GEOMETRIC METHODS IN QUANTUM MECHANICS

Mauro Spera - UCSC Brescia

Lecture IV Geometric (pre) quantization (Weil-Kostant theorem)

International Doctoral Program in Science

Brescia, Capitolium

GEOMETRIC QUANTIZATION

An introduction

$\boxed{\text{The basic idea}}$ "quantization"

$$(X, \omega) \overset{Q}{\dashrightarrow} \mathcal{H}_X \qquad \text{actually } P(\mathcal{H}_X)$$

symplectic manifold $\omega \in \Lambda^2(X)$ $d\omega = 0$
ω non degenerate
phase space of a classical dynamical system ⟿ pure states of the classical system

Hilbert space
(pure) states of the corresponding quantum system

projective space

If G is a Lie group acting symplectically on (X, ω) (i.e. $g^*\omega = \omega$ $\forall g \in G$) then Q should give a unitary representation of G on $\mathcal{H}_X$

G. Q. : construct Q exploiting the geometry of X
roughly : $\mathcal{H}_X$ manufactured from the space of sections of a complex line bundle $L \to X$

The celebrated Borel-Weil theorem is encompassed by G.Q.
BW yields the irreducible representations of semisimple Lie groups as spaces of holomorphic sections on suitable homogeneous Kähler manifolds

[The coherent state map plays a special role]

Prologue : outline of geometric quantization

- (M, ω) symplectic manifold — Classical phase space

  ω closed ($d\omega = 0$), non degenerate 2-form

Darboux : locally $\omega = dp \wedge dq$

$= d\vartheta \quad (\equiv d(p\,dq))$

ϑ : symplectic potential

$\dot{\lambda} = \{\lambda, H\}$

Hamilton

- $[\omega] \in H^2(M, \mathbb{Z})$ (de Rham) $\qquad \int_\Sigma \omega \in \mathbb{Z} \quad \forall \Sigma,\ \partial\Sigma = 0$ "integral flux"

integrality condition

$\Leftrightarrow$ **prequantization**

$\Rightarrow$ (Weil - Kostant)

$\exists \quad (L, \nabla, h)$ s.t. $\Omega_\nabla = -2\pi i\,\omega$

complex line bundle — connection — hermitian metric

compatible with

$L \to M$

$c_1(L) = [\omega]$

Chern class

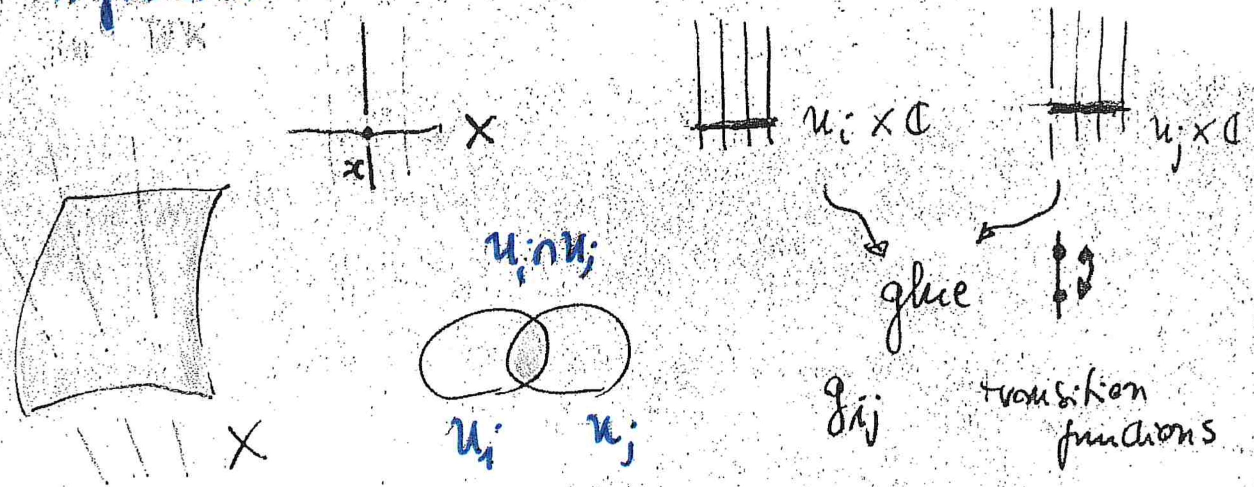
$\rightsquigarrow$ **prequantum** Hilbert space $\overline{\Gamma(L)}^{L^2}$

completion

"wave functions"

smooth sections of $L \to M$

* ingredients L_x L



$L \to X$ line bundle

$g_{ij}\, g_{jk}\, g_{ki} = 1$ on triple, non void, intersections

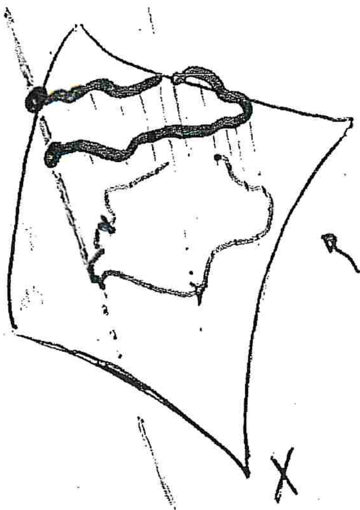
cocycle condition

☆ Cohomological description

Čech ~ $\check{H}^1(X, \underline{S}^1) \cong H^2(X, \mathbb{Z})$

metric...

Čech, singular, or de Rham up to torsion



$\nabla = d + \vartheta$ connection (covariant derivative)

$\Downarrow$ local expression

parallel transport

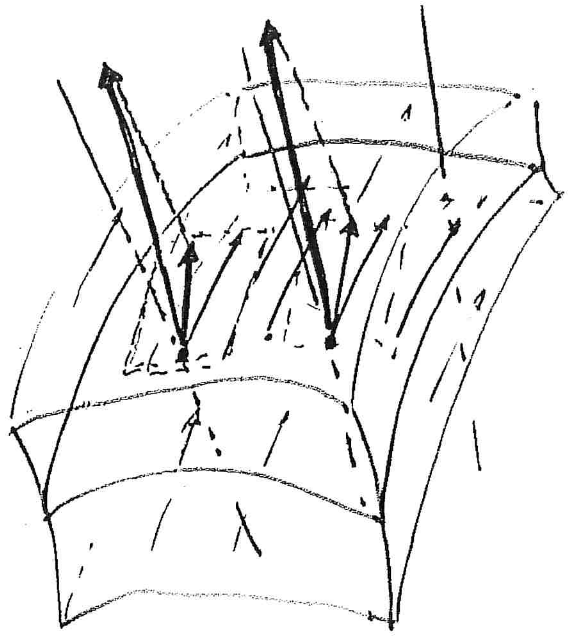
$\nabla^2 = \Omega$

$\Omega(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$

curvature

$\Omega = -2\pi i\, w$

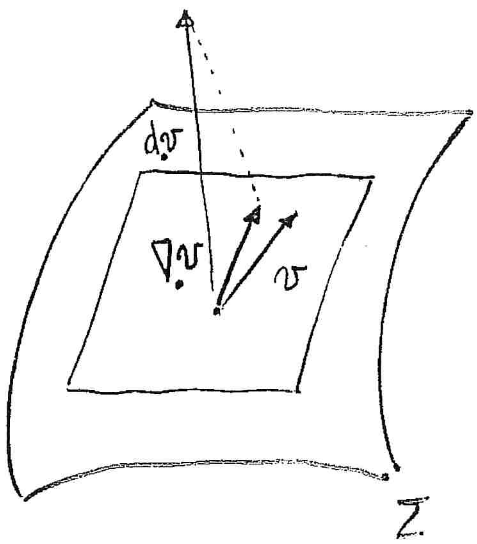
The Levi-Civita connection



projection onto tangent space

$$\nabla = \underline{P}\, d$$

★ geometric interpretation of the covariant derivative (Levi-Civita)



★ uniquely determined on any Riemannian manifold

- metricity
- torsion-free

parallel transport

$$\nabla_{\dot\gamma} v = 0$$

$$\nabla = d + \mathcal{A}$$

locally

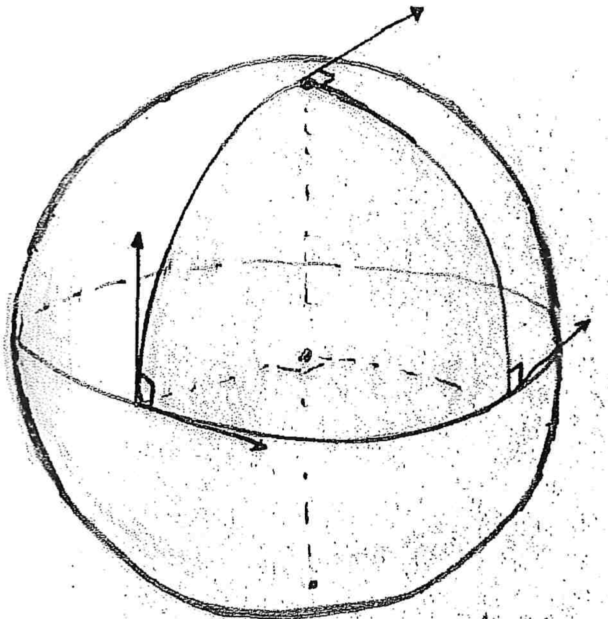
★ Curvature and topology

Chern - Weil

$$d\vartheta = \Omega$$

Stokes: $\displaystyle\int_{\mathcal{C}} \vartheta = \int_{D} \Omega$, $\quad \mathcal{C} = \partial D$

parallel transport on a sphere



viewed as a complex line bundle

$$c_1(T\Sigma) \in \mathbb{Z}$$

Gauss - Bonnet

Gaussian curvature

$$\frac{1}{2\pi}\int_{\Sigma} K = \chi(\Sigma) = 2 - 2g$$

Euler - Poincaré

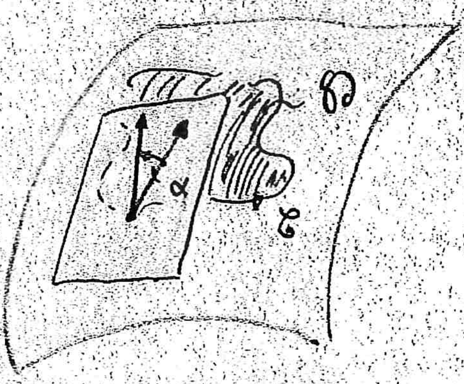
analysis / differential geometry

topology

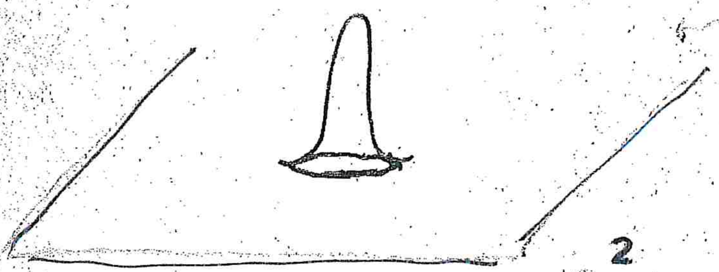
Σ

" $\nabla = d + \vartheta$ "

$$e^{i\alpha} = e^{\int_{\mathcal{C}} \vartheta} = e^{\int_{D} \Omega}$$



the prequantum space is too big



$\mathbb{R}^2$

phase space of the harmonic oscillator

arbitrarily localized wave functions are **forbidden**

★ Heisenberg Uncertainty Principle

need of a **polarization**

★ complex

holomorphic sections
(M Kähler)

The above situation are automatically ruled out

oscillator:
Bargmann - Fock representation

· Spin
· Kepler
· .....

projective embedding
(Kodaira)

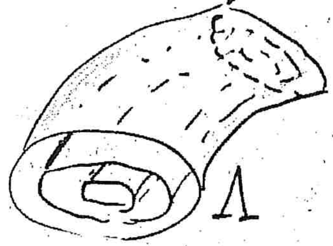
coherent states

(Simple) Lie group representation theory:

Borel - Weil

★ real

In particular, in the integrable case



Λ

Lagrangian submanifold

(cf $\psi = \psi(q)$)

look for **covariantly constant sections of** $\nabla|_\Lambda$ (flat...)

$\nabla s = 0$

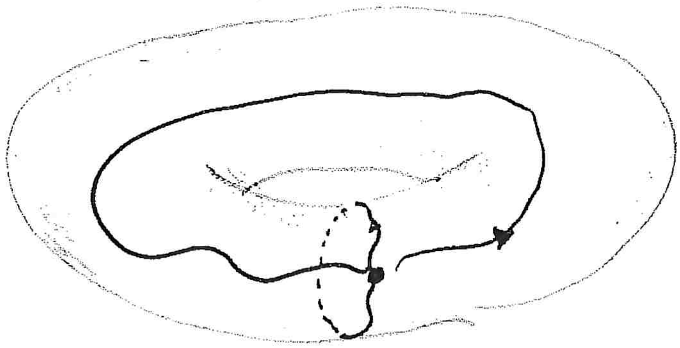
$s = s_0 \, e^{\frac{iS}{\hbar}}$

semi classical wave function

$$\Rightarrow \int_{\gamma} \vartheta \in 2\pi\mathbb{Z}$$

trivial holonomy — closed loop

* Bohr-Sommerfeld (without Maslov's correction



$$\frac{1}{2\pi}\int_{\gamma_i} \vartheta = n_i + \nu_i, \qquad \nu_i = \frac{\beta_i}{4} \qquad \beta_i \in \mathbb{Z}$$

* multivaluedness of the semiclassical wave function .. Keller, Maslov

$\Downarrow$

Knot theory

Polarizations

(M, ω) symplectic manifold

polarization : subbundle of $T^{\mathbb{C}}M$, F

$\operatorname{rk} F = \frac{1}{2} \dim M$

two natural choices:

(1) $\bar{F} = F$ real polarization

(2) $\bar{F} = F^{\perp}$ complex polarization

$\rightsquigarrow$ integrable almost complex structure I $\quad I^2 = -\mathbb{1}$

$F = T^{0,1}$

local descriptions $\mathbb{R}^{2n} \cong \mathbb{C}^n$

(1) $\langle \frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_n} \rangle$ or $\langle \frac{\partial}{\partial p_1} \dots \frac{\partial}{\partial p_n} \rangle$

(2) $\langle \frac{\partial}{\partial \bar{z}_1} \dots \frac{\partial}{\partial \bar{z}_n} \rangle$ ...... holomorphic polarization

(1) Lagrangian fibration (for c.i. systems)

$\pi : M \longrightarrow B$

B

$\pi^{-1}(b_{gen}) = \mathbb{T}^n$

(2) (ω, I) yields a Kähler metric with $[\omega] \in H^2(M, \mathbb{Z})$

Hodge metric

Some details BS quantization

$\chi : \pi_1(T) \to U(1)$ character

yielding the gauge class of ∇

T : Lagrangian submanifold **Bohr-Sommerfeld**

if $\chi = 1$ i.e $\exists$ covariantly constant

section of $(L, \nabla)|_T$

BS - fibres (tori) discrete

and, when B is compact $BS \subset B$ finite

B

cov. constant section

$$\boxed{Q_F(M) \equiv \mathcal{H}_\pi = \bigoplus_{b \in BS \subset B} \mathbb{C} \cdot s_b}$$

quantum Hilbert space

Similarly T BS of level k if $\chi^k = 1$

i.e. $\exists$ cov. constant section of $(L^k, \nabla_k)|_T$

Kähler quantization

$Q_F(M_I) = \ker \nabla^{0,1}$ Holomorphic sections

Check independence of I (complex structure)

(Hitchin's theory)

Amplification

★ $L \to M$ complex line bundle

connection

$$\nabla : \Omega^0(L) \longrightarrow \Omega^0(L) \otimes_{\mathcal{C}^\infty(M)} \Lambda^1(M)$$

- linearity

$$\nabla(\alpha s_1 + \beta s_2) = \alpha \nabla s_1 + \beta \nabla s_2$$

- Leibniz rule

$$\underset{\substack{\cap \\ \mathcal{C}^\infty(M)}}{\nabla(f} \underset{\substack{\cap \\ \Omega^0(L)}}{s)} = df\,s + f\,\underset{\substack{\shortparallel \\ A s}}{\nabla s}$$

local form $\nabla = d + A$

$$\nabla(fs) = (df + fA)s$$

covariant derivatives

$$\underset{\substack{\cap \\ \mathfrak{X}(M)}}{X} \longmapsto \underset{\substack{\text{linear} \\ \text{operator}}}{\nabla_X} : \Omega^0(L) \to \Omega^0(L)$$

$$\nabla_X(fs) = \underset{\substack{\shortparallel \\ X(f)}}{df(X)}\,s + f\,\nabla_X s$$

local description $M = \bigcup_\alpha \mathcal{U}_\alpha$ $\quad L = \bigcup_{\alpha \in A} \mathcal{U}_\alpha \times \mathbb{C}$

$\{g_{\alpha\beta}\}$ transition functions

$$s \rightsquigarrow \{s_\alpha\}$$

$s_\beta = g_{\beta\alpha} s_\alpha$ on non void overlappings

$$\nabla s_\beta = \nabla(g_{\beta\alpha} s_\alpha) = dg_{\beta\alpha} s_\alpha + g_{\beta\alpha} A_\alpha s_\alpha$$

$$= (dg_{\beta\alpha} + g_{\beta\alpha} A_\alpha) s_\alpha$$

$$\nabla s_\beta = A_\beta s_\beta = A_\beta g_{\beta\alpha} s_\alpha$$

$$A_\beta = A_\alpha + g_{\beta\alpha}^{-1} d g_{\beta\alpha}$$

$$A_\beta = A_\alpha + d \log g_{\beta\alpha}$$

If $|g_{\alpha\beta}| = 1$ $\log g_{\alpha\beta} = i f_{\alpha\beta}$

↑ choice of a branch

$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$ yields
cocycle identity $f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} \in 2\pi \mathbb{Z}$

given $\langle\ \rangle$ on $L \to M$ (metric), ∇ is compatible with it if $d\langle s, s'\rangle = \langle \nabla s, s'\rangle + \langle s, \nabla s'\rangle$

* Curvature First extend $\nabla : \Omega^0(L) \otimes_{C^\infty(M)} \Lambda^k(M)$
via $\qquad \to \Omega^0(L) \otimes_{C^\infty(M)} \Lambda^{k+1}(M)$

$$\nabla(\omega \otimes s) = d\omega\, s + (-1)^{\partial\omega}\, \omega\, \nabla s$$

$$R_\nabla \equiv \nabla^2 : \Omega^0(L) \to \Omega^0(L) \otimes \Lambda^2(M)$$

tensoriality

$$\nabla^2(fs) = \nabla(df\, s + f\nabla s) = \underbrace{d^2 f}_{0}\, s - \cancel{df\, \nabla s} + \cancel{df\, \nabla s} + f\nabla^2 s = f\nabla^2 s$$

$$\boxed{R_\nabla \in \underbrace{\mathrm{End}(L)}_{\substack{= L\otimes L^* \\ = \text{trivial}}} \otimes \Lambda^2 \cong \Lambda^2(M)}$$

in terms of covariant derivatives

$$R_\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

obstruction to get a Lie algebra representation
$X \to \nabla_X$

* R_∇ is closed

indeed

$$\nabla^2 s_\alpha = \nabla(\nabla s_\alpha) = \nabla(A_\alpha s_\alpha) =$$

$$= dA_\alpha s_\alpha + A_\alpha \nabla s_\alpha = dA_\alpha s_\alpha + A_\alpha \wedge A_\alpha s_\alpha$$

$$= \underbrace{(dA_\alpha + A_\alpha \wedge A_\alpha)}_{\text{Cartan}} s_\alpha = dA_\alpha s_\alpha \qquad \forall \alpha$$

R_∇ locally exact $\Rightarrow$ R_∇ closed

The set of all connections on $L \to M$ is actually an affine space modelled on $\Lambda^1(M)$

(a connection ∇^0 always exists (partition of unity...)

$$\nabla = \nabla^0 + \underset{\in \Lambda^1(M)}{a}$$

Then one also has, immediately:

$$R_\nabla = R_{\nabla^0} + da$$

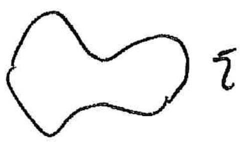
$$\Rightarrow \quad \frac{[R_\nabla]}{2\pi i} \in H^2(M, \mathbb{R}) \text{ is independent of } \nabla$$

$$\quad \| \quad c_1(L) \quad \text{1st Chern class of } L \to M$$

Theorem (i) $c_1(L)$ does not depend on ∇ (clear)

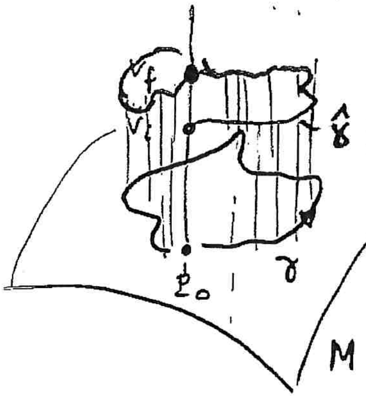
(ii) $c_1(L)$ is integral:

$$\boxed{\int_\Sigma \frac{R_\nabla}{2\pi i} \in \mathbb{Z}}$$



for all closed surfaces in M

ad (ii)



v_i, v_f in the same fibre

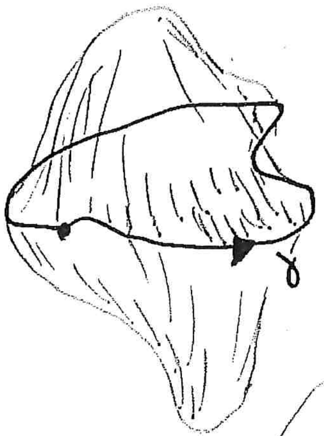
$\leftarrow$ horizontal lift

+ parallel transport

$$v_f = e^{\int_\gamma A} v_i$$

$|\ \ | = 1$ if the connection is metric

Now the crucial point is the following :

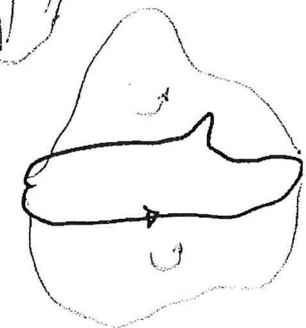


if $\gamma = \partial \Sigma$

then (Stokes)

$$v_f = e^{\int_\Sigma R_\nabla} v_i$$

hence Σ closed



$$e^{\int_\Sigma R_\nabla} = 1$$

whence the assertion

Conversely, one has the **Weil - Kostant theorem**

$$[\omega] \in H^2(X, \mathbb{Z}) \qquad X = \bigcup_{\alpha \in \Omega} \mathcal{U}_\alpha$$

good covering

$$d\omega = 0 \qquad \omega_\alpha = \omega_\beta$$

on non-empty overlaps

$$\Rightarrow \omega_\alpha = dA_\alpha$$

$$d(A_\alpha - A_\beta) = 0 \Rightarrow A_\alpha - A_\beta = df_{\alpha\beta}$$

Then

$$0 = (A_\alpha - A_\beta) + (A_\beta - A_\gamma) + (A_\gamma - A_\alpha) = d(\underbrace{f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha}}_{c_{\alpha\beta\gamma} \in 2\pi i \mathbb{Z}})$$

$$[c] \in \check{H}^2(X, \mathbb{Z}) \qquad \text{so } \delta c = 0$$

$$\underbrace{e^{f_{\alpha\beta}}}_{g_{\alpha\beta}} \underbrace{e^{f_{\beta\gamma}}}_{g_{\beta\gamma}} \underbrace{e^{f_{\gamma\alpha}}}_{g_{\gamma\alpha}} = e^{c_{\alpha\beta\gamma}} = 1$$

$\{g_{\alpha\beta}\}$: transition functions of a complex line bundle $L \to X$

$$[g] \in H^1(X, \underset{\mathbb{C}^*}{S^1}) \cong H^2(X, \mathbb{Z})$$

i.e. $P \xrightarrow{S^1} X$ principal S^1-bundle

$$\underset{[c]}{[\omega]} = c_1(L)$$

$$A_\alpha - A_\beta = g_{\alpha\beta}^{-1} d g_{\alpha\beta}$$

$$= d \log g_{\alpha\beta}$$

$$\boxed{\begin{array}{ll} A & \text{connection} \\ \omega & \rightsquigarrow \text{curvature} \end{array}}$$

$$(\Omega = -2\pi i \omega)$$

Quantization (Dirac's recipe)

$$\{f, g\} = \ell \xrightarrow{Q} [\underset{\substack{\shortparallel \\ Q(f)}}{\hat{f}}, \hat{g}] = i\hbar \hat{\ell}$$

curvature

geometric $\Omega = -2\pi i\, w$

symplectic potential

$$\nabla_X := X - \frac{i}{\hbar} \langle \underset{\substack{\wr \\ d\vartheta = w}}{\vartheta}, X \rangle$$

$$Q(f) := i\hbar \nabla_{X_f} + f = i\hbar X_f + \langle \vartheta, X_f \rangle + f$$

$$[Q(f), Q(g)] = i\hbar\, Q(\ell)$$

ok at prequantum level

obstructions arise

★ need polarization preservation

Also: quantization should be independent of the choice of a polarization