GEOMETRIC METHODS IN QUANTUM MECHANICS

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Lecture IX : Hydrodynamical aspects of QM

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GEOMETRIC ASPECTS OF GENERALIZED SCHRÖDINGER EQUATIONS

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Moment map, gauge-theoretic, Fisher information aspects

1. Moment maps

M: smooth manifold
ω: closed, non degenerate 2-form

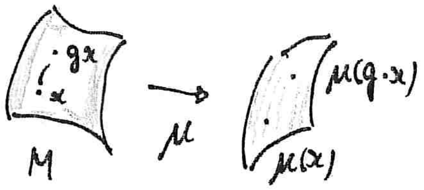
(M, ω) symplectic manifold acted upon by G (Lie group), symplectically: $\mathcal{L}_{\xi^\#}\, \omega = 0$

($\xi^\#$ fundamental vector field associated to $\xi \in \mathfrak{g} = \mathrm{Lie}(G)$)

$$\boxed{\mu : M \longrightarrow \mathfrak{g}^*}$$

(G-equivariant) moment map

$$\mu(g \cdot x) = \underbrace{Ad^*(g)}_{\text{coadjoint action}}\, \mu(x) \qquad x \in M,\ g \in G$$



a G-orbit in M is mapped to a coadjoint orbit

μ yields $\boxed{\lambda_x(x) := \langle \mu(x), x \rangle}$ with $\boxed{d\lambda_x = \imath_x \omega}$

(under mild conditions)

$$\boxed{\Lambda = \{ \lambda_x \mid x \in \mathfrak{g} \}}$$ Hamiltonian algebra (Raselli - Regge current algebra)

$\{ \lambda_x, \lambda_v \} = \lambda_{[x,v]}$ (+ cocycle measuring lack of equivariance)

Poisson bracket induced by ω

$x \longmapsto \lambda_x$ comoment map

Hamilton equations $\boxed{\dot{\lambda}_x = \{ \lambda_x, H \}}$

2. An infinite dimensional example (cf Donaldson 99, Penna - S. 92)

S : (compact) manifold with fixed volume form σ

(M,ω) : symplectic manifold

$$\boxed{\mathcal{M} = \{ f : S \to M \mid f \text{ in a fixed homotopy class} \}}$$

$T_f\mathcal{M}$ = sections of $f^*(TM) \to S$ tangent space of $\mathcal{M}$ at f

$$\boxed{\Omega(v,w) := \int_S \omega(u,v)\,\sigma}$$ (abuse of notation)

symplectic form

G : volume preserving diffeomorphisms of S acts on $\mathcal{M}$ via composition on the right and preserves Ω. Assume $f^*([\omega]) = 0$ in $H^2(S)$ $[\ ]$: de Rham class. Hence $f^*\omega = da$

Set $\xi \in \mathfrak{g}$: "divergence-free" v. fields

$$(a,\xi) := \int_S a(\xi^\#)\,\sigma \quad \text{and get:}$$

$$\boxed{\underset{\underset{\mathcal{M}}{\cap}}{\mu(f)} := \{ \xi \mapsto (a,\xi) \} \in \mathfrak{g}^*} \quad \boxed{\textbf{moment map}}$$

3. The Schrödinger equation (SE)

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi \equiv H\psi}$$

ψ : wave function; m : mass; $V(x)$: potential (energy); H : quantum Hamiltonian

$\hbar = \frac{h}{2\pi}$

point particle of mass m in 3-space

★ SE à la Madelung - Bohm

$$\boxed{\begin{cases} \dfrac{\partial \rho}{\partial t} = -\operatorname{div}(\underbrace{\rho \nabla S}_{j}) & \text{continuity equation} \\[2ex] \dfrac{\partial S}{\partial t} = -\left[\dfrac{1}{2m}\|\nabla S\|^2 + V(x) \underbrace{- \dfrac{\hbar^2}{2m}\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}}_{Q} \right] & \text{generalized Hamilton-Jacobi} \end{cases}}$$

j : probability current

Q : quantum potential

If $\boxed{\mathcal{H} := \langle \psi | H \psi \rangle = \int \psi^\dagger \left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi \, d^3x = \int \left\{ \frac{\hbar^2}{2m} \|\nabla\psi\|^2 + V(x)|\psi|^2 \right\} d^3x}$

the SE can be reformulated as a Hamilton equation (Bohm)

$$\boxed{\begin{cases} \frac{\partial \rho}{\partial t} = \frac{\delta \mathcal{H}}{\delta S} \\ \frac{\partial S}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \rho} \end{cases}}$$

functional derivatives

★ Geometrize:

$$\boxed{\mathcal{M} = \{ \Phi : \mathbb{R}^3 \to \mathbb{C} \; ; \; x \mapsto (\rho(x), \$(x)) \}} \quad \omega = d\rho \wedge d\$$$

S^3

polar coordinates

$$\boxed{\Omega = \int_{\mathbb{R}^3} d^3x \, \delta\rho(x) \wedge \delta\$(x) = \int_{\mathbb{R}^3} \langle W, \cdot \times \cdot \rangle}$$

$\left\{ \begin{array}{l} G = \mathrm{Sdiff}(\mathbb{R}^3) \\ \mathfrak{g} = \{ b \mid \mathrm{div}\, b = 0 \} \end{array} \right.$ + conditions at ∞

importance of G: statistics (cf. Goldin et al)

$$\boxed{W = \nabla\rho \times \nabla\$} = \mathrm{curl}\,(\underbrace{\rho\nabla S}_{j})$$

vorticity

or $w^b = dj^b$

$j = \rho\nabla\$$

$j^b = \rho\, d\$$

so $\quad a \leftrightarrow j^b \qquad \xi \longleftrightarrow b$

$$\boxed{\begin{array}{lll} \Lambda = \{ \lambda_b \mid \mathrm{div}\, b = 0 \} & \lambda_b = \int j \cdot b = \int w \cdot B & \mathrm{curl}\, B = b \\ \text{SE:} \quad \dot{\lambda}_b = \{ \lambda_b, \mathcal{H} \} & \{ \lambda_b, \lambda_c \} = \lambda_{[b,c]} & \text{RR} \end{array}}$$

moment map

Arnold hydrodynamical bracket

$$\boxed{\mu(\Phi) = \{ b \mapsto \lambda_b \} \rightsquigarrow j_\Phi \rightsquigarrow W_\Phi \in \mathfrak{g}^*_{\mathfrak{g}}}$$

$[j]$.

For a charged spinless particle $H = -\frac{\hbar^2}{2m}\left(\nabla - \frac{iq}{\hbar c}A\right)^2 - q\Phi =$ electric potential

$= \frac{1}{2\pi}\left(p - \frac{q}{c}A\right)^2 - q\Phi \quad , \quad j \mapsto j^A = j + \frac{q}{\hbar c}\rho A \; , \; w^A = \mathrm{curl}\, j^A$

RR Hamiltonians $\lambda^A_b = \int j^A \cdot b$

$$\boxed{\{ \lambda^A_b, \lambda^A_c \} - \lambda^A_{[b,c]} = -\int \rho \underset{\substack{\| \\ \mathrm{curl}A}}{B} \cdot b \times c}$$

magnetic field

magnetic obstruction to equivariance

4. The Pauli equation

(cf. Bohm et al)

$$\boxed{i\hbar \frac{\partial \Psi}{\partial t} = \frac{1}{2m}\left[(\sigma\cdot(p-qA))^2 + q\Phi\right]\Psi \equiv H\Psi}$$

or

Ψ: spinor wave function

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{1}{2m}\left\{\left[(p-qA)^2 - q\hbar\, \sigma\cdot B\right] + q\Phi\right\}\Psi$$

use Cayley-Klein parameters $\begin{bmatrix} a_1 + i a_2 \\ a_3 + i a_4 \end{bmatrix}$ $\{a_1,a_2\} = \{a_3,a_4\} = 1$, $\{a_1,a_3\} = \{a_2,a_4\} = 0$

$$\boxed{\Psi = \sqrt{\rho}\, e^{iS/\hbar}\chi = \sqrt{\rho}\, e^{iS/\hbar}\frac{1}{\sqrt{2\hbar}}\begin{bmatrix} a_1+ia_2 \\ a_3+ia_4 \end{bmatrix}}$$

$$\chi \equiv \begin{bmatrix} \cos\frac{\theta}{2}\, e^{iW/\hbar} \\ \sin\frac{\theta}{2} \end{bmatrix}$$

symplectic form on $\mathcal{M} = C_c^\infty(\mathbb{R}^3, \underbrace{\mathbb{R}^2\times\mathbb{R}^2}_{M} = \mathbb{R}^4)$

$$\Omega = \int \delta a_1\wedge\delta a_2 + \delta a_3 \wedge \delta a_4 = \int \delta\rho\wedge\delta S + \delta\rho_3\wedge\delta W$$

ρ, S, ρ_3, W — canonical variables, with $\rho_3 = \rho\cos^2\frac{\theta}{2}$

Hamilton:

$$\left\{\begin{array}{ll} \dfrac{\partial\rho}{\partial t} = \dfrac{\delta\mathcal{H}}{\delta S} & \dfrac{\partial S}{\partial t} = -\dfrac{\delta\mathcal{H}}{\delta\rho} \\ \dfrac{\partial\rho_3}{\partial t} = \dfrac{\delta\mathcal{H}}{\delta W} & \dfrac{\partial W}{\partial t} = -\dfrac{\delta\mathcal{H}}{\delta\rho_3} \end{array}\right.$$

$\mathcal{H} = \int \Psi^\dagger H \Psi$

$$\boxed{j^\flat = \rho\, dS + \rho_3\, dW \qquad w^\flat = d\rho\wedge dS + d\rho_3\wedge dW}$$

"spin fluid" $v_f^\flat = \xi\, d\eta$ $\quad w_f^\flat = d\xi\wedge d\eta$ $\qquad \xi = \cos^2\frac{\theta}{2}$, $\eta = W/\hbar$

$$\mu : \mathcal{M} \to \mathfrak{g}^*$$

$$\Phi = \{x \mapsto (\rho(x), S(x), \rho_3(x), W(x))\} \longmapsto \mu(\Phi) := j$$

actually $[j]$

$$\dot{\lambda}_b = \{\lambda_b, \mathcal{H}\}$$

← Poisson stemming from Ω

6. Gauge geometry of the SE

$$\psi = \sqrt{\rho}\, e^{iS/\hbar}, \qquad \sqrt{\rho} = |\psi|$$

$$\boxed{A = \psi^{-1} d\psi = d \log \psi = \underbrace{\tfrac{1}{2} d \log \rho}_{\mathbb{R}^+} + \underbrace{\tfrac{i}{\hbar} dS}_{U(1)}}$$

$$\boxed{\begin{aligned} G &= \mathbb{C}^* = \mathbb{C} \setminus \{0\} \\ &\cong \mathbb{R}^+ \times U(1) \\ \mathfrak{g} &= \mathbb{R} + i\mathbb{R} \end{aligned}}$$

Maurer - Cartan (MC) gauge field

flat connection on the trivial line bundle over (a region of) $\mathbb{R}^3$

Weyl gauge equivalence

$$(g, \varphi) \sim (\lambda g, \varphi - \tfrac{1}{2} d \log \lambda)$$

metric; 1-form governing parallel transport of lengths; $\lambda > 0$ conformal factor

$\rightsquigarrow$ canonical connection (cf. Levi-Civita)

Present case $(\mathbb{R}^3, (g_0, \frac{1}{2} d \log \rho))$

g_0: standard metric

One finds (Santamato '85)

$$\boxed{R_W = -4 \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \propto Q}$$

R_W: Weyl scalar curvature; Q: quantum potential

Bohm interpretation

$$\boxed{F = -\nabla \mathcal{V} = -\nabla V - \underbrace{\frac{\hbar^2}{8m} \nabla R_W}_{-\nabla Q}}$$

force acting on the particle

Reconstruction of SE

Start from the MC - field A

1. Re A = Weyl field governing parallel transport of lengths $\mathbb{R}^+$
2. Im A : particle content $p = \nabla S$
3. Require continuity

particle acted upon by a potential $V + Q$, $Q \propto$ Weyl scalar curvature

4. The two equations yield SE

Weyl field: "pilot wave" $\longrightarrow$ osmotic velocity in stochastic QM (Nelson; Guerra-Marra)

Geometric origin of Q : Clue to linearity of SE

5. generalized Schrödinger equations (GSE)

$$\boxed{i\dot{\psi} = \nabla^{\dagger}\nabla\psi = -\nabla^{2}\psi \equiv H\psi}$$

rough laplacian

typical fibre V carrying a f.d. unitary representation of G compact, simple → section of a complex G-vector bundle $\mathcal{E} \to \mathbb{R}^3$ with connection ∇

$\mathcal{S}$: compactly supported sections of $\mathcal{E} \to \mathbb{R}^3$

$$\Omega = i\int \delta\psi^{\dagger} \wedge \delta\psi \qquad \text{symplectic form}$$

$$j^{b} = \mathrm{Im}\,(\psi^{\dagger}\nabla\psi) \qquad \text{probability current}$$

$$\nabla = d + \underbrace{A \cdot I}_{U(1)} + \underbrace{C}_{G}$$

$$\mu : \mathcal{S} \ni \psi \longrightarrow [j] \in \mathrm{Lie}(\mathrm{Sdiff}(\mathbb{R}^3))$$

moment map (equivariant if $A = C = 0$)

Hamiltonian structure:

$$\left\{ \begin{aligned} \frac{\partial\psi}{\partial t} &= -i\,\frac{\delta\mathcal{H}}{\delta\psi^{\dagger}} \\ \frac{\partial\psi^{\dagger}}{\partial t} &= +i\,\frac{\delta\mathcal{H}}{\delta\psi} \end{aligned} \right. \qquad \mathcal{H} = \int \psi^{\dagger} H \psi$$

$$\boxed{\dot{\lambda}_b = \{\lambda_b, \mathcal{H}\}}$$

∇ flat $\Rightarrow$ generalized Aharonov-Bohm & Aharonov-Casher effects emerging via the corresponding monodromy representation

see also below

7. Gauge geometry of the Pauli Equation (PE)

$$\boxed{\begin{array}{l} G = \mathbb{R}^{+} \times U(1) \times SU(2) \\ \text{MC gauge field:} \quad A = \frac{1}{2}\, d \log \rho + g^{-1} dg \qquad \rho \in \mathbb{R}^{+} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad g \in U(1) \times SU(2) \end{array}}$$

$$\psi = \sqrt{\rho}\, e^{iS/\hbar}\, \tilde{g}\, \psi_0 \equiv \sqrt{\rho}\, g\, \psi_0$$

ψ_0 ↖ reference unit spinor

Spin representation

$V = \mathbb{C}^2 \qquad V \setminus \{0\} \to P(V) \cong S^2$

$SU(2)$ - homogeneous line bundle $\mathcal{L} \to S^2$

$\mathcal{L}_y = \mathbb{C}\, \chi(y) \quad y \in S^2 \qquad \nabla \chi := \langle \chi | d\chi \rangle \chi$

Chern - Bott (Berry) connection

$\langle \chi | d\chi \rangle = \chi^{+} d\chi = \ldots\, i\, v_f^{b}$ ~ velocity of the "spin fluid"

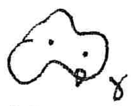
$\Omega_\nabla = i\, w_f^{b}$ ~ vorticity ⇝ monopole bundle

Upshot: The trivial rank 2 vector bundle over $\mathbb{R}^3$, with structure group G and equipped with the MC connection (flat) is reduced to a $U(1)$-line bundle with non-trivial curvature given by the vorticity of the spin fluid

8. Topological aspects

$\psi = |\psi|\, e^{iS/\hbar}$

$$\oint_\gamma dS \in 2\pi\hbar\, \mathbb{Z} = h\mathbb{Z}$$



(ψ must be well-defined) Bohr - Sommerfeld

Aharonov-Bohm Effect à la MC

Shift of interference pattern due to a solenoid

$$\psi^{-1} \nabla^{A} \psi := \psi^{-1} \left(d - \frac{iq}{\hbar c} A\right) \psi = \frac{1}{2}\, d \log \rho + i \left[\frac{dS}{\hbar} - \frac{q}{\hbar c} A \right]$$

⇝ quantization condition

$$\oint \left(dS - \frac{q}{c} A\right) \in h\mathbb{Z}$$

the phase variation of the wave functions must compensate the holonomy of A ~ ψ section of a suitable line bundle

9. Geometric quantum mechanics & Fisher information

Let $(H, \langle | \rangle)$ a complex separable Hilbert space and $\mathbb{P}(H)$ be its associated projective space (pure states of a quantum system). Following Marmo et al notation

$$\boxed{h(\psi,\psi) = \frac{\langle d\psi | d\psi\rangle}{\langle \psi|\psi\rangle} - \frac{\langle d\psi|\psi\rangle\langle\psi|d\psi\rangle}{\langle\psi|\psi\rangle^2}}$$

h: quantum Fisher information (hermitian) = Fubini-Study hermitian metric on $\mathbb{P}(H)$

Chern-Bott connection on $\mathcal{O}(-1) \to \mathbb{P}(H)$ (tautological bundle on $\mathbb{P}(H)$, associating to a non zero vector ψ the line $[\psi]$ it represents

$$\boxed{\nabla\psi := -\langle\psi|d\psi\rangle\,\psi}$$

with curvature $\boxed{\Omega = d\langle\psi|d\psi\rangle = \langle d\psi|d\psi\rangle}$

Dual of $\mathcal{O}(-1)$: hyperplane section bundle $\mathcal{O}(1)$

Borel-Weil theory & coherent states (aside)

V finite dimensional vector space carrying an irreducible representation of a compact, simple Lie group G, with associated projective space $\mathbb{P}(V)$. Then V becomes the total space of a complex, G-homogeneous line bundle $\mathcal{L} \to Y$ with $Y \hookrightarrow \mathbb{P}(V)$ compact Kähler manifold $Y = G/H$

Namely, if $|0\rangle \in V$ is a (regular) highest weight vector, consider the vectors $U(g)|0\rangle$, $g \in G$ (coherent state vectors). The G-orbit in $\mathbb{P}(V)$ $(\{[U(g)|0\rangle], g\in G\})$ is the coherent state manifold Y and H = isotropy group of $|0\rangle$. The fibre $\mathcal{L}_y$ is the complex line corresponding to $y = [U(g)|0\rangle]$ for some g determined up to $h \in H$. Upshot: V becomes the space of holomorphic sections of the dual bundle $\mathcal{L}^* \to Y$, which carries a natural hermitian metric and an ensuing Chern-Bott connection. The $SU(2)$ case leads to monopole bundles (cf. the section on the Pauli equation).

11. Comparison with Fusca, Khesin - Misiołek - Modin

* Main theorem of KMM: Madelung transform : Kähler morphism between the cotangent bundle of the space of smooth probability densities equipped with the (Sasaki)- Fisher - Rao metric, and an open subset of the infinite dimensional complex projective space of smooth wave functions, equipped with the Fubini - Study metric

Amplification

M compact connected n-manifold, μ volume form $\int_M \mu = 1$

$$\boxed{\mathrm{Dens}^s(M) = \{\rho \in H^s(M) \mid \rho > 0 , \int_M \rho\mu = 1\}}$$ (Hilbert or Fréchet manifold)

Sobolev $s > \frac{n}{2}$ or $s = \infty$

$$\boxed{T\,\mathrm{Dens}^s(M) = \mathrm{Dens}^s(M) \times H^s_0(M)} \qquad T^*\mathrm{Dens}^s(M) = \mathrm{Dens}^s(M) \times H^s(M)/\mathbb{R}$$

$H^s_0(M) = \{c \in H^s(M) \mid \int_M c\mu = 0\}$

regular part

pairing: $(\dot\rho, [\theta]) \mapsto \int_M \theta\dot\rho\mu$ indep of θ since $\int_M \dot\rho\mu = 0$

Madelung transform : $\Phi : (\rho, \theta) \mapsto \psi = \sqrt{\rho}\, e^{i\frac{\theta}{2}}$ caveat

$$\Omega^{T^*\mathrm{Dens}}_{(\rho,[\theta])}(\dot\rho_1, [\dot\theta_1], \dot\rho_2, [\dot\theta_2]) = \int_M (\dot\theta_1\dot\rho_2 - \dot\theta_2\dot\rho_1)\mu$$ symplectic structure on $T^*\mathrm{Dens}^s$

Fisher - Rao $\quad g_\rho(\dot\rho, \dot\rho) = \frac{1}{4}\int_M \dot\rho^2/\rho\ \mu$ (diff invariant)

Sasaki - Fisher - Rao (lift to $T^*\mathrm{Dens}^s$) $\quad g^*_{(\rho,[\theta])}((\dot\rho,\dot\theta),(\dot\rho,\dot\theta)) = \frac{1}{4}\int_M \left(\frac{\dot\rho^2}{\rho} + \dot\theta^2\rho\right)\mu$

Kähler structure of $T^*\mathrm{Dens}^s(M)$ $\quad J_{(\rho,[\theta])}(\dot\rho, \dot\theta) = \left(\dot\theta\rho, -\frac{\dot\rho}{\rho}\right)$ cf. j cf. Weyl

Inverse Madelung Transform $\quad M : H^s(M, \mathbb{C}) \to \Omega^1(M) \times \mathrm{Dens}^s(M)$

$$\psi \longmapsto (m, \rho) = (2\,\mathrm{Im}(\psi^\dagger d\psi), \psi^\dagger\psi)$$

$\mathfrak{s} = \mathfrak{X}(M) \ltimes H^s(M, \mathbb{R}) = \mathrm{Lie}(S = \mathrm{Diff}\,M \ltimes H^s(M))$

$M : H^s(M, \mathbb{C}) \to \mathfrak{s}^*$ is (up to scaling by 4) a moment map

cf. the previous pages

10. Fisher information and coherent states

Let $X = \mathbb{R}^3$ (Lebesgue measure), $H = L^2(X, V) \cong L^2(X) \otimes \bar{V}$ consisting of wave functions $\psi = \sqrt{\rho}\, e^{is} \chi$, $\chi \in \bar{V}$, $\langle \chi | \chi \rangle = 1$ $\left(\parallel \chi^\dagger \chi\right)$

($\bar{V}$ as before). Also $\langle \psi | \psi \rangle = \int_X \rho \chi^\dagger \chi = \int_X \rho = 1$

(Let ρ smooth and rapidly vanishing at infinity)

Let f a generic tensor, let $\Theta = Y$ be a parameter space (the coherent state manifold); define

$$\boxed{E[f] \equiv E_\rho[f] := \int_X \rho f}$$

Take the pull-back of $h_X(\psi, \psi)$ to $\Theta = Y$, the quantum Fisher information

$$g_X = \mathrm{Re}\left\{ \frac{\langle d\psi | d\psi \rangle}{\langle \psi | \psi \rangle} - \frac{\langle d\psi | \psi \rangle \langle \psi | d\psi \rangle}{\langle \psi | \psi \rangle^2} \right.$$

reads (S '18)

$$\boxed{g_X = \int_X \rho \left\{ \frac{1}{4} \left[\frac{d\rho}{\rho}\right]^2 + ds^2 + d\chi^\dagger d\chi \right\} - \left\{ \int_X \rho \left(ds + (-i) \chi^\dagger d\chi\right) \right\}^2}$$

"internal" part

$$= \underbrace{E\left[\frac{1}{4}\left(\frac{d\rho}{\rho}\right)^2\right]}_{\text{"classical" Fisher information}} + E[ds^2] + E[g_{FS}([\chi])) - 2\, \underbrace{E[ds]}_{\int \rho\, ds}\, \underbrace{E[(-i)\chi^\dagger d\chi]}_{\int \rho (-i \chi^\dagger d\chi)}$$

$-\mathcal{V}_j$

"coupling" term

$\bar{V}$ 1-dim $\to$ Marmo & al

cf. Reginatto Fisher-information term of the Pauli Lagrangian

$$L_B = \int \rho \left\{ \underbrace{\frac{\|\nabla \rho\|^2}{\rho^2}}_{\text{classical Fisher}} + \underbrace{\|\nabla \theta\|^2 + \sin^2\theta\, \|\nabla \omega\|^2}_{\text{internal part}} \right\} d^3x\, dt$$

cf. page 4