GEOMETRIC METHODS IN QUANTUM MECHANICS

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Lecture III
Geometric quantum mechanics

International Doctoral Program in Science

Brescia,
Capitolium

Kibble, Provost-Vallée, Brody-Hughston, Ashtekar-Schilling, Cirelli-Pizzocchero, Manini-Gatti, M.S., BSS..

- Geometric quantum mechanics

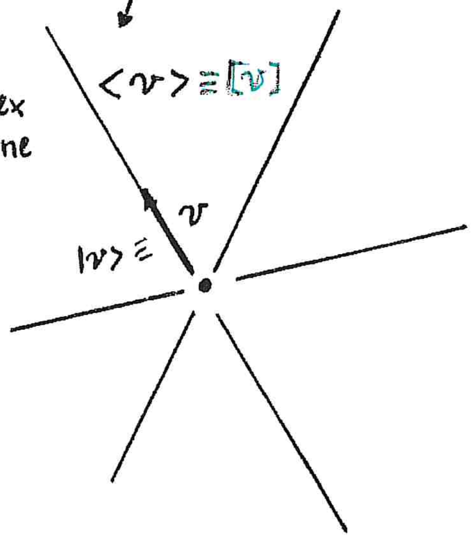
linearity

$(V, \langle \mid \rangle)$ complex Hilbert space $\dim_{\mathbb{C}} V = n+1$

$\hbar = 1$ throughout

★ $\mathbb{P}(V)$ associated projective space $(\dim_{\mathbb{C}} \mathbb{P}(V) = n)$

points : pure states in quantum mechanics



V

Dirac's bra-ket notation

$$\underset{\substack{\uparrow \\ 0}}{[v]} = \frac{|v\rangle\langle v|}{\|v\|^2}$$

orthogonal projector onto $\langle v \rangle$

if $\|v\| = 1$ $\qquad v = \sum_{i=0}^{n} \alpha_i e_i \qquad |v\rangle\langle v| \leftrightarrow (\bar{\alpha}_i \alpha_j)$

orthonormal basis

density matrix

unitary group

$$\boxed{★ \quad \mathbb{P}(V) \cong \frac{U(V)}{U(V') \times U(1)}}$$

isotropy

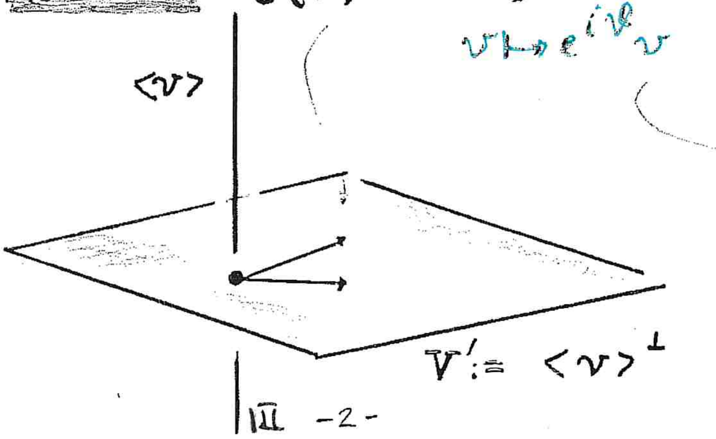
$U(V)$ - homogeneous Kähler manifold

$\wr\!\wr$

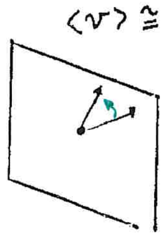
$$\frac{U(n+1)}{U(n) \times U(1)}$$

(simplest Grassmannian)

$v \mapsto e^{i\theta_v} v$

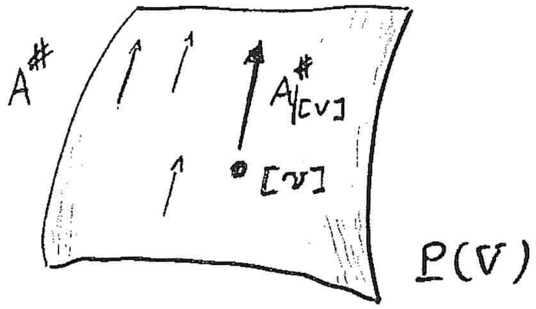


V



$\mathfrak{u}(V) \cong \mathfrak{u}(n+1)$ Lie algebra of $U(n+1)$

$\equiv$ skew-hermitian matrices ~ **observables in Q.M.**



$A^\#$: fundamental vector field associated to $A \in \mathfrak{u}(V)$

$$A^\#|_{[v]} := |v\rangle\langle Av| + |Av\rangle\langle v|$$

complex structure J : $J|_{[v]} A^\#_{[v]} = |v\rangle\langle iAv| + |iAv\rangle\langle v|$

$J^2 = -1$

★ Fubini-Study metric & Kähler form

$\mathrm{tr}\,(|v\rangle\langle w|) = \langle w|v\rangle$

• $g_{[v]}(A^\#|_{[v]}, B^\#|_{[v]}) =$

$$= \mathrm{Re}\{\langle Av|Bv\rangle + \langle v|Av\rangle\langle v|Bv\rangle\}$$

• $\omega|_{[v]}(A^\#|_{[v]}, B^\#|_{[v]}) = g_{[v]}(J|_{[v]} A^\#|_{[v]}, B^\#|_{[v]})$

$$= \frac{i}{2}\langle v[A,B]v\rangle$$

★ variance of A in $[v]$
(dispersion)

Fubini-Study length

$$\Delta_{[v]} A = \|Av - \langle v|Av\rangle v\| = \|A^\#_{[v]}\|_{FS}$$

$$\sqrt{\overline{A^2} - \bar{A}^2} \quad := \sqrt{g_{[v]}(A^\#_{[v]}, A^\#|_{[v]})}$$

$$= \|J|_{[v]} A^\#|_{[v]}\|_{FS}$$

Cauchy - Schwarz for g $\Rightarrow$

Heisenberg Uncertainty Principle

$$\boxed{\Delta_{[v]}(A)\, \Delta_{[v]}(B) \geq \frac{1}{2} \left| \langle v | [A,B]\, v \rangle \right|}$$

geometric interpretation of the uncertainty structure

- symplectic geometry

via the Killing - Cartan metric

$(A,B) = -\frac{1}{2}\, \mathrm{Tr}(AB)$

$$\mu : \mathbb{P}(V) \longrightarrow u(V)^* \cong u(V)$$

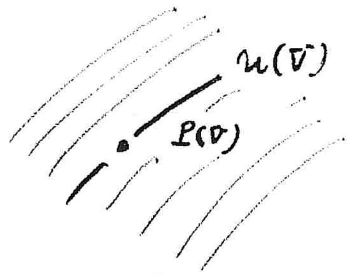
moment map

$$\boxed{\mu([v]) := -i\, |v\rangle\langle v|}$$

- Hamiltonian algebra

$$\boxed{\mu_A([v]) = (\mu, A) = \frac{i}{2} \langle v | A v \rangle} \qquad A \in u(V)$$

$\Rightarrow$ ω $\equiv$ Kirillov symplectic form on $\mathbb{P}(V)$, looked upon as a coadjoint orbit of $U(V)$



Also

$$d\mu_A = i_{A^\#}\, \omega$$

Poisson bracket

$$\boxed{\{\mu_A, \mu_B\} := \omega(A^\#, B^\#) = \mu_{[A,B]}}$$

$$\Big(\text{caveat:} \quad \underset{\substack{\uparrow \\ \text{vector field} \\ \text{commutator}}}{[A^\#, B^\#]} = - \underset{\substack{\uparrow \\ \text{Lie algebraic} \\ \text{commutator}}}{[A, B]}^\# \Big)$$

local potential

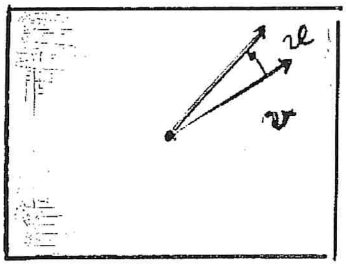
$$u = -i \langle v | dv \rangle$$

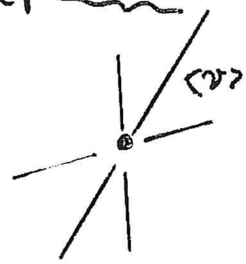
Chern-Bott connection form on $O(1) \to P(V)$

"Berry's connection"

Hyperplane section bundle, dual to $O(-1)$, tautological line bundle

★ geometrically: infinitesimal angle variation of v, relative to the complex plane it generates, determined by a norm preserving displacement





cf. the geometric quantization prescription (Weil-Kostant)

◆ Toral actions & integrability

Quantum Hamiltonian non degenerate spectrum

$$H = \sum_{j=0}^{n} \lambda_j \underbrace{|e_j\rangle\langle e_j|}_{P_j}$$

$\lambda_i \neq \lambda_j$ if $i \neq j$

$$0 = \lambda_0 < \lambda_1 < \dots < \lambda_n \quad \Rightarrow \quad H = \sum_{j=1}^{n} \lambda_j P_j$$

Schrödinger equation

$$\boxed{\frac{\partial |v\rangle}{\partial t} = -iH|v\rangle}$$

$v = \sum_{i=0}^{n} \alpha_i e_i$

$\sum_{i=0}^{n} |\alpha_i|^2 = 1$

$(\|v\| = 1)$

$$h([v]) := \frac{\langle v | H v \rangle}{\langle v | v \rangle} = \dots = \sum_{j=1}^{n} \lambda_j |\alpha_j|^2$$

mean value of H in the state $[v]$

$\equiv$

★ "classical" Hamiltonian on $P(V)$

$\Rightarrow \qquad h([v]) = \mu_{(-2iH)}$

★ Critical points of h : $[e_j]$ eigenstates   cf $\Delta_{[v]} A = \|A^{\#}_{[v]}\|$

$$h([v]) = \lambda_j + \sum_{k=0}^{n} (\lambda_k - \lambda_j)\underbrace{(x_k^2 + y_k^2)}_{|\alpha_k|^2}$$

h : perfect Morse function

index $[e_j] = 2j$ $b_{2j} = 1$

negative eigenvalues of the Hessian   Betti number

recall

$$\mathbb{P}^n = \underbrace{\mathbb{P}^n \setminus \mathbb{P}^{n-1}} \cup \underbrace{\mathbb{P}^{n-1} \setminus \mathbb{P}^{n-2}} \cup \dots \cup \underbrace{\overset{S^2}{\mathbb{P}^1} \setminus \mathbb{P}^0} \cup \overset{pt}{\mathbb{P}^0}$$

as a CW complex

$\mathbb{A}^n = \mathbb{C}^n$ $\mathbb{A}^{n-1} = \mathbb{C}^{n-1}$ $\mathbb{A} = \mathbb{C}$ 0-cell

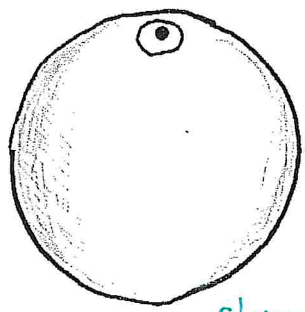
2n-cell   2(n-1) cell   2-cell

$\updownarrow$ $\updownarrow$ $\updownarrow$

ω^n ω^{n-1} ω

Fubini-Study

Poincaré-Cartan integral invariants



$\mathbb{P}^1 \cong S^2$

stereographic projection

$$v = \sum_{j=0}^{n} \alpha_j e_j \longmapsto \sum_{j=0}^{n} \alpha_j e^{i\beta_j} e_j$$

$\alpha_j \neq 0 \quad \forall j$ temporarily

a global phase change leaves $[v]$ invariant

$\boxed{\text{effective action of } G = \mathbb{T}^n \text{ on } \mathbb{P}(V)}$ (set $\beta_0 = 0 \ldots$)

$$(\bar{\alpha}_i \alpha_j) \longmapsto (\bar{\alpha}_i \alpha_j e^{i(\beta_j - \beta_i)})$$

density matrices

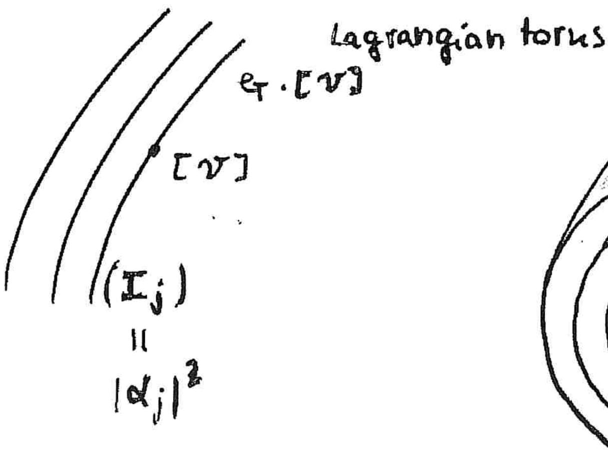
generators of the G action : $iP_j \quad , j = 1, 2, \ldots, n$

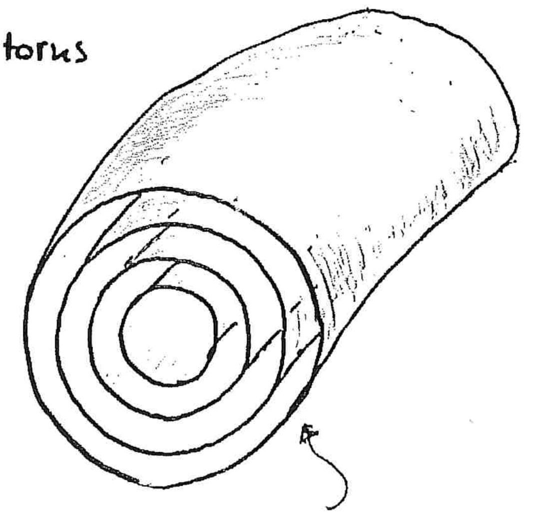
$\mathcal{H}_j = \mu_{(-2iP_j)}$ Hamiltonians

★ n first integrals in involution = action variables

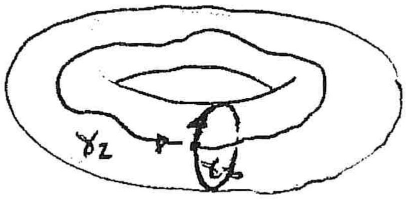
= transition probabilities (cf. Cirelli - Pizzocchero as well)

★★ complete integrability on an open dense set in $\mathbb{P}(V)$ (+ isotropic tori of lower dimension





Lagrangian tori



homology basis

local potential

$$\left[I_j = \frac{1}{2\pi} \int_{\gamma_j} \vartheta = \ldots = |\alpha_j|^2 \right]$$

action variable

homology basis

transition probabilities $|\langle e_j | \psi \rangle|^2$

$$\gamma_j \; : \quad [0, 2\pi) \ni \beta_j \mapsto \left[\sum_{n \neq j} \alpha_n e_n + \alpha_j e^{i\beta_j} e_j \right]$$

Schrödinger's evolution takes place on leaves!

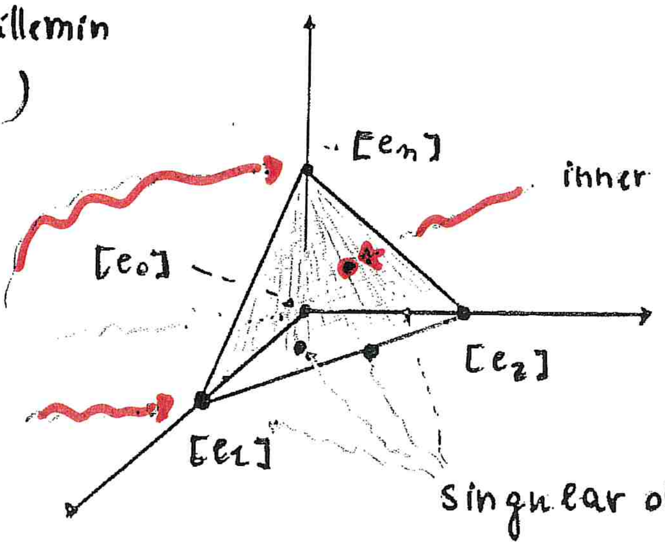
geometrically : convex polytope in $\mathbb{R}^n$ ($\cong$ Lie G)

$\equiv$ standard n-simplex Δ_n

$$0 \leq \sum_{j=1}^{n} I_j = 1 - |\alpha_0|^2 \leq 1$$

geometry of toral orbits

(Atiyah, Guillemin Sternberg ...)



inner points $\cong$ generic orbits

critical points

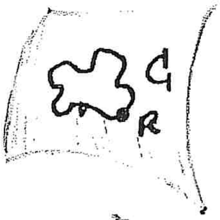
singular orbits (isotropic tori)

I_j globally defined ...

◇ Consequences of complete integrability

◆ $\boxed{\text{Berry} = \text{Hannay}}$

adiabatic, cyclic evolution



$\mathcal{C} : t \mapsto \mathcal{C}(t)$

(eigenvectors go to eigenvectors...)
quantum adiabatic theorem

$\mathcal{R}$

parameter space

$H = H(R)$

family of Hamiltonians

$$\boxed{e_j(\mathcal{C}(T)) = e^{i\int_{\mathcal{C}} \overbrace{-i\langle e_j(R) | d_R e_j(R)\rangle}^{d\vartheta_j}} e_j(\mathcal{C}(0)) \equiv e^{i\Delta\vartheta_j^B} e_j(\mathcal{C}(0))}$$

$(e_0(\mathcal{C}(t)) \equiv e_0 \quad \forall t \in [0,T])$

$\boxed{\ast \text{ Berry's geometric phase}}$

=> "classically" we have a migration of Lagrangian and isotropic tori taking place on $\mathcal{R} \times \mathbb{P}(V) \to \mathcal{R}$ (trivial fibration)

=> geometrical framework for Hannay's angles

(Montgomery's connection: " averaging is holonomy

$$\Rightarrow \boxed{\Delta\vartheta_j^H = \Delta\vartheta_j^B}$$

Hannay's angles

(Cushman)

Indeed $\langle d\vartheta_j \rangle_G = d\vartheta_j$ $\Rightarrow$ $\Delta\vartheta_j^H = \int_{\mathcal{C}} d\vartheta_j = \Delta\vartheta_j^B$

average over G

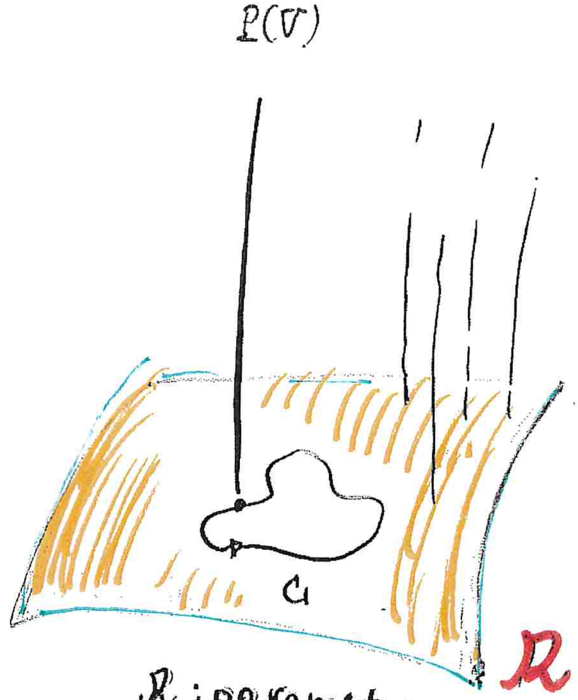
$P(A\psi)$

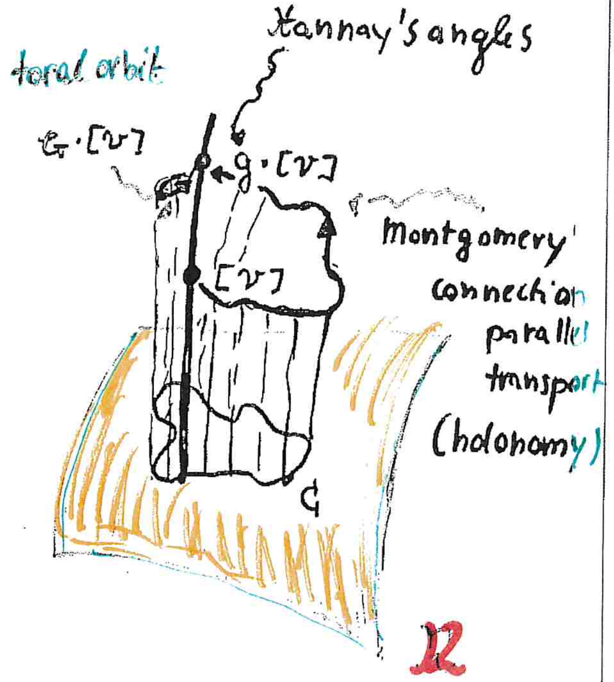
Also notice: $\nabla_{A^H}\psi = \langle\psi|A\psi\rangle\psi$

Chern - Bott

$P = |\psi\rangle\langle\psi|$

$\nabla = P d$





$\mathcal{R}$: parameter space

$\dim \mathcal{R} \geq 2$

$$\underbrace{\sum \alpha_i e_i}_{v} \longmapsto \sum \alpha_i e^{i\beta_i} e_i$$

remain constant (adiabaticity)

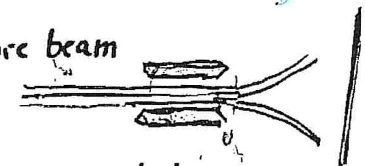
→ experiments have detected interference terms (off-diagonal elements in the density matrix)

⚠ caveat : this is different from Berry's calculation of his phase in semiclassical approximation of classically integrable systems : our calculation is totally within quantum mechanics

◆ Remarks on quantum measurement

◆ measurement of H

Bohm's "orthodox" approach (predating "Bohmian mechanics")
based on (generalizations of)
the Stern-Gerlach experiment
(discovery of spin)



$$\sum_j \alpha_j e_j \xrightarrow{\text{measurement}} \sum_j \alpha_j e^{i\beta_j} e_j$$

generic superposition of stationary states

stay the same

uncontrollable relative phase-shifts
(Heisenberg Uncertainty Principle)

=> one must average over a (long) series of experiments => formally we have a classical adiabatic perturbation
(I_j = constant)

+ toral action

=> (upon working with density matrices

$$\boxed{\frac{\partial \rho}{\partial t} = -i\,[H, \rho]} \quad \text{von Neumann's equation}$$

$\rho = \rho^*$, $\rho \geq 0$ $\quad Tr\rho = 1$

$$w_\rho(A) := Tr(\rho A)$$

state induced by ρ)

one arrives at the following averaging theorem

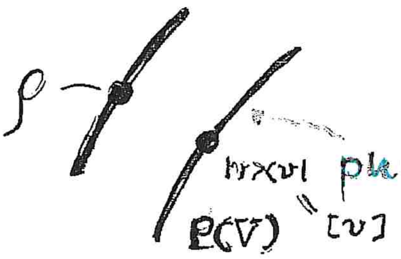
generalized convex combinations

$$\underbrace{\lim_{T\to+\infty}}_{\text{time}} \underbrace{\frac{1}{T}\int_0^T e^{-iHt}\,[\underbrace{(\bar{\alpha}_i\alpha_j)}_{[v]}]\,dt}_{\text{time average}} = \underbrace{\int_G g\cdot[(\bar{\alpha}_i\alpha_j)]\,dg}_{\text{averagin over "fast" angle variables}}$$

(valid in the non degenerate case)

$$= \underbrace{(|\alpha_j|^2\delta_{ij})}_{*} \equiv \rho$$

geometrically: passage to a different coadjoint orbit



* diagonal density matrix describing a mixture of states in proportions $|\alpha_j|^2 = I_j$

expressing the probability of finding the "particle" in the stationary state $[e_j]$ with energy λ_j

- on the "collapse" of the wave function — crux of QM

$$v = \Psi = \sum \alpha_j e_j \quad \xrightarrow[\text{?}]{\text{measurement}} \quad e_J \equiv \Psi'$$

wave function cf. Schrödinger's representation

in the Copenhagen interpretation this is not analyzed, deliberately

(with probability) $|\alpha_J|^2$

the superposition shrunk to e_J

this is not compatible with combined Q.M. of the system + measuring apparatus

* decoherence approach (Zeh, Zurek, Joos ...)

system + apparatus + environment

Here : just a geometric model for collapse in terms of geometric invariant theory, by relaxing unitarity whilst keeping linearity

Let $\mathbb{P}(V)$ be acted upon by $GL(V)$ (full linear group)

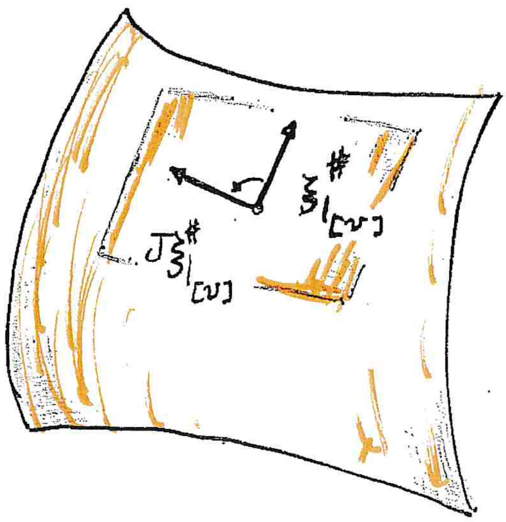
complexification of the toral action introduces "dissipative" (i.e. skew-hermitian) "observables"

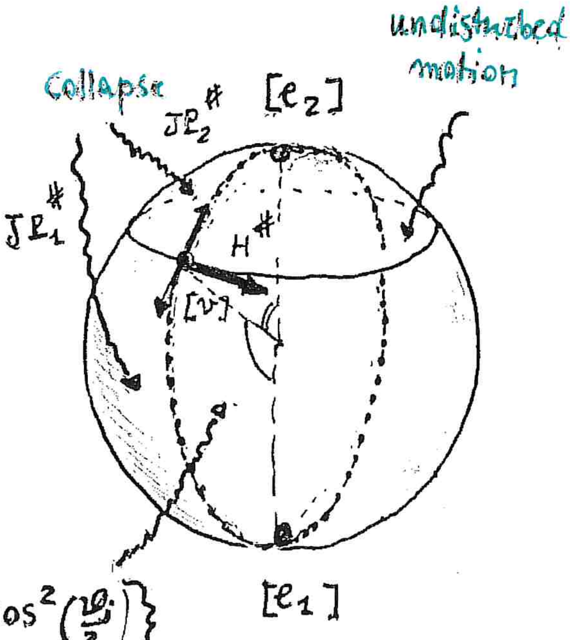
($J\xi^{\#}$ considered alongside with $\xi^{\#}$)

note this

e.g.

$$\underset{t\to+\infty}{\lim} \underbrace{e^{t\rho_j}}_{\text{gradient flow (Morse)}} \cdot [v] = [e_j] \qquad (\alpha_j \neq 0)$$





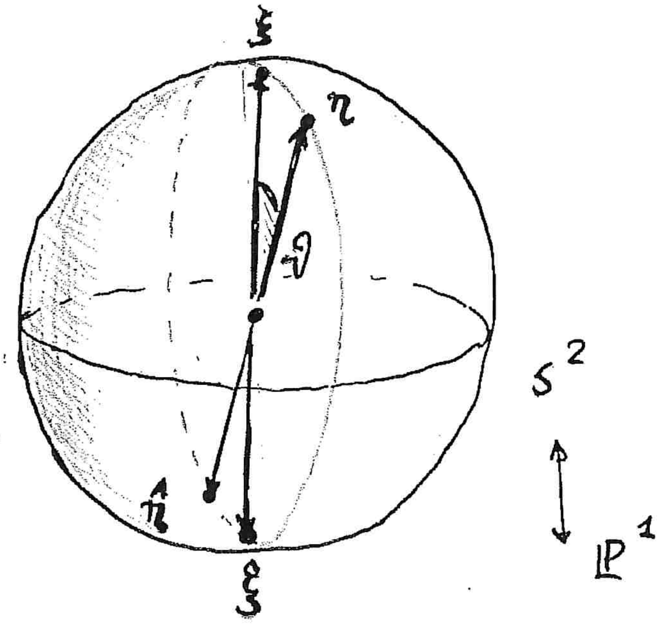
recall $\Delta_{[v]} A = \| A^{\#} \|_{FS}$

Ahronov-Anandan

$$|\alpha_j|^2 = \text{suitable cross ratio} = \underbrace{\cos^2\left(\tfrac{\vartheta_j}{2}\right)}_{\text{geodesic distance}}$$

$\mathbb{P} \cong S^2$

some detailes



$$\mathcal{R} = \frac{\xi^j \bar\eta_j \eta^k \bar\xi_k}{\bar\xi_j \xi^j \bar\eta_k \eta_k} \quad \left(= \frac{\langle\xi|\eta\rangle\langle\eta|\xi\rangle}{\langle\xi|\xi\rangle\langle\eta|\eta\rangle} \right.$$

$$\| \qquad\qquad = \frac{|\langle\xi|\eta\rangle|^2}{\langle\xi|\xi\rangle\langle\eta|\eta\rangle} \left. \right)$$

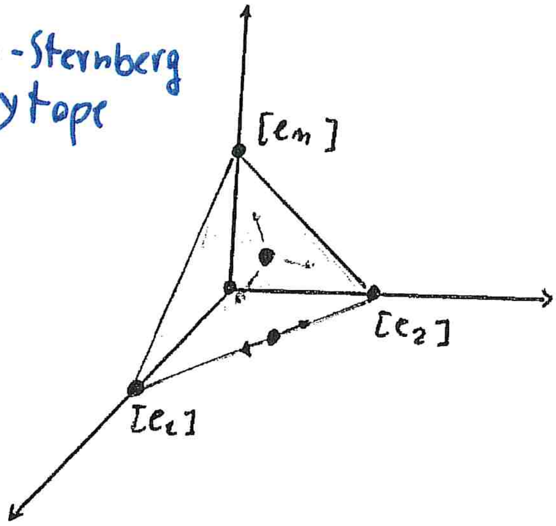
$$CR(\xi, \eta, \hat\xi, \hat\eta)$$

$$= \cos^2\left(\frac{\vartheta}{2}\right)$$

* Geometric picture of collapse

points of the symplex forced onto vertices (stationary states)

Atiyah – Guillemin-Sternberg polytope

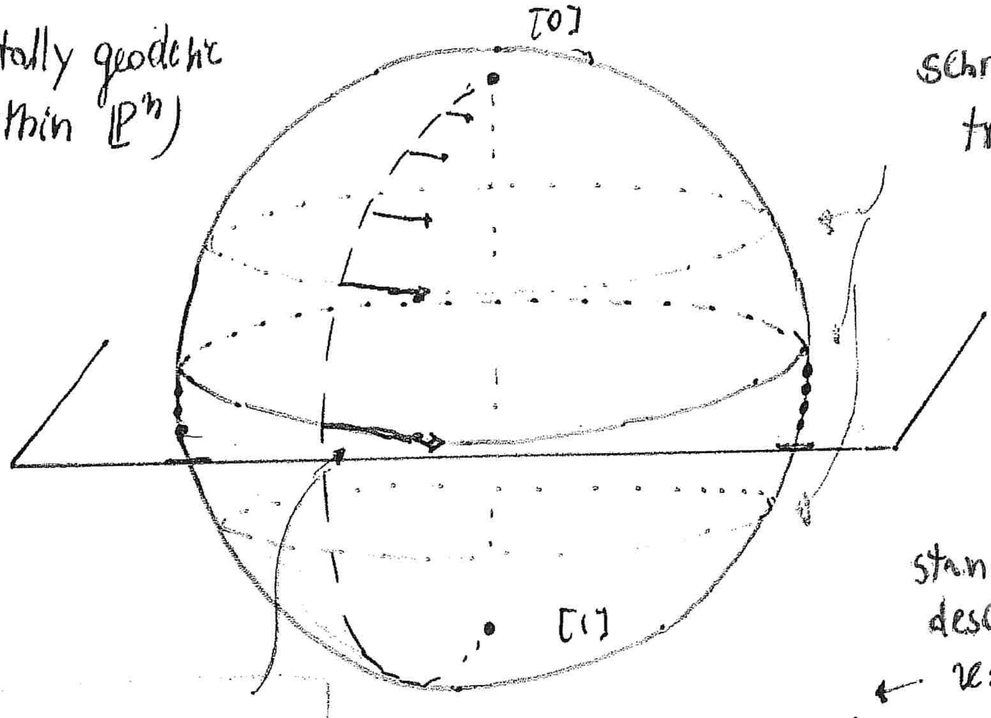


* collapse destroys adiabaticity (probability conservation)

The Bloch Sphere

1-qubit space

(totally geodesic within $\mathbb{P}^n$)



Schrödinger trajectories

standard description

ϑ: colatitude

$$|\psi\rangle = \cos\frac{\vartheta}{2}\,|0\rangle + e^{i\varphi}\sin\frac{\vartheta}{2}\,|1\rangle$$

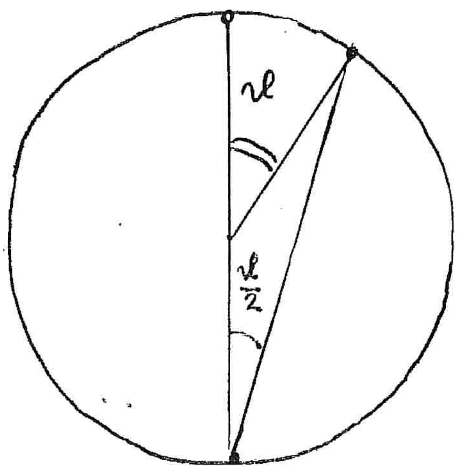
longitude

★ Jacobi field (B-S, RMP 2006)

$$J'' + KJ = 0$$
$$\parallel$$
$$4$$

$[R = \frac{1}{2}]$

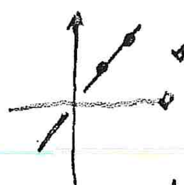
geodesic variations



$$\mathbb{P}^1 \cong S^3/S^1 \cong S^2$$

$$\mathbb{C}^2 \setminus \{0\} /\sim \;=\; S(\mathbb{C}^2)/\sim$$

phase equivalence



$$\frac{z_1}{z_0} = \tan\frac{\vartheta}{2}\cdot e^{i\varphi}$$

homogeneous coordinates $[z_0, z_1]$

unit sphere in $\mathbb{R}^4 \cong \mathbb{C}^2$

$$|z_0|^2 + |z_1|^2 = 1$$

$$S^3 \xrightarrow{S^1} S^2 \quad \text{Hopf fibration}$$

Entanglement

quantum mechanics

$$|\Psi\rangle \in \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2 = \mathcal{H}$$

$[\Psi]$ entangled if $\not\exists\, \overset{\mathcal{H}_1}{\psi_1} \quad \overset{\mathcal{H}_2}{\psi_2}$

$$|\Psi\rangle \neq |\Psi_1\rangle \otimes |\Psi_2\rangle$$

★ Geometric formulation

$$\mathbb{P}(\mathcal{H}_1) \times \mathbb{P}(\mathcal{H}_2) \xhookrightarrow{S} \mathbb{P}(\mathcal{H})$$

$$([v], [w]) \longmapsto [v \otimes w]$$

★ **Segre embedding**

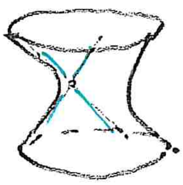
disentangled states: **intersection of Quadrics**

[see e.g. B-S, RMP 2006]

$$\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3 \qquad \text{example}$$

$$([z_0, z_1], [w_0\, w_1]) \mapsto (\underbrace{z_0 w_0}_{x_0}, \underbrace{z_1 w_0}_{x_1}, \underbrace{z_0 w_1}_{x_2}, \underbrace{z_1 w_1}_{x_3})$$

reguli



$$\boxed{x_0 x_3 = x_1 x_2}$$

Entanglement criteria

(A. Benvegnù, — 2006)

$\bar{V} \equiv$ qubit space $= \langle |0\rangle, |1\rangle \rangle \cong \mathbb{C}^2$

$\mathbb{P}(V^{\otimes n}) \longrightarrow [z_\gamma]$ homogeneous coordinates

projective space
dim $= 2^n - 1$

$\gamma = 0, 1, \ldots 2^n - 1$
(binary form)

$\alpha\, 0_k$

$\beta\, 1_k$

α — 0 — k

binary digits of $\alpha = 0, 1, \ldots 2^{n-1} - 1$

$X \subset \mathbb{P}^{2^n - 1}$

disentangled states

$\mathbb{C}^2$
$\Downarrow$
$[\xi_1 \otimes \xi_2 \ldots \xi_{2^n}]$

$\star$ Segre embedding

$\parallel$

common zero locus of

$$\boxed{Q_{\alpha, \beta, k} = z_{\alpha 0_k} z_{\beta 1_k} - z_{\alpha 1_k} z_{\beta 0_k} = 0}$$

$\alpha, \beta = 0, 1 \ldots 2^{n-1} - 1$, $k = 1, 2, \ldots n - 1$
$\alpha \neq \beta$

$\rightarrow$ quadric hypersurfaces

Chern-Bott calculations

$\|\psi\| = 1$

$$\langle \psi | d\psi \rangle (A^{\#})\Big|_{[4]} = \langle \psi | A \psi \rangle$$

$$\boxed{\nabla_{A^{\#}} \psi = \langle \psi d \psi \rangle \psi = \langle \psi | A \psi \rangle \psi}$$

$$\nabla_{B^{\#}} \nabla_{A^{\#}} \psi = B^{\#} \langle \psi | A \psi \rangle \psi + \langle \psi | A \psi \rangle \nabla_{B^{\#}} \psi$$

$$= \langle B\psi | A \psi \rangle + \langle \psi | A B \psi \rangle \psi$$

$$+ \langle \psi | A \psi \rangle \langle \psi | B \psi \rangle \psi$$

$$= \langle \psi | -BA \psi \rangle + \langle \psi | AB \rangle \psi + \langle \psi | A \psi \rangle \langle \psi B \psi \rangle \psi$$

$$= \{ \langle \psi [A, B] \psi \rangle + \langle \psi | A \psi \rangle \langle \psi B \psi \rangle \} \psi$$

$$\nabla_{A^{\#}} \nabla_{B^{\#}} \psi = \{ - \langle \psi [[A, B] \psi \rangle + \langle \psi | A \psi \rangle \langle \psi B \psi \rangle \} \psi$$

$$[\nabla_{A^{\#}}, \nabla_{B^{\#}}] = -2 \langle \psi | [A, B] \psi \rangle$$

$$\nabla_{[A,B]^{\#}} \psi = \langle \psi | [A, B] \psi \rangle$$

$$\nabla_{[A^{\#}, B^{\#}]} \psi = - \langle \psi [A, B] \psi \rangle$$

$$\parallel$$

$$\nabla_{-[A^{\#}, B^{\#}]} \psi = - \nabla_{[A^{\#}, B^{\#}]} \psi$$

$$\boxed{\Omega = [\nabla_{A^{\#}}, \nabla_{B^{\#}}] - \nabla_{[A^{\#}, B^{\#}]} = - \langle \psi | [A, B] \psi \rangle}$$

$$-2 \qquad + 1 \qquad = -1$$