GEOMETRIC METHODS IN QUANTUM MECHANICS

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Lecture V: Hamiltonian systems & (holomorphic) geometric quantization: general aspects

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Hamiltonian systems $(\underbrace{M,\omega}_{\text{symplectic manifold}}, \underset{\text{Hamiltonian}}{H})$

Hamiltonian vector field: $\underset{\text{symplectic gradient}}{X_H}$ s.t. $i_{X_H}\omega = dH$

X_H exists since ω is non degenerate

X_H is of course symplectic: $\mathcal{L}_{X_H}\omega \overset{\text{cartan } \mathcal{L}=di+id}{=} (di_{X_H} + i_{X_H}d)\omega$

$= d(i_{X_H}\omega) = d^2H = 0$. If M is simply connected every symplectic v.f. is hamiltonian

* Poisson brackets: $\{f,g\} := \omega(X_f, X_g)$

$(\mathcal{C}^\infty(M), \{\ \})$ (Poisson)-Lie algebra

If G acts symplectically on M, one has

$$\mathcal{L}_{\xi^\#}\omega = 0 \qquad d(i_{\xi^\#}\omega) = 0$$

fundamental vector field pertaining to $\xi \in \mathfrak{g}$ i.e. $\xi^\#$ symplectic

Under suitable Lie algebraic cohomological / topological conditions a hamiltonian action is induced:

$\exists !\ \lambda_\xi$ fulfilling $i_{\xi^\#}\omega = d\lambda_\xi$

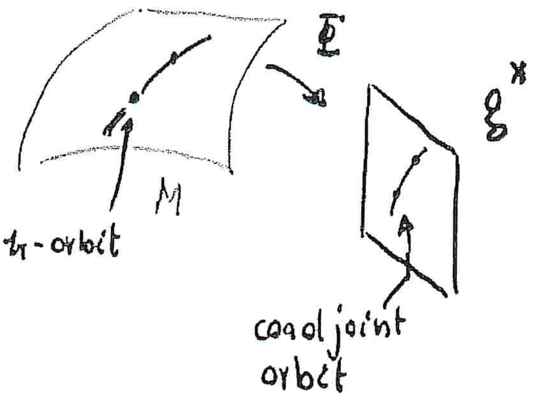
$\Lambda = \{\lambda_\xi \,/\, \xi \in \mathfrak{g}\}$ becomes a Lie algebra $\cong \mathfrak{g}$

$$\{\lambda_\xi, \lambda_\eta\} := \omega(\xi^\#, \eta^\#) = \ldots = \lambda_{[\xi,\eta]}$$

* Hamiltonian algebra (in vortex theory: Rasetti-Regge current algebra)

This turns out to be equivalent to the existence of an equivariant moment map $\Phi : M \longrightarrow \mathfrak{g}^*$:

i.e. Φ intertwines the G-action on M with the coadjoint action : $\Phi(g \cdot x) = Ad^*(g)\, \Phi(x)$



Then

$$\lambda_\xi(x) = \langle \Phi(x), \xi \rangle$$

For a coadjoint orbit itself :

$$\lambda_\xi(f) = \langle f, \xi \rangle$$

$f \in \mathcal{O}_{f_0}$

$$\boxed{M_f := \Phi^{-1}(f) / G_f}$$

* Marsden - Weinstein quotient Mayer (symplectic reduction)

under suitable assumptions

M_f becomes a symplectic manifold

integrality of $\mathcal{O}_f$

G_f : stabilizer of f

$$\rho_f : \xi \in \mathfrak{g}_f \longmapsto 2\pi i \langle f, \xi \rangle \in i\mathbb{R}$$

ρ_f : infinitesimal character (i.e., vanishes on $[\mathfrak{g}_f, \mathfrak{g}_f]$)

f integral if $\exists$ $\chi_f : G_f \to S^1$ (global character) such that $d\chi_f = \rho_f$

(ρ_f exponentiates to a character of G_f)

★ Symplectic quotients & geometric invariant theory (Guillemin-Sternberg '82)

$X_{classical}$	compact symplectic manifolds acted on by a compact Lie group G (symmetries)
$X_{quantum}$	associated quantum Hilbert space (carrier of a repr. of G)

G simple

$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$

X_G "fixed pts" of G in $X_{classical}$

$X_G = X /\!/ G$ Marsden - Weinstein symplectic quotient

$(X_{quantum})_G$ "fixed pts" of G in $X_{quantum}$

"Theorem": $(X_{quantum})_G = (X_G)_{quantum}$ (★)

★ quantization commutes with reduction

Assumptions: A: X prequantizable

B: $\exists$ positive definite complex polarization, G - invariant

$X_{quantum}$: obtained via GQ

$\xi \mapsto \nabla_{\xi^\#} + 2\pi i \phi^\xi \qquad \phi^\xi(x) = \langle \Phi(x), \xi \rangle$

Then:

Theorem a) X_G inherits prequantum data from X, together with a positive definite complex polarization

b) (★) holds, upon taking $(X_G)_{quantum}$ = the Hilbert space manufactured from (prequantizing X_G

fully general (Meinrenken)

Crucial example of MW-reduction

$\mathbb{R}^{2n} \ni (x, y)$ $\quad x = (x_1 \dots x_n)$
$\quad y = (y_1 \dots y_n)$

$$\begin{pmatrix} x'^k \\ y'^k \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix}}_{\in\, SO(2) \cong U(1)} \begin{pmatrix} x^k \\ y^k \end{pmatrix}$$

$\Omega = \sum_{k=1}^{n} dx_i \wedge dy_i$

symplectic form

canonical action
(preserves Ω)

momentum map $J : \mathbb{R}^{2n} \to \mathfrak{so}(2)^* \cong \mathbb{R}$

$$J(x, y) = \frac{1}{2}(x^2 + y^2) \qquad x^2 = \sum x_i^2$$

harmonic oscillator

for $\mu \in \mathfrak{so}(2)^*$, $J^{-1}(\mu) \cong S^{2n-1}$ $\quad$ isotropy : $SO(2)$

reduced space : $\quad S^{2n-1} / SO(2)$

in complex coordinates

$z = x + iy \qquad z \mapsto e^{i\lambda} z \qquad \Omega = \frac{1}{2} \operatorname{Im}(d\bar{z} \wedge dz)$

$J(z) = \frac{1}{2}|z|^2$

reduced space : $\quad \mathbb{CP}^{n-1}$ $\quad$ complex projective space

Let O_ρ the coadjoint orbit pertaining to ρ irreducible repres. of G

(**Borel-Weil à la GS : integral orbits** in $\mathfrak{g}^*$ (of G) $\leftrightarrow$ irreps of G)

$$\boxed{\begin{array}{l}\text{Theorem : } \rho \text{ appears in } X_{quantum} \\ \quad \Leftrightarrow \quad O_\rho \text{ is in the image of} \\ \quad \Phi : X_{cl} \longrightarrow \mathfrak{g}^* \quad \text{moment map}\end{array}}$$

$\mathfrak{g} \ni \xi \mapsto \nabla_{\xi^\#} + 2\pi i \phi^\xi$

Mumford's quotient

G acts on X in a hamiltonian fashion

compact Lie

$\Phi : X \longrightarrow \mathfrak{g}^*$ moment map

$\langle \Phi(x), \xi \rangle = \phi^\xi(x)$

Let G act freely on $X_0 = \Phi^{-1}(0)$

Let $X_G \equiv X /\!/ G = \Phi^{-1}(0) / G$ ← isotropy group of 0

the Marsden-Weinstein quotient

$\dim X_G = \dim X - 2 \dim G$

Let $X_S = \{ g \cdot x \ / \ x \in X_0, \ g \in G^{\mathbb{C}} \} \equiv G^{\mathbb{C}} X_0$

saturation of X_0 w.r.to $G^{\mathbb{C}}$ $\equiv \{$ stable pts under the action $\}$ (Mumford)

Now $X_S \subset X$ and $G^{\mathbb{C}}$ acts freely on it. Hence

$$\boxed{X_G \equiv X_S / G^{\mathbb{C}}}$$

exhibits the complex structure of X_G

symplectic quotient (M-W)

Mumford quotient geometric invariant theory

* symplectic manifold (M, ω) (dim $M = 2n \dots$)

ω closed $d\omega = 0$

ω non degenerate (ω^n volume form)

(M, ω) (pre)quantizable: $[\omega] \in H^2(M, \mathbb{Z})$

$$\int_C \omega \in \mathbb{Z}$$

C ← integral 2-cycle

polarization : integrable (involutory) field

$F : x \longmapsto F_x$ complex tangent planes

F_x maximal isotropic w.r.t. ω

$x \longmapsto F_x + \bar{F}_x$ integrable, of constant dimension

$-$: complex conjugation in $T^{\mathbb{C}}_x M$

$F \to D$ $\quad D_x = F_x \cap \bar{F}_x \cap T_x M$

$F \to E$ $\quad E_x = (F_x + \bar{F}_x) \cap T_x M$

fiberings of M require: $i\,\omega(\xi, \bar{\xi}) \geq 0 \quad \forall \xi \in F$

this generalizes Lagrangian fibrations (F real)

and Kähler structures (D 0-dimensional)

in the latter case $F_x \cap \bar{F}_x = \{0\}$ and $F_x + \bar{F}_x = T^{\mathbb{C}}_x M$

by Newlander & Nirenberg $\exists$ unique complex structure s.t. $F_x = \text{span}\{$anti-holomorphic vector fields at $x\}$

$(z^1 \dots z^n)$ local complex coordinates

$$\omega = \sum a_{ij}\, dz^i \wedge dz^j + \sum b_{ij}\, dz_i \wedge d\bar{z}_j + \sum c_{ij}\, d\bar{z}^i \wedge d\bar{z}^j$$

F gen by $\frac{\partial}{\partial \bar{z}^j}$ $\Rightarrow$ max isotr iff $c_{ij} = 0$

ω is real $\Rightarrow$ ($\bar{F}$ max isotropic $\Rightarrow a_{ij} = 0$)

$$\Rightarrow \omega = \sum b_{ij}\, dz^i \wedge d\bar{z}^j \quad \text{(type (1,1))}$$

ω, symplectic form, is called a **Kähler form** with respect to a complex structure F on M if it is of type $(1,1)$ and if the tensor

$$(\xi, \eta) \longmapsto \omega(J\xi, \eta)$$

is a Riemannian metric $\left(J_x \text{ s.t. } F_x = Eig_{-i}(J_x) \quad \bar{F}_x = Eig_{+i}(J_x) \right)$

i.e. $i\,\omega(\xi, \bar{\xi}) \geq 0 \quad \forall \xi \in F$

★ Kähler quantization (M, ω, F)

(i) prequantize : $L \to M$ $\quad \nabla$ with curvature ω

(ii) $H = \{ s \,/\, \nabla_{\xi} s = 0 \quad \forall \xi \in F \}$

covariantly constant sections

$\rightsquigarrow$ holomorphic sections of a hol. line bundle

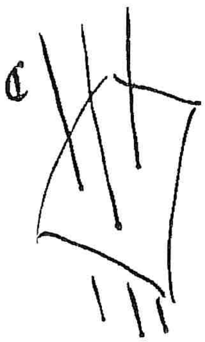
L complex line bundle $\quad \{U_i\}$ open covering of M

$\{ l_{ij} \}$ C^∞-complex functions on $U_i \cap U_j$ $(\neq 0)$
transition functions

$l_{ii} = 1 \quad l_{ij} l_{jk} = l_{ik}$ on non void triple intersections $U_i \cap U_j \cap U_k$
cocycle condition

s : section of L $\quad s : \{ s_i \}$ C^∞-complex functions $\quad s_i = l_{ij} s_j$ on overlaps

∇ connection $\sim$ $\{ \alpha_i \}$ C^∞ complex 1-forms



$$\alpha_j - \alpha_i = \frac{1}{2\pi\sqrt{-1}}\, d \log l_{ij}$$

Ω curvature form : $\quad \Omega = d\alpha_i$ on U_i
globally defined

$\nabla_{\xi} s \rightsquigarrow \{ t_i \} \quad t_i = \xi s_i + 2\pi\sqrt{-1} \langle \alpha_i, \xi \rangle s_i$

ξ F-field : $\xi_x \in F_x \quad \forall x$

Now

$$\alpha_i = \underset{(1,0)}{\beta_i} + \underset{(0,1)}{\gamma_i} \qquad d\alpha_i = \omega \quad (1,1) \quad \longleftarrow$$

$$\Rightarrow \quad \bar\partial \gamma_i = 0$$

Poincaré - Dolbeault : $\gamma_i = \bar\partial \varphi_i$

Then $\alpha_i - d\varphi_i = \beta_i - \partial\varphi_i \in \Lambda^{(1,0)}$

Thus : $\langle \alpha_i - d\varphi_i , \xi \rangle = 0 \quad \forall \xi \in$ F-fields

antiholomorphic

Set :

$$\Phi_i = \exp(2\pi\sqrt{-1}\,\varphi_i)$$

$$\tilde{s}_i = \Phi_i s_i$$

$$\tilde{\ell}_{ij} = \Phi_i \ell_{ij} \Phi_j^{-1}$$

$$\tilde{\alpha}_i = \alpha_i - d\varphi_i$$

new connection with curvature $= d\alpha_i = \omega$

Therefore :

$$\xi \tilde{s}_i = \xi(\Phi_i s_i) = \Phi_i \xi s_i + 2\pi\sqrt{-1} \langle \xi, d\varphi_i \rangle \Phi_i s_i$$

$$= \Phi_i(\xi s_i + 2\pi\sqrt{-1}\langle \xi, \alpha_i\rangle s_i)$$

F - field

Upshot: $\nabla_\xi s = 0 \quad \forall \xi$ F-field iff $\{\tilde{s}_i\}$ is a holomorphic section of $\tilde{L} = \{\tilde{\ell}_{ij}\}$

the $\tilde{s}_i$ are holomorphic, so is $\tilde{\ell}_{ij}$ $\quad (\tilde{s}_i = \tilde{\ell}_{ij}\tilde{s}_j)$

$\square$

The Chern-Bott connection

∇ connection on $E \to M$ complex vector bundle on M, complex manifold

$e = (e_1 \dots e_m)$ local frame

$$\nabla e_i = \sum \theta_{ij} e_j$$

$$\sigma = \sum \sigma_i e_i \qquad \nabla \sigma = \sum_i d\sigma_i e_i + \sum \sigma_i \nabla e_i$$

$$= \sum_j \left(d\sigma_j + \sum_i \sigma_i \theta_{ij} \right) e_j$$

Change of frame $\quad e'_i = g_{ij} e_j$

$$\nabla e'_i = \sum dg_{ij} e_j + g_{ij} \nabla e_j$$

$$= \sum g_{ik} \theta_{kj} e_j + \sum dg_{ij} e_j$$

$$\boxed{\theta_{e'} = dg \cdot g^{-1} + g \theta g^{-1}}$$

1. ∇ compatible with the complex structure: $\nabla'' = \bar\partial$
2. ∇ Compatible with a hermitian metric:

$$d(\xi, \eta) = (\nabla \xi, \eta) + (\xi, \nabla \eta)$$

Chern-Bott: $\exists !\ \nabla$ compatible with both structures

Let $e = (e_1 \dots e_n)$ be a holomorphic frame

set $h_{ij} = \langle e_i, e_j \rangle$

$$dh_{ij} = d(e_i, e_j) = \underbrace{\sum \theta_{ik} h_{ij}}_{(1,0)} + \underbrace{\sum \bar\theta_{jk} h_{ik}}_{(0,1)}$$

Thus: $\quad \partial h_{ij} = \sum \theta_{ik} h_{kj} \qquad \partial h = \theta h$

$\qquad\quad \bar\partial h_{ij} = \sum \bar\theta_{jk} h_{ik} \qquad \bar\partial h = h^t \bar\theta$

$$\Rightarrow \quad \theta = \partial h \cdot h^{-1} \qquad \text{unique solution}$$

in a unitary frame h is skew-hermitian $\quad h = -\bar h^t$

Kähler manifolds

M complex manifold

(M, h) hermitian if h is a hermitian metric

$h = h_{i\bar{j}}\, dz_i\, d\bar{z}_j$ locally

$\underbrace{h_{i\bar{j}}}$ hermitian positive

$$\omega := \frac{i}{2} h_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$$

type $(1,1)$

(M, h) Kähler if ω is closed

$$d\omega = 0$$

(Kähler $\Rightarrow$ symplectic)

locally (Dolbeault) $\omega = i\, \partial\bar{\partial} f$, $\mathcal{C}^\infty$

f Kähler potential

basic example: $\mathbb{P}^N$ with the Fubini-Study metric (crucial in quantum mechanics!)

If ω is integral and "sufficiently positive"

one finds a holomorphic $L \to M$

(equipped with a canonical (Chern - Bott)

connection (s.t. $\nabla^{(0,1)} = \bar{\partial}$)

which is **very ample**, i.e. allows

a **projective embedding**

suitable N

$$M \hookrightarrow \mathbb{P}^N$$

$$z \longmapsto [\, s_0(z), \dots s_N(z) \,]$$

base-point free

($\neq$ coherent state map in QM)

$\neq$ **Kodaira**

M compact

$\neq$ **algebraic equations** (Chow)

* The Kodaira - Nakano vanishing theorem

If $L \to M$ is a positive (holom.) line bundle, then

$$\boxed{H^p(M, \Omega^q(L)) = 0 \quad \text{for } p+q > n}$$

Bochner type proofs, via Weitzenböck

thus, for a negative line bundle,

$$H^p(M, \Omega^q(L)) = 0 \quad \text{if } p+q < n$$

special case $p = q = 0$

$$H^0(M, \mathcal{O}(L)) = 0$$

L possesses a metric & connection with curvature $\frac{2}{i}$ (negative (1,1))

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\left(\frac{1}{|s|^2}\right) = \frac{1}{4} \underbrace{\left(\frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial y_i \partial y_j} \right)}_{\text{negative definite}} \log \frac{1}{|s^2|} + \cdots$$

negative definite hermitian

assume $H^0 \neq 0$

s assumes a maximum at x_0

minimum at x_0

by the maximum principle

$\therefore$ the two matrices $\frac{\partial^2}{\partial x_i \partial x_j} \log \frac{1}{|s^2|}$ and $\frac{\partial^2}{\partial y_i \partial y_j} \log \frac{1}{|s|^2}$ must be positive semi definite

$\Downarrow$

Contradiction

M Riemann surface

$$c_1(L) = \int_M \frac{\sqrt{-1}}{2\pi} \Theta < 0$$

curvature

$$\| $$

$\deg D$ divisor pertaining to s

$\vee\!|$

0

$\Rightarrow$ contradiction

$$\deg L = \sum_i k_i$$

zeros of any hol. section counted with multiplicity

* The Kodaira Embedding Theorem

A compact complex manifold M is an algebraic variety (i.e. embeddable in projective space (via Chow))

if and only if it has a closed, positive $(1,1)$-form ω whose cohomology class $[\omega]$ is rational

If $[\omega] \in H^2(M,\mathbb{Q})$ then for some k, $[k\omega] \in H^2(M,\mathbb{Z})$

consider $H^1(M,\mathcal{O}^*) \to H^2(M,\mathbb{Z}) \xrightarrow{i_*} H^2(M,\mathcal{O})$

$i_*[k\omega] = 0 \implies \exists\ L \to M$ hol. line bundle with $c_1(L) = [k\omega]$ L is positive

$H^0(M,\Omega^q(E)) \cong H^{p,q}_{\bar\partial}(E)$

Alternative statement

Let M be a compact complex manifold and $L \to M$ be a positive line bundle. Then, for $k \geq k_0$

$$i_{L^k}: M \to \mathbb{P}^N$$

is well defined and it is an embedding

(via coherent state map)

Hitchin's Theory

$L \to M$ symplectic manifold with a Kähler polarization

$I \in \Omega^0(M; \mathrm{End}\, T)$

Newlander Nirenberg complex manifold
$$\left\{\begin{array}{l} I^2 = -1 \\ [IX, IY] = [X,Y] + I[IX,Y] + I[X,IY] \end{array}\right.$$

$$\omega(X, IY) = -\omega(IX, Y)$$

$\omega(X, IX)$ positive definite

$g(X,Y) = \omega(X, IY)$ kähler metric

$\omega \in \Omega^{1,1}(M)$ Kähler form

Now L becomes a holomorphic line bundle

$$\nabla^{0,1} : \Omega^0(M; L) \to \Omega^{0,1}(M, L)$$

$$\nabla^{0,1} = (1 + iI)\nabla$$

on $(0,1)$ forms I acts as $-i$

locally: $$\sum_i \underbrace{\left(\frac{\partial f}{\partial \bar z} + \theta_i f\right)}_{=0} d\bar z_i$$

← a local solution, by Dolbeault, exists iff $\bar\partial(\underbrace{\Sigma\, \theta_i d\bar z_i}) = 0$

$(0,2)$ comp of the curvature, i.e. ω, of type $(1,1)$

Thus $\nabla^{0,1} s = 0$

has a local non vanishing solution. If $\tilde s$ and s are two local solutions,

$$\tilde s = g \cdot s \implies \frac{\partial g}{\partial \bar z_i} = 0 \implies g \text{ holom.}$$

$$H^0(L) = \{ s \mid \nabla^{(0,1)} s = 0 \}$$ quantum Hilbert space

$$V_I = \{ s \in \Omega^0(M,L) : \overbrace{(1+iI)\nabla}^{\nabla^{0,1}} s = 0 \}$$

↑ check independence upon variations of I

$\nabla^{0,1} : \Omega^{0,p} \to \Omega^{0,p+1}$ Dolbeault complex

$H^p(M,L)$ cohomology groups

$H^p(M;L) \underset{\text{Serre}}{\cong} H^{n-p}(M, KL^{-1}) = 0$ if $p > 0$

if $\underset{\uparrow \text{holomorphic } n\text{-forms}}{K} L^{-1}$ is negative i.e.

(Kodaira - Nakano)

$-c_1(KL^{-1}) = \underset{\substack{\downarrow \\ \frac{\omega}{2\pi} \\ \text{Kähler form}}}{c_1(L)} - c_1(K)$ is represented by a Kähler form

$L \to L^k$ (level shift) gives $k c_1(L) - c_1(K)$

$\Rightarrow$ ok for k sufficiently big

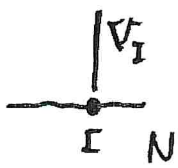
Riemann-Roch-Hirzebruch

$$\sum_{p=0}^{n} (-1)^p \dim H^p(M,L) = \text{topological invariant}$$

$\Rightarrow$ $\dim V_I$ does not change

N connected finite dimensional parameter space of Kähler polarization

$V \to N$ vector bundle



⟹ find a $\overset{\text{flat}}{\text{connection}}$ on V

identifications of $P(V_I)$

projectively flat central curvature

$t \mapsto I_t$ path of Kähler polarizations

$$\nabla_t^{0,1} s_t = 0 \qquad \text{family of smooth solutions } s_t$$

$$(1 + i I_t) \nabla s_t = 0$$

differentiate:

$$i \dot{I} \nabla s + (1 + iI) \nabla \dot{s} = 0$$

connection: C^∞ section $\kappa(s, \dot{I})$ of L (bilinear dependence on s and $\dot{I}$) fulfilling

$$i \dot{I} \nabla s + (1 + iI) \nabla \kappa = 0$$

$\Rightarrow$ need to solve or $i \dot{I} \nabla^{1,0} s + \nabla^{0,1} \kappa = 0$

$$\frac{\partial s}{\partial t} = \kappa\left(s, \frac{\partial I}{\partial t}\right)$$

From $I^2 = -1$ one has $\dot{I} I + I \dot{I} = 0 \Rightarrow$ $(-i)$-eig $\to$ $(+i)$-eig

$$\dot{I} \in \Omega^{0,1}(M; T)$$

holomorphic tangent bundle
(1,0) tang. vectors

$$\dot{I} = \sum a^i_j \frac{\partial}{\partial z_i} \otimes d\bar{z}_j$$

also: $\bar{\partial} \dot{I} = 0 \in \Omega^{0,2}(M, T)$ (integrability)

and $\dot{I} = \sum G^{ij} \omega_{j\bar{k}} \frac{\partial}{\partial z_i} \otimes d\bar{z}_k$ $\qquad \omega = \sum \omega_{i\bar{k}} dz_i \wedge d\bar{z}_k$

$G^{ij} = G^{ji}$

$G = \sum G^{ij} \frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial z_j}$ (symmetric) section of $T \otimes T$

Also, by integrability (and closure of ω)

$$\sum \frac{\partial G^{ij}}{\partial \bar z_\ell} \omega_{j\bar k} \, d\bar z_k \wedge d\bar z_\ell = 0$$

local solutions can be found:

$$\bar\partial (\dot I \nabla^{1,0} s) = 0 \in \Omega^{0,2}(M, L)$$

write

$$i \dot I \nabla^{1,0} s = \sum i G^{ij} \omega_{j\bar k} \nabla_i s \otimes d\bar z_k$$
$$= \sum \alpha^i_k \frac{\partial s}{\partial z_i} \otimes d\bar z_k + \sum \beta_k s \otimes d\bar z_k$$

$\mathcal{D}^k(L)$ = holomorphic linear diff. operator of order k on L $\cong \mathrm{Hom}(J^k(L), L)$ (k-jets of holom. sections)

$$0 \to \mathcal{D}^{k-1}(L) \to \mathcal{D}^k(L) \xrightarrow{\sigma} S^k T \to 0.$$

symbol map

$\Rightarrow$ $i \dot I \nabla^{1,0}$ is an element of $\Omega^{0,1}(M; \mathcal{D}^1(L))$

$$\bar\partial (i \dot I \nabla^{1,0}) = 0 \in \Omega^{0,2}(M, \mathcal{D}^1(L))$$

Cohomological interpretation

$$A^p = \Omega^{0,p}(M, \mathcal{D}^1(L)) \oplus \Omega^{0,p-1}(M, L)$$

$$d_s : A^p \to A^{p+1} \qquad d_s(D, u) = (\bar\partial D, \bar\partial u + (-1)^{p-1} Ds,$$

from $\bar\partial s = 0$ one gets $d_s^2 = 0$ $\rightsquigarrow$ cohomology groups

$$\sigma: \mathbb{H}^{p\,1}_s(M, \mathcal{D}^1(L)) \xrightarrow{\cong} H^1(M, T)$$

$$d_s (i \nabla \dot I \nabla^{1,0} u) = 0 \qquad \begin{aligned} &\sigma(\dot I \nabla^{1,0}, u) \\ &= [\dot I] \end{aligned}$$

Kodaira - Spencer class

Theorem

M compact symplectic manifolds

$L \to M$ complex line bundle

∇ connection $\Omega = \omega$

family of kähler polarization

(i). $[\omega] : H^0(M, T) \xrightarrow{\text{iso}} H^1(M, \mathcal{O})$

(ii) $\forall\ s \in H^0(M, L)$ and $\dot{I}$ $\exists$

$$A(\dot{I}, s) \in \mathbb{H}^1_s(M, \mathcal{D}'(L))$$

$$-i\,\sigma\,A(\dot{I}s) = [\dot{I}] \in H^1(M, T)$$

Kodaira - Spencer

$\boxed{\text{Then } A \text{ defines a connection on the projective bundle}}$

comment: $\omega = \sum \omega_{ij}\, dz_i \wedge d\bar{z}_j \in \Omega^{0,1}(M, T^*)$

$[\omega] : H^0(M, T) \to H^1(M, \mathcal{O})$ cup product

$$[\omega]\left(\sum x^i \frac{\partial}{\partial z_i}\right) = \sum x^i \omega_{i\bar{j}}\, d\bar{z}_j$$

Important example

"Heat connection"

in theta function theory