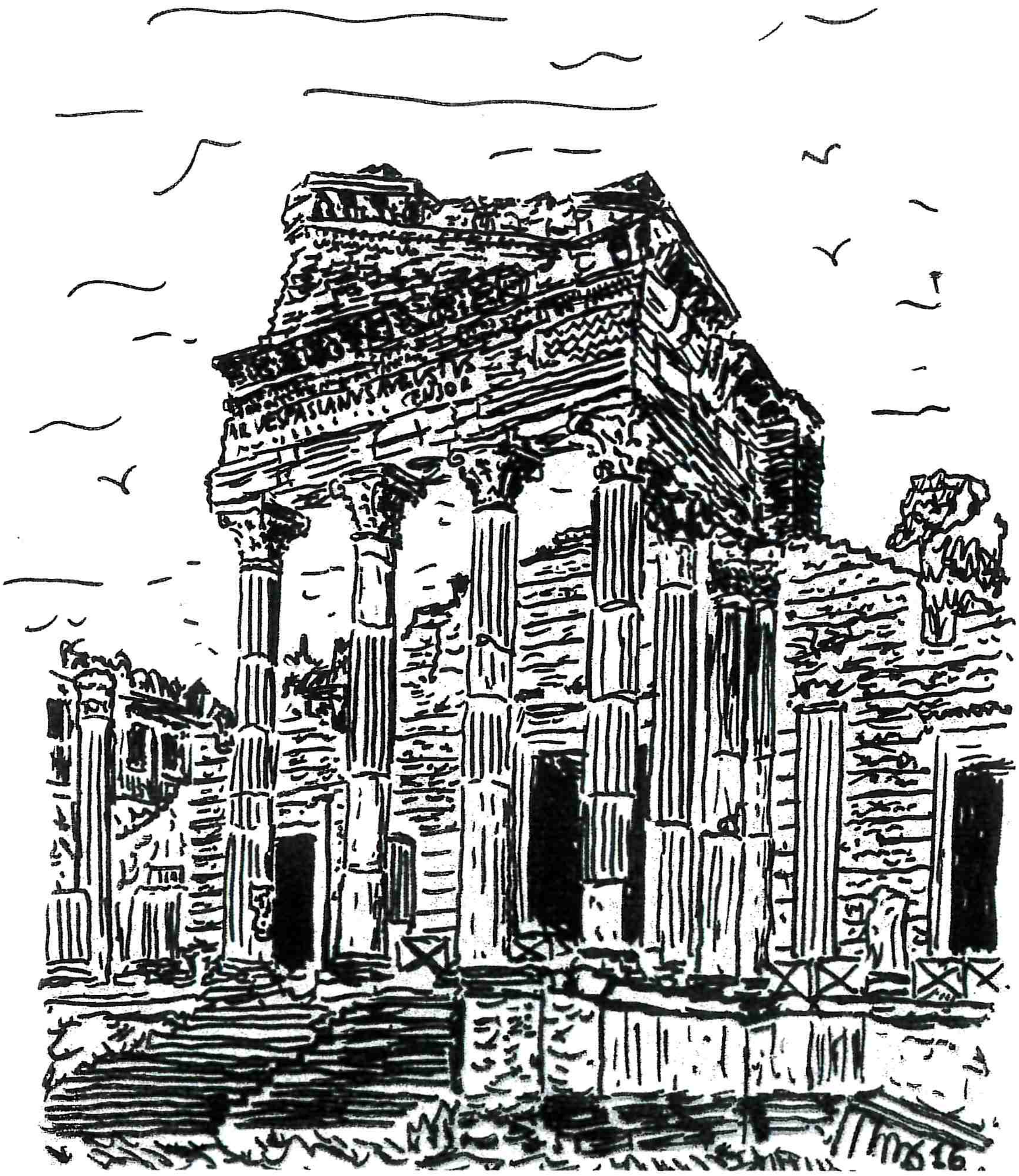
Lecture II Interlude: connections, Dirac monopole, Bohm-Aharonov effect, Berry's phase

International Doctoral Program in Science



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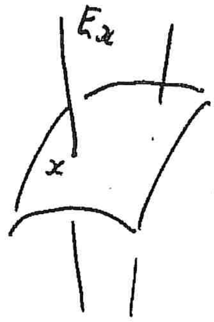
4 Connections

M n-dimensional manifold

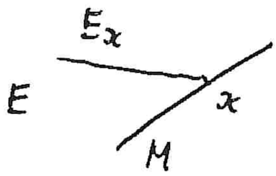
$E \longrightarrow M$ vector bundle



$\dim E_x = r$
rank

$$\Lambda^p(E) = \underset{\text{p-forms}}{\Lambda^p} \otimes_{\mathbb{R}} \Gamma(E)$$

$\Gamma(E) = \Lambda^0(E)$ sections



∇ connection

$$\nabla : \Lambda^0(E) \longrightarrow \Lambda^1(E) \quad \text{linear}$$

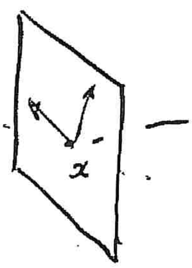
$$\nabla(f\sigma) = df\,\sigma + f\nabla\sigma \quad \text{Leibniz rule}$$

$f \in \Lambda^0$ (function)

$s = (s_1 \dots s_r)$ local frame over U

$$s_i \in \Lambda^0(E|_U)$$

$(s_1(x) \dots s_r(x))$ basis of E_x fibre $\forall x \in U$

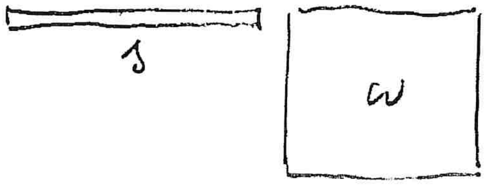


then $\exists\ \omega = (\omega_{ij}) \in M_r(\Lambda^1|U)$

s.t.

$$\nabla s = s\,\omega$$

connection form



$$\xi = \sum_{i=1}^{r} \xi_i s_i \qquad \text{section}$$

smooth functions $\Lambda^0|_U$

$$\xi \mapsto \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \end{pmatrix}$$

local expression

then:

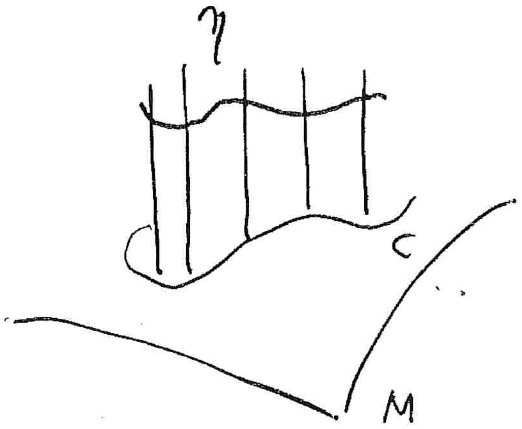
$$\nabla \xi = d\xi + \omega \xi = (d+\omega)\xi$$

local expression of ∇

upon contraction with $X \in \mathfrak{X}(M)$ vector field

$$(\nabla_X \xi)(x) := (\nabla \xi)(X)(x) \in E_x$$

covariant derivative of ξ along X



η : section of E along c (projects onto c)

- ξ is parallel if (tout court)

$$\boxed{\nabla \xi = 0}$$

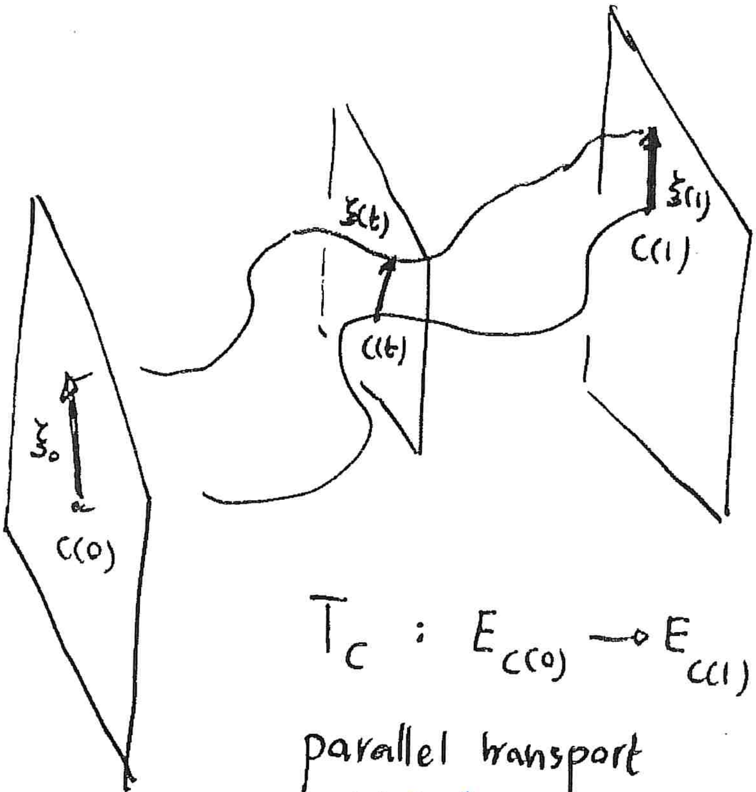
- η is parallel if (covariantly constant) (along c)

$$\boxed{\nabla_{\dot c} \eta = 0}$$

$$d\xi = -\omega\,\xi$$

$$\frac{d\xi_i}{dt} = -\sum_{j=1}^{r} w_{ij}(\dot c(t))\,\xi_j$$

1st order system of diff. equations (Cauchy-Lipschitz) $\Rightarrow$ $\exists!$ sol. given an initial condition



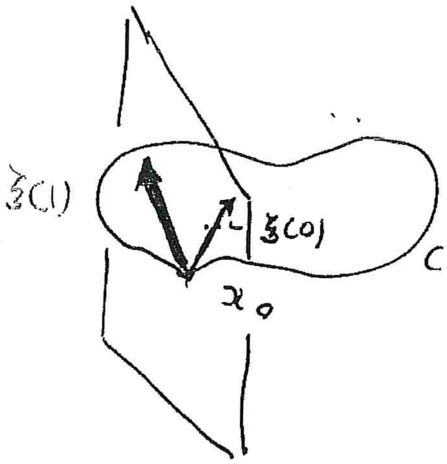
Given ξ_0, $\exists!$ parallel section along c, the parallel transport of ξ_0 along c (horizontal lift)

$$T_c : E_{c(0)} \to E_{c(1)}$$

parallel transport operator (Levi-Civita 1917)

$$T_c(\xi_0) = \xi(1)$$
$$\xi_0 \equiv \xi(0).$$

If $c(0) = c(1) = x_0$ (c closed) the T_c's give rise to the holonomy group of ∇ at x_0



Extend:

$$\nabla : \Lambda^p(E) \longrightarrow \Lambda^{p+1}(E)$$

$$\nabla(\varphi\sigma) = (\nabla\sigma) + \sigma\, d\varphi$$

Curvature: $R = \nabla^2$ (Λ^0-linear (module property))

$$\nabla^2(f\sigma) = \nabla(\sigma\, df + f\nabla\sigma)$$

$$= \underbrace{\nabla\sigma \wedge df + \overset{\text{1-form}}{df} \wedge \nabla\sigma}_{=0} + f\nabla^2\sigma = f\nabla^2\sigma$$

$s\,\Omega = \nabla^2 s$ $\quad$ Ω : curvature form

2-form with values in $\mathrm{End}(E)$

$$s\,\Omega = \nabla(s\omega) = \nabla s \wedge \omega + s\, d\omega$$

$$= s\omega \wedge \omega + s\, d\omega = s(d\omega + \omega\wedge\omega)$$

$$\Rightarrow \boxed{\Omega = d\omega + \omega \wedge \omega}$$

$\star$ Cartan's formula (structure equation)

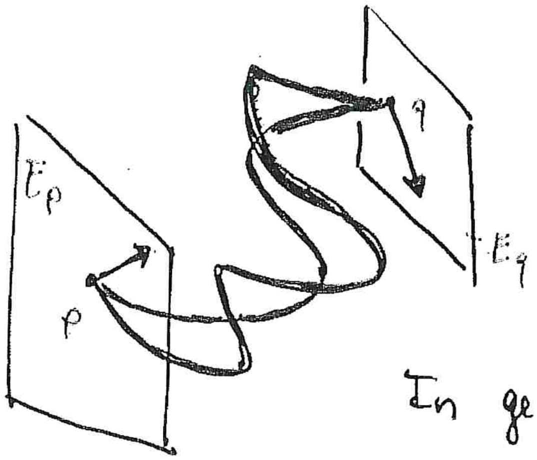
Therefore, Ω is the obstruction to the existence of a nontrivial parallel section

∇ is called flat if $\Omega = 0$

$$d\omega + \omega\wedge\omega = 0$$

Bianchi: $\nabla^3 = 0$ $\quad$ $d\Omega = \Omega\wedge\omega - \omega\wedge\Omega$

★ **Theorem** (crucial) ∇ flat connection on $E \to M$. Then, p, q fixed, $T_c : E_p \to E_q$ only depends on $[c]$ homotopy class of c



$\Rightarrow$ no dependence if M is simply connected

In general (∇ flat)

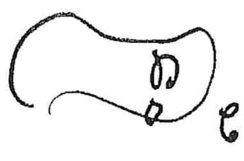
★ $T_c : E_{x_0} \circlearrowright$ defines a **representation** (via $\mathrm{End}(E_{x_0})$) of $\pi_1(M, x_0)$ fundamental group of M (based at x_0)

(various proofs, e.g. via product integrals (Volterra, Schlesinger)

in general if $E = L$, line bundle, then

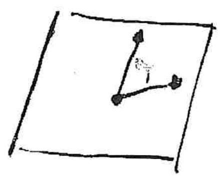
$$\Omega = d\omega + \underbrace{\omega \wedge \omega}_{= 0} = d\omega$$

and this reduces to Levi-Civita's formula (essentially, Stokes' theorem)



$$\int_{\mathcal{C}} \omega = \int_{D} d\omega = 0$$

angle of parallelism (if lengths are preserved...) (flatness)



The Bohm-Aharonov effect

$$\boxed{i\hbar\dot{\psi} = H\psi}$$

Schrödinger equation for a charged (non relativistic) particle

$$H = -\frac{\hbar^2}{2m}\Big(\underset{\text{gradient}}{\nabla} - \frac{iq}{\hbar c}\underline{A}\Big)^2 - q\Phi$$

Hamiltonian

4-potential: $A_\mu = (\underset{\text{scalar pot.}}{\Phi}, \; -\underset{\text{vector pot.}}{\underline{A}})$

U(1) - gauge theory:

$$\underline{A} \longmapsto \underline{A} + \nabla\chi$$
$$\Phi \longmapsto \Phi - \dot{\chi}$$

$$\psi \longmapsto \underbrace{e^{-\frac{iq}{\hbar c}\chi}}_{\text{phase}}\psi$$

$$i\hbar c D_0\psi = -\frac{\hbar^2}{2m}D_k D^k\psi$$ gauge invariant form

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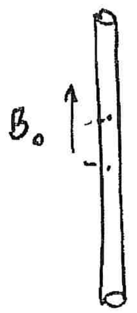
$x^0 = ct, \; x^\mu = (x^0, \underline{x})$

$D_\mu = \partial_\mu + i a_\mu$

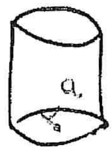
$a_0 = \frac{q}{\hbar c}\Phi$

$a_k = -\frac{q}{\hbar c}A_k$

Now take a solenoid



$B = 0$



flux

$\Phi_0 = \pi a^2 B_0$

$\underline{B} = (0, 0, B_0)$

$\underline{B} = \text{curl}\,\underline{A}$

solve Poisson...

$$\underline{A}(x,y,z) = \frac{B_0 a^2}{2}\cdot\begin{cases} r\frac{B_0}{2} & r < a \\ \frac{\Phi_0}{2\pi r} & r > a \end{cases}$$

continues...

Cylindrical coordinates

Hamiltonian (classical)

$$H = \frac{1}{2m}\left(\underline{p} - \frac{q}{c}\underline{A}\right)^2 = \frac{1}{2m}\left[p_r^2 + \frac{1}{r^2}\Big(\overbrace{p_\phi - \frac{q\Phi_0}{2\pi c}}^{p_\phi'}\Big)^2 + p_z^2\right]$$

a canonical transformation eliminates Φ_0

but quantum mechanically

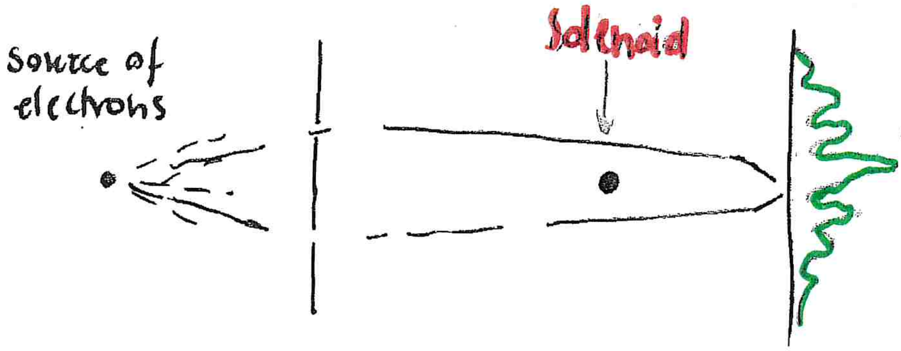
$$\psi \longmapsto \psi' = e^{-iq\Phi_0\varphi/\hbar c}\,\psi$$

eliminates Φ_0 from the Schrödinger equation

but Φ_0 affects the boundary conditions:

$$\psi(2\pi) = \psi(0) \quad \text{but} \quad \psi'(2\pi) = e^{-iq\Phi_0/\hbar c}\,\psi'(0)$$

this resulting in an interference pattern



$$\underset{\text{phase shift}}{\Delta\varphi} = -\frac{q\Phi_0}{\hbar c} = -\frac{q}{\hbar c}\int \underline{A}\cdot d\underline{\ell}$$

$$\Phi_0 = \int_\Sigma \underline{B}\, d\underline{\sigma}$$

$\underline{B}$ vanishes outside the solenoid,

$\underline{A}$ does not

limiting situation: $M = \mathbb{R}^2 \setminus \{0,0\}$

$B_z(x) = \Phi_0\,\delta(x)$

$$\underline{A} = \frac{\Phi_0}{2\pi}\frac{(-y,x)}{r^2}$$

Biot-Savart again

$\pi_1(M) \cong \mathbb{Z}$

$$\int_C \underline{A}\cdot d\underline{\ell} = n\,\Phi_0$$

"topological phase"

Upshot: on the trivial complex line bundle $L \xrightarrow{\mathbb{C}} M$, $L = M \times \mathbb{C}$

$\underline{A}$: flat connection with non trivial holonomy

$\underline{B}$: curvature ($= 0$ on M)

(singular on $\mathbb{R}^2$)

one has a representation of $\pi_1(M) \cong \mathbb{Z}$

(emergence of a topological phase)

Differential geometry plays a crucial role throughout modern physics

see also below: gauge approach to QM

★ The magnetic monopole

Dirac, 1931
Wu & Yang, 1975

$L \to \mathbb{R}^3$ trivial line bundle
(there is no canonical trivialization)

$$\hat{P} = -i\hbar \nabla = -i\hbar \left(d + \frac{ie}{\hbar c} A \right)$$

generalized momentum operator

vector potential for a magnetic field B: $B = dA$

$$R = i \frac{e}{\hbar c} B$$

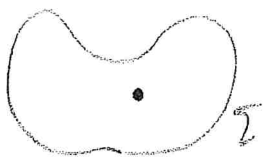
looked upon as a 2-form (Hodge operator on $\mathbb{R}^3$ understood)

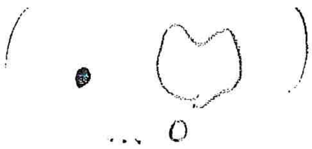
work in $\boxed{\mathbb{R}^3 \setminus \{0\}}$ single magnetic charge concentrated on the origin

$$\mathcal{M} := \frac{1}{4\pi} \iint_\Sigma B \, d\sigma \quad \left(= \frac{-i\hbar c}{4\pi e} \iint_\Sigma R \right)$$

monopole intensity

Σ containing 0





Theorem (Dirac) monopole charge quantization

$$\mathcal{M} = \frac{c\hbar}{2e} n \qquad n \in \mathbb{Z}$$

$$H^*(\mathbb{R}^3 \setminus \{0\}) = H^*(S^2) = \begin{cases} 0: & \mathbb{R} \\ 1: & 0 \\ 2: & \mathbb{R} \end{cases}$$

cf Gauss' theorem

$$\iint_\Sigma E \propto q$$

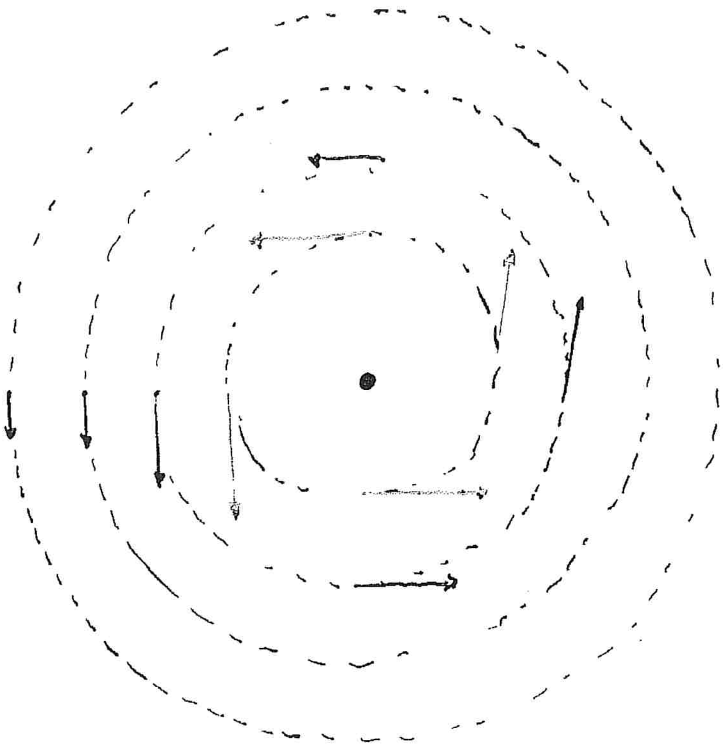
enclosed charge

$E \leftrightarrow \underline{E} \cdot d\underline{\sigma} = \underline{E} \cdot \underline{n} \, d\sigma$

2-form flux

$\operatorname{div} \vec{E} = \rho$ Gauss

$\operatorname{div} \vec{B} = 0$ absence of monopoles



differential geometry of linking numbers

$$\int_{C} v = \int_{D} \delta \cdot dx \wedge dy$$

currents...

Dirac's δ

velocity
$\equiv$ connection form

(concentrated) vorticity
$\equiv$ curvature

$w = dv$

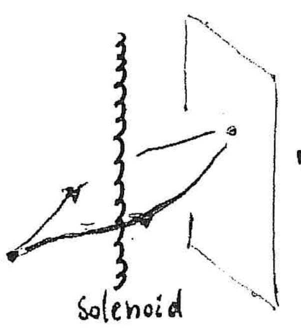
$\delta v = 0$ divergence-free

"topological Stokes' formula"

flat connection. on $E \to M$
$\rightsquigarrow$ representation of the fundamental group of M
$\pi_1(M, *)$

via parallel transport

★ **topological information** still carried (concentrated curvature)



Aharonov - Bohm

A vector potential

$F = dA$ (loc.)

electromagnetic field

Explicit formula for parallel transport

along a curve, working locally, upon setting

$$\eta(t) = A(t)\,\underset{\eta(0)}{\overset{\shortparallel}{\eta_0}}$$

one gets:

$$(d+\omega)\,\eta(t) = 0 \qquad (d+\omega)\,A(t)\,\eta_0 = 0$$

$$\Rightarrow \qquad dA + \omega A = 0 \quad \text{i.e.} \quad \begin{cases} \dot{A} + \omega A = 0 \\ A(0) = Id \end{cases}$$

A local expression of parallel transport

Chen's iterated path integrals

if $C = -\omega$, we obtain

$$A(t) = I + \sum_{m=1}^{\infty} \int_{0 \le t_1 \le .. t_m \le t} C(t_m)\,..\,C(t_1)\,dt_1 \dots dt_m$$

$$\equiv P \exp \int_0^t C(u)\,du \equiv e^{:\int_0^t C(u)\,du:}$$

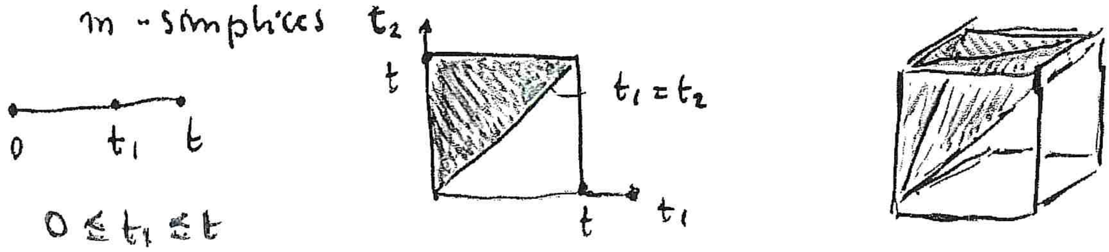
time ordered exponential

normal (or Wick) ordering

The given system is non autonomous, so in general $[C(t), C(s)] \neq 0 \quad t \neq s$

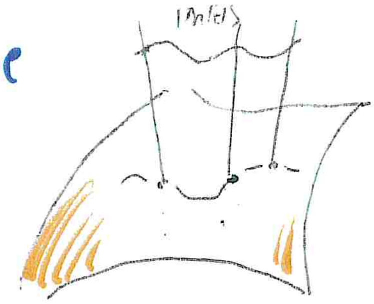
if $C(t) \equiv C$, then $A(t) = \exp(tC)$

m - simplices



$0 \le t_1 \le t$ $\qquad$ $0 \le t_1 \le t_2 \le t$ $\qquad$ $0 \le t_1 \le t_2 \le t_3 \le t$

* Adiabaticity & Berry phase
in a nutshell



Born - Fock

(infinitely) adiabatically slowly varying Hamiltonian $H = H(t)$

[discrete, non degenerate spectrum]

$$(*) \quad H(t)\,|n(t)\rangle = E_n(t)\,|n(t)\rangle$$

$$\langle n(t)\,|\,m(t)\rangle = \delta_{nm}$$

phase freedom

$$|n(t)\rangle \mapsto |n'(t)\rangle = e^{i\gamma_n(t)}\,|n(t)\rangle$$

Let

$$\Psi(0) = |n(0)\rangle \quad \text{for some } n$$

Then

$$\Psi(t) = \sum_m c_m(t)\, \underbrace{e^{-\frac{i}{\hbar}\int_0^t E_m(\tau)\,d\tau}}_{\text{dynamical phase}}\, |m(t)\rangle$$

and

$$\dot{c}_m(t) = -c_m(t)\,\langle m|\dot{m}\rangle - \sum_{k\neq m} c_k\,\langle m|\dot{k}\rangle\, e^{-\frac{i}{\hbar}\int_0^t (E_k(\tau)-E_m(\tau))\,d\tau}$$

Differentiation of $(*)$ yields

$$\dot{H}\,|k\rangle + H\,|\dot{k}\rangle = \dot{E}_k\,|k\rangle + E_k\,|\dot{k}\rangle \quad \Rightarrow$$

$$\langle m\,|\,\dot{k}\rangle = \frac{1}{E_k - E_m}\,\underbrace{\langle m|\dot{H}|k\rangle}_{0} \qquad m \neq k$$

adiabaticity

$$\leadsto \quad \langle m\,|\,\dot{k}\rangle = 0 \qquad \& \qquad \dot{c}_m = -c_m\,\langle m|\dot{m}\rangle$$

$$c_m(0) = \delta_{nm}$$

Therefore $c_m(t) = 0$ for $m \neq n$ &

$$\Psi(t) = c_n(t)\, e^{-\frac{i}{\hbar}\int_0^t E_n(\tau)\,d\tau} \cdot |n(t)\rangle$$

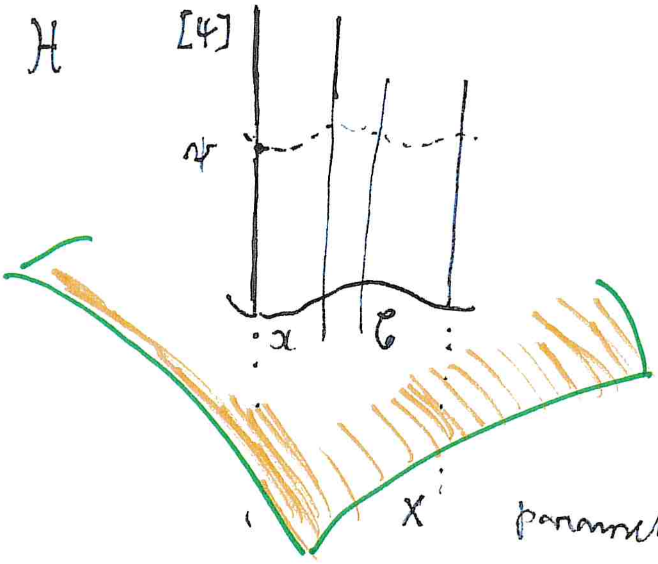
i.e. $\Psi(t) \in n$-eigenspace of $H(t)$

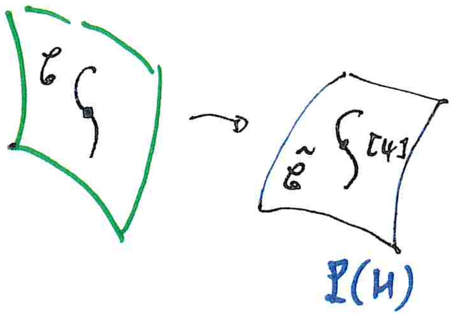
Also $\qquad c_n(t) = e^{i\phi_n(t)}$ $\qquad$ ↖ ignored for almost 50 years!

$$\dot{\phi}_n(t) = i\langle n|\dot{n}\rangle$$

if $\quad |\tilde{n}(t)\rangle = e^{i\phi_n(t)}\,|n(t)\rangle$

then $\quad \langle \dot{\tilde{n}}|\tilde{n}\rangle = 0 \qquad$ (Born-Fock gauge)





Let $\quad P = |\psi\rangle\langle\psi| \qquad$ projection onto the line $[\psi]$

Born-Fock $\quad$ (Berry-Simon connection) $\qquad$ "Levi-Civita" $\quad$ "Grassmann"

$$\nabla = P\,d$$

$$\nabla\psi = |\psi\rangle\langle\psi\, d\,\psi\rangle$$

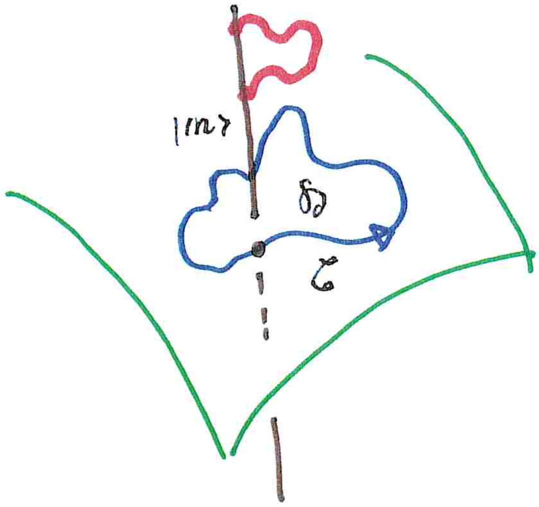
$$\frac{\nabla}{dt}\psi = |\psi\rangle\langle\psi|\dot{\psi}\rangle \qquad \nabla s \equiv \theta s$$

$$\nabla(fs) = df\,s + f\nabla s \equiv df\,s + f\,\theta s = (df + \theta f)s$$

connection form

Adiabatic evolution $\equiv$ parallel transport w.r.to ∇

$$\frac{\nabla}{dt}\psi = 0 \quad \text{along } \mathcal{C}$$



Cyclic evolution

⇣

Berry phase

$$\int_{C} \langle n|dn\rangle \underset{\text{Stokes}}{=} \int_{\Omega} \langle dn|\wedge|dn\rangle$$

$$C = \partial\Omega$$

Curvature

$$\dot{\phi}_n = i\langle n|\dot{n}\rangle$$

purely imaginary

physicists' notation

$$A^{(n)} := i\langle n|dn\rangle$$

$$A^{(n)} = -\mathrm{Im}\langle n|dn\rangle$$

$$\gamma_n(C) = \phi_n(\underset{\text{large}}{T}) = \oint A^{(n)}$$

Berry phase (coming from Aharonov-Anandan phase on $P(\mathcal{H})$)

Total phase

$$\boxed{\gamma = \underbrace{-\frac{1}{\hbar}\int_0^T E_n(\tau)\,d\tau}_{\text{dynamical}} + \underbrace{\gamma_n(C)}_{\text{geometric}}}$$

$$F^{(n)} = dA^{(n)} = -\mathrm{Im}\langle dn|\wedge|dn\rangle$$

$$= \frac{1}{2}F_{ij}^{(n)}\,dx^i\wedge dx^j = \frac{1}{2}\Big[-\mathrm{Im}\big(\langle\partial_i n|\partial_j n\rangle - (i\leftrightarrow j)\big)\Big)$$

For completeness, we record the following formula for F_n

$$1 = \sum_m |m\rangle\langle m|$$

$$F^{(n)} = -\mathrm{Im} \sum_m \langle dn|m\rangle \wedge \langle m|dn\rangle$$

$$= -\mathrm{Im} \sum_{m\neq n} \langle dn|m\rangle \wedge \langle m|dn\rangle$$

($\langle n|dn\rangle \in i\mathbb{R}$) $$= -\mathrm{Im} \sum_{m\neq n} \overline{\langle m|dn\rangle} \langle m|dn\rangle$$

But $\langle m|dn\rangle = \dfrac{\langle m|dH|n\rangle}{E_n - E_m}$
$m\neq n$

(small...)
adiabatic approximation

$$\Rightarrow \boxed{F^{(n)} = -\mathrm{Im} \sum_{m\neq n} \frac{\langle n|dH|m\rangle \wedge \langle m|dH|n\rangle}{(E_m - E_n)^2}}$$

One finds $\sum_n F^{(n)} = 0$