GEOMETRIC METHODS IN QUANTUM MECHANICS

Mauro Spera - UCSC Brescia

Lecture VII: Applications (I)

International Doctoral Program in Science

Brescia, Capitolium

Examples of symplectic manifolds

- cotangent spaces

  M manifold $\quad X = T^*M \quad$ phase space

  local coordinates $\quad (q_1 \dots q_n, \; p_1 \dots p_n)$
  positions $\quad$ momenta

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$$

Darboux: any (X,ω) has a symplectic atlas
$X = \dot{\bigcup_\alpha} \mathcal{U}_\alpha \qquad \omega|_{\mathcal{U}_\alpha} = \sum_{i=1}^{n} dq_i \wedge dp_i$

$M = \mathbb{R} \qquad T^*\mathbb{R} \cong \mathbb{R}^2 \cong \mathbb{C} \qquad \omega = \frac{i}{2}\, dz \wedge d\bar{z}$
(Kähler form)

Schrödinger
Bargmann-Fock

- $\left(\mathbb{C}^{n+1}, \; \frac{i}{2} \sum_{i=1}^{n+1} dz_i \wedge d\bar{z}_i \right)$

  [phase space pertaining to $n+1$ harmonic oscillators]

- $\mathbb{P}^n$ projective space $\mathbb{C}^{n+1} \setminus \{0\} / \sim$

MW reduction $\quad$ straight lines through the origin in $\mathbb{C}^{n+1}$

$[z_0, z_1, \dots, z_n]$ homogeneous coordinates

ω = Fubini-Study Kähler form

$\mathcal{U}_0 = \{(z_0, \dots z_n) \in \mathbb{C}^{n+1} / z_0 \neq 0\} \qquad \zeta_i = \frac{z_i}{z_0}$

$\omega|_{\mathcal{U}_0} = i \partial\bar{\partial} f_0 \qquad f_0 = \log\left(1 + \sum_{i=1}^{n} |\zeta_i|^2\right)$

(local) Kähler potential

physically: fixed energy submanifold of a system of $n+1$ harmonic oscillators

$[\omega]$ generates the cohomology of $\mathbb{P}^n$

- $M \overset{i}{\hookrightarrow} \mathbb{P}^N$

  compact complex manifold embedded in $\mathbb{P}^N$

  $i(M)$ is then an algebraic manifold

$(M, i^*\omega)$ symplectic manifold

F.S. Kähler

example: $M = \mathbb{P}^1 \cong S^2$ (the 2-sphere) $N = 3$

$(z_0, z_1) \longmapsto (z_0^3, z_0^2 z_1, z_0 z_1^2, z_1^3)$

(rational normal curve)

$M = \mathbb{P}^1 \times \mathbb{P}^1 \cong S^2 \times S^2$

energy manifold for the Kepler system

$$S[(w_0, w_1), (\zeta_0, \zeta_1)] = (\underbrace{w_0\zeta_0}_{x_{00}}, \underbrace{w_0\zeta_1}_{x_{01}}, \underbrace{w_1\zeta_0}_{x_{10}}, \underbrace{w_1\zeta_1}_{x_{11}}) \subset \mathbb{P}^3$$

Segre map

$$\boxed{x_{00}x_{11} = x_{01}x_{10}}$$

$\mathbb{P}^1 \times \mathbb{P}^1$ embeds in $\mathbb{P}^3$ as a quadric

(Segre embedding) $S^* \omega_{\mathbb{P}^3} = \omega_{\mathbb{P}^1} + \omega_{\mathbb{P}^2}$

- Algebraic tori (Abelian varieties)

$$X = \mathbb{C}^n / \Lambda \qquad \Lambda : \text{lattice of maximal rank } 2n$$

X is embedded in $\mathbb{P}^N$ if and only if the <u>Riemann conditions</u> are satisfied:

$(\lambda_1 \dots \lambda_{2n})$ integral basis for Λ $\qquad \lambda_i = \sum_\alpha \omega_{\alpha,i}\, e_\alpha$ $\qquad (e_1, \dots e_n)$ basis of $\mathbb{C}^n$

$$\Lambda = \Omega\, e$$

$$\Omega = (\Delta_\delta, Z) \qquad \operatorname{Im} Z > 0 \qquad Z = Z^t$$

$n \times 2n$

$$\Delta_\delta = \begin{pmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{pmatrix} \qquad \delta_i \in \mathbb{N} \qquad \delta_i \mid \delta_{i+1}$$

$\delta_i \equiv 1$: principal polarization

$$\omega = \sum_{j=1}^n dq_j \wedge dp_j = \frac{i}{2} \sum_{\alpha,\beta} W_{\alpha\beta}\, dz_\alpha \wedge d\bar{z}_\beta$$

$$W = (\operatorname{Im} Z)^{-1}$$

$\rightsquigarrow$ theta function theory recast as G.Q.

★ coadjoint orbits of Lie groups

Kirillov
Kostant, Souriau

G Lie group $\mathfrak{g}$ Lie algebra of G $\mathfrak{g}^*$ dual

G acts on $\mathfrak{g}$ (adjoint representation)

matrix case: $Ad(g)X = gXg^{-1}$, $X \in \mathfrak{g}$ $g \in G$

infinitesimally

$$ad(u)\,v = [u\ v] \qquad u\ v \in \mathfrak{g}$$

coadjoint action

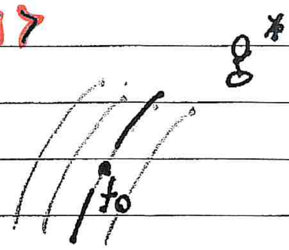
$$\langle ad^*(u)f, v\rangle := -\langle f, [u,v]\rangle \qquad f \in \mathfrak{g}^*$$

$$\langle Ad^*(g)f, v\rangle := \langle f, Ad(g^{-1})\,v\rangle \qquad u, v \in \mathfrak{g}$$

$\langle\, ,\rangle$ duality $g \in G$

★ KKS form

$$\omega_f(ad^*(u)f, ad^*(v)f) := \langle f, [u,v]\rangle$$



ω is Ad^*-invariant

symplectic form on $M_{f_0} \cong G/G_{f_0}$ isotropy group

★ coadjoint orbit

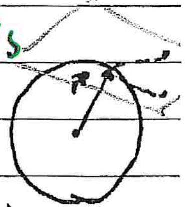
Example: $G = SO(3)$ $\mathfrak{g} = so(3) \cong \mathbb{R}^3$ as a vector space

$(\mathfrak{g}, [\ ]) \underset{\text{Lie alg.}}{\cong} (\mathbb{R}^3, \times)$ vector product

$$\mathfrak{g} = \left\langle \underset{e_1}{\begin{pmatrix} 0&0&0\\0&0&1\\0&-1&0 \end{pmatrix}}, \underset{e_2}{\begin{pmatrix} 0&0&1\\0&0&0\\-1&0&0 \end{pmatrix}}, \underset{e_3}{\begin{pmatrix} 0&1&0\\-1&0&0\\0&0&0 \end{pmatrix}} \right\rangle$$

$[e_i, e_j] = \varepsilon_{ijk} e_k$  $e_1 \wr \underline{i}$ $\underline{j}$ $\underline{k}$

$S^2 \cong SO(3)/SO(2)$ any $g \in SO(3)$ is a rotation around an axis
$SO(2)$: rotations on plane $\perp$ axis



$\mathfrak{g}^* \cong \mathfrak{g}$ via Tr: coadjoint orbits: spheres

KKS form: $\omega_{\vec{r}}(ad^*(u)\vec{r}, ad^*(v)\vec{r}) = \langle \vec{r}, \vec{u} \times \vec{v}\rangle$
$= \det(\vec{r}, \vec{u}, \vec{v})$

Heisenberg group

$$(\underbrace{x_1, x_2}_{x}, s) \cdot (\underbrace{y_1, y_2}_{y}, t) = (x_1+y_1, x_2+y_2, t+s+B(x,y))$$

$$B(x,y) = (x_1 y_2 - x_2 y_1)$$

or any symplectic form

$$\begin{pmatrix} 1 & x_1 & s \\ 0 & 1 & x_2 \\ & & 1 \end{pmatrix}$$

isotropy of $f = e_3$
$= \{ \lambda e_3 \}$

coadjoint orbits
$\cong$ planes $\mathbb{R}^2 \simeq \mathbb{C}$

$e_1 = iP$ $\quad [e_1, e_2] = e_3$

$e_2 = iQ$ $\quad [e_j, e_3] = 0$

$e_3 = iI$ $\quad [x, y] = B(x,y)\, e_3$

★ B = Kirillov symplectic form

★ Bargmann - Fock = Kähler geometric quantization of the harmonic oscillator

- The **Abel-Jacobi** map

$\wp = \wp(z)$ Weierstraß $\wp$-function

$$\psi : \underbrace{\mathbb{C}/\Lambda}_{\tau} \longrightarrow \mathbb{P}^2$$

$$z \longmapsto [1, \wp(z), \wp'(z)]$$

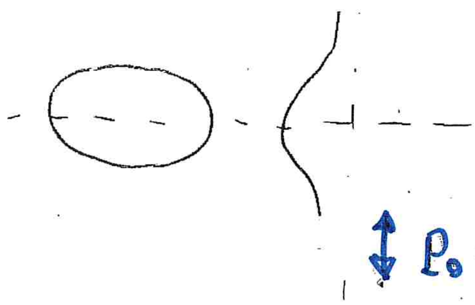
$$\psi = A^{-1} \qquad A(p) = \int_{p_0}^{p} \frac{dx}{y} \quad \text{modulo periods}$$

$p_0 = [0,0,1]$

ψ : Kodaira embedding for $L = L_D = L_{3p_0}$

$h^0(L_D) = 3$ (Riemann-Roch)

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3$$

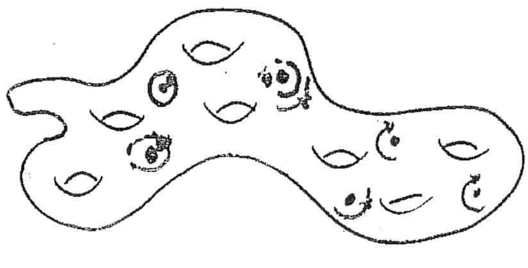


elliptic cubic

$P_0 : [0,0,1]$

.. vortex theory on Riemann surfaces

vorticity divisor



* **Spin** intrinsic angular momentum

$\equiv$ representation theory of $SU(2)$

$SU(2)$ unitary 2×2 matrices with **$\det = 1$**

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \qquad a, b \in \mathbb{C}$$

fundamental representation
acting on $\mathbb{C}^2$

$$|a|^2 + |b|^2 = 1$$

$$\Rightarrow \quad SU(2) \cong S^3$$

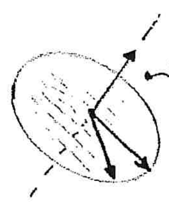
intimately connected with $SO(3)$ special orthogonal group

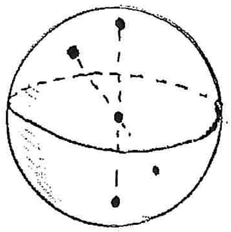
indeed $\underbrace{SU(2)}_{\text{universal covering group of } SO(3)} \xrightarrow{2:1} SO(3)$

$$\underset{\substack{\cong \\ \mathbb{R}P^3}}{SO(3)} = SU(2) \big/ \mathbb{Z}_2 \cong \pi_1(SO(3))$$

fundamental group

Euler:





closed 3-d ball
of radius π
with antipodal
boundary points identified

★ details

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \leadsto \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} \equiv X = X^*$$

hermitian

$$X \mapsto UXU^* = UXU^{-1} \equiv X'$$

is a rotation [adjoint representation of SU(2)]

explicitly consider the following basis of $\mathfrak{su}(2)$
(orthogonal w.r.t. the KC metric)

$\mathfrak{su}(2)$
Lie algebra of SU(2)
≡ skew hermitian, traceless 2×2 matrices

$$\left(i\sigma_x ,\ i\sigma_y ,\ i\sigma_z \right)$$

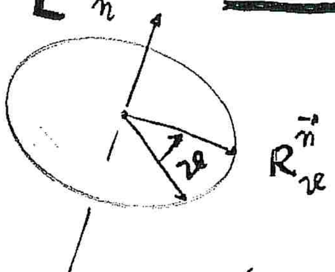
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli matrices

$$\Rightarrow X = x\sigma_x + y\sigma_y + z\sigma_z$$

$$\left[\ R^{\vec{n}}_{\vartheta} = e^{i\frac{\vartheta}{2}\vec{\sigma}\cdot\vec{n}}\ \right] \qquad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

$$= \mathbb{1}\cos\frac{\vartheta}{2} + \vec{\sigma}\cdot\vec{n}\,\sin\frac{\vartheta}{2}$$



quaternions
I J K
$\wr$ $\wr$ $\wr$
$\underline{i}$ $\underline{j}$ $\underline{k}$ ✗

notice that $\pm U$ induce the same rotation...

$\vartheta = 0 \Rightarrow$ r.h.s $= \mathbb{1}$
$\vartheta = 2\pi \Rightarrow$ r.h.s $= -\mathbb{1}$

vectors in $\mathbb{C}^2$
≡ **spinors**

fundamental representation ≡

spin $\frac{1}{2}$ representation

$$\frac{\sigma_z}{2} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

diagonal

$\pm \frac{1}{2}$: outcomes of a measurement of the spin along an arbitrary axis (in this explicit representation, z) cf. Stern-Gerlach

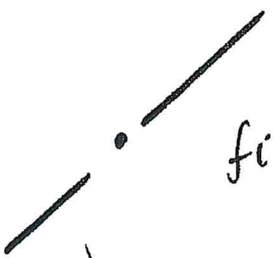
$$[\sigma_x, \sigma_y] = i\,\sigma_z$$

do not commute ∴ spin along a different axis is indeterminate

4 Geometric reinterpretation:

SU(2) acts on first degree homogeneous polynomials in z_0, z_1 (Coordinates in $\mathbb{C}^2$) in a natural way. The latter provide the

holomorphic sections of $\mathcal{O}(1) \equiv$ hyperplane section bundle = dual to the tautological line bundle on $\mathbb{P}^1 \cong S^2$ (Riemann Sphere)



fibre at $[v] \equiv$ the line $\langle v \rangle$

$$H^0(\mathcal{O}(1))$$

$h^0(\mathcal{O}(1)) = 2$

dimension

in general:

spin $\frac{j}{2}$ representation $j \in \mathbb{N}$

on degree j homogeneous polynomials in z_0, z_1 $\equiv$

holomorphic sections of $\mathcal{O}(j)$

$$= \underbrace{\mathcal{O}(1) \otimes \mathcal{O}(1) \otimes \ldots \otimes \mathcal{O}(1)}_{n \text{ times}}$$

symmetric tensor product

$$h^0(\mathcal{O}(j)) = j+1 \qquad = 2\frac{j}{2}+1$$

$j=0$ scalar (trivial) representation

this is a special instance of the celebrated Borel-Weil theorem

cf. geometric quantization

$$\boxed{\begin{array}{ll} S^2 \cong \mathbb{P}^1 & \text{"classical" phase space} \\ H^0(\mathcal{O}(j)) & \text{"wave functions"} \end{array}}$$

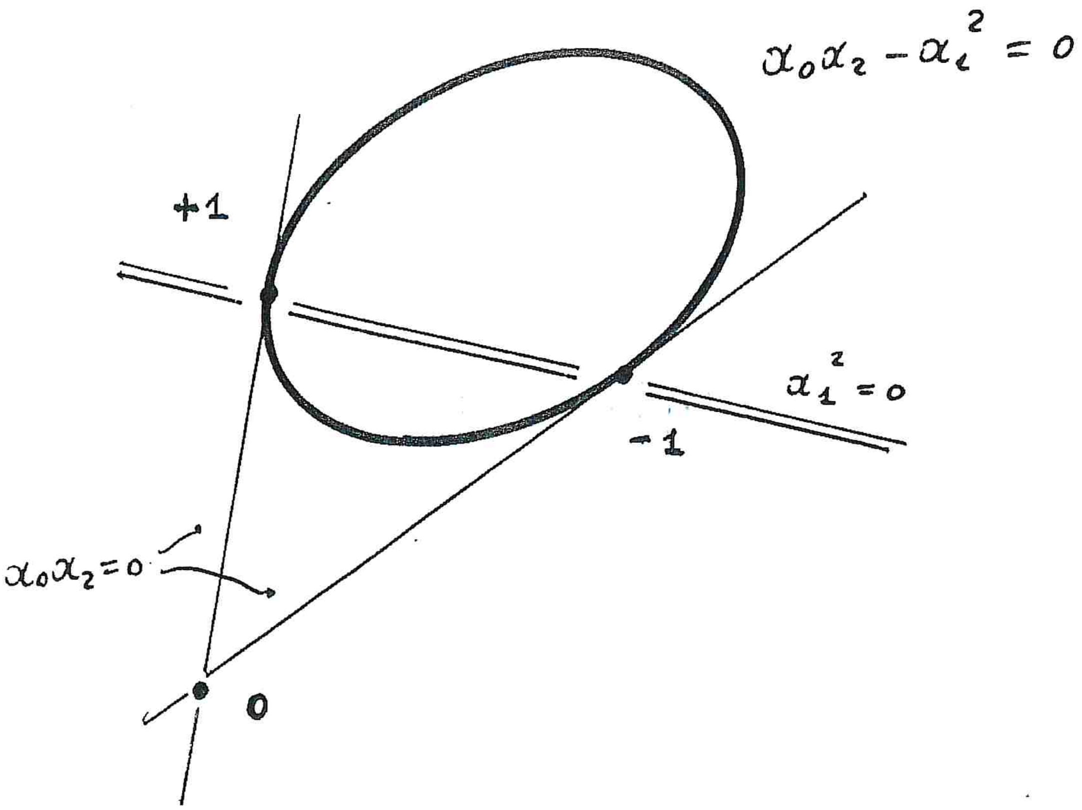
$\leadsto$ "top model" recovered!

deep physical significance: Coherent states

★ Geometry of the spin 1 representation
adjoint representation of $SO(3)$

$$\rho\left(\frac{\sigma_z}{2}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\mathbb{P}^2$



$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$$ normal rational curve Veronese

$$(z_0, z_1) \longmapsto (\underbrace{z_0^2}_{x_0}, \underbrace{z_0 z_1}_{x_1}, \underbrace{z_1^2}_{x_2})$$

$$x_0 x_2 = x_1^2 \qquad \text{conic}$$

★ Kodaira embedding via $\mathcal{O}(2) \to \mathbb{P}^1$

holomorphic sections: homogeneous polynomials of degree 2 in z_0, z_1

* The Segre map

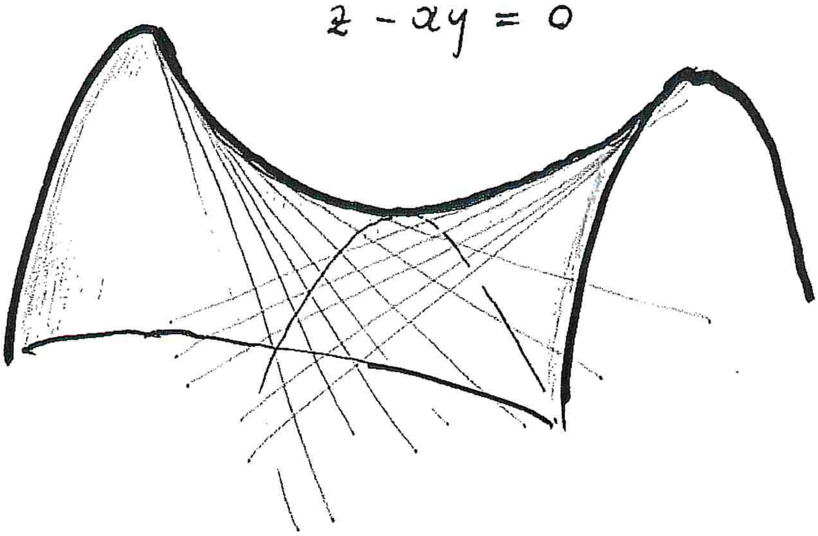
$$S: \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

$$\begin{array}{ccccc} & 1 & x & y & z \\ ((z_0, z_1), (w_0, w_1)) \longmapsto & (\underbrace{z_0 w_0}_{x_{00}}, & \underbrace{z_0 w_1}_{x_{01}}, & \underbrace{z_1 w_0}_{x_{10}}, & \underbrace{z_1 w_1}_{x_{11}}) \end{array}$$

$$x_{00} x_{11} - x_{01} x_{10} = 0$$

+ quadric (copies of $\mathbb{P}^1$: reguli)

$$z - xy = 0$$



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Example The Kepler system (after Kunstanheimo-Shiefel)

$$\Big(\underbrace{T^*(\mathbb{R}^3 \setminus \{0\})}_{M}, \underbrace{\Sigma\, dq_i \wedge dp_i}_{\omega}, \underbrace{\tfrac{1}{2}\Sigma p_i^2 - \frac{1}{(\Sigma q_i^2)^{\frac{1}{2}}}}_{H} \Big)$$

$$\omega = d\vartheta \qquad \vartheta = -\Sigma\, p_i dq_i \qquad \text{(exact)}$$

upon conformal regularization get

$$(M, \omega, H) \qquad M = \Pi_0/_\sim = \{ \alpha = (\alpha_1 \ldots \alpha_4) \in \mathbb{C}^4 / |\alpha_1|^2 + |\alpha_2|^2 = |\alpha_3|^2 + |\alpha_4|^2 \}$$

"null twistors" $/\sim$

$$\alpha \sim \beta \quad \text{iff} \quad \alpha = \gamma\beta, \ |\gamma| = 1$$

$$\omega = d\vartheta \qquad \vartheta = \mathrm{Im}\, \Sigma\, \bar{\alpha}_i d\alpha_i \Big|_M$$

$$H = -\tfrac{1}{2}\left(|\alpha_1|^2 + |\alpha_2|^2\right)^{-2}$$

Physical interpretation: consider

$$(M_0, \omega_0, H_0) = \left(\mathbb{C}^4, \tfrac{1}{2}\Sigma\, d\alpha_i \wedge d\bar{\alpha}_i, \Sigma|\alpha_i|^2\right)$$

4-oscillator system

Kepler system: constrained oscillators (also Moser)

upon reduction:

$$\underset{\substack{\uparrow \\ \text{Kepler energy}}}{M_{\mathcal{E}}} \cong M_{0,\frac{E}{2}} \times M_{0,\frac{E}{2}} \simeq \underset{\text{2-oscill}}{\mathbb{P}^1} \times \underset{\text{2-oscill}}{\mathbb{P}^1} \overset{S}{\hookrightarrow} \underset{\text{4-oscill}}{\mathbb{P}^3}$$

$S^2 \times S^2$

$$\frac{E}{2}(\omega_1 + \omega_2) = (-2\mathcal{E})^{-\frac{1}{2}}(\omega_1 + \omega_2)$$

geometric quantization of oscillator + Kepler

$\mathbb{P}^n$ $\qquad L \to \mathbb{P}^n \qquad c_1(L) = [\omega]$

$L = H = \mathcal{O}(1)$ hyperplane section bundle
dual to the tautological line bundle

$* \quad h^0(H) = n+1$ holomorphic sections

patches $\mathcal{U}_j = \{(z_0 \dots z_n) / z_j \neq 0\}$

$g_{kh} = \frac{z_h}{z_k} \qquad s \sim \{s_r\} \qquad s_k = g_{kh} s_h$

basis $s^i \quad i=0..n \qquad s^i = \{s^i_k\} \qquad s^i_k = \frac{z^i}{z^k}$

orthogonal w.r.t. the natural metric

$$h^0(pH) = \binom{p+n}{n}$$ p-th symmetric tensor power

$z_i \sim s^i \longleftrightarrow a_i^+ |0\rangle$

$\quad \qquad \qquad \qquad \qquad \quad = 1 \leftarrow$ trivial line bundle

$[a_i, a_j^+] = \delta_{ij} \qquad a_i^+ \sim z_i \qquad a_i \sim \frac{\partial}{\partial z_i}$

$$N = \sum_{i=0}^{n} a_i^+ a_i = \sum_{i=0}^{n} z_i \frac{\partial}{\partial z_i}$$ (Euler operator)

Fock space $\mathcal{F} = \sum^{\oplus} H^0(pH)$

$$H_0 = \sum_{i=0}^{n} a_i^+ a_i + \frac{n+1}{2}$$ ← zero pt energy

sections of pH : states with $E = p + \frac{n+1}{2}$

Theorem $N \geq 2$ (G. Gaeta, — '88)

$H^0(L_N)$, with $L_N = S^*(N-1)H = (N-1)S^*H$,

is the (negative) Kepler energy eigenspace, with energy

$$E_N = -\frac{1}{2N^2}$$

basis: $\left\{ z_0^{n_0} z_1^{n_1} w_0^{n_0'} w_1^{n_1'} \;\middle|\; n_0+n_1 = n_0'+n_1' \right\}$

$\boxed{h^0(L_N) = N^2}$

$H^0(L_N)$ can be viewed as a subspace of

$H^0(2(N-1)H)$: we are selecting oscillator states of total quantum energy $2(N-1)+2 = 2N$ subject to (*)

L_N carries a representation of $\underbrace{\mathrm{Spin}(4)}_{= \widetilde{SO(4)}} \cong SU(2)\times SU(2)$

GEOMETRIC QUANTIZATION & LANDAU LEVELS revisited

(A. Galasso & M.S 2016)

Holomorphic geometric quantization of the harmonic oscillator

$$M = \mathbb{R}^{2n} \cong \mathbb{C}^n \qquad n \geq 1 \qquad (\hbar = 1) \qquad z_j = x_j + i y_j$$

$$\tilde{\omega} = \sum_{j=1}^{n} dx_j \wedge dy_j = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j = \frac{i}{2} \partial \bar{\partial} \mathcal{Y}$$

Kähler form

$$\mathcal{Y} := \sum_{j=1}^{n} \bar{z}_j z_j =: 2\tilde{h}$$

$\tilde{h}$ oscillator Hamiltonian

Kähler potential

$L = M \times \mathbb{C} \longrightarrow M$ trivial complex line bundle

$s_0 \equiv 1$ trivialising section


M

Hermitian metric $(1, 1) := e^{-\mathcal{Y}}$

$$(s = \tilde{s} \cdot 1, \; s' = \tilde{s}' \cdot 1) = \bar{\tilde{s}} \, \tilde{s}' \, e^{-\mathcal{Y}}$$

$$\tilde{\mathcal{H}} = L^2(M, \mu) \qquad \mu = e^{-\mathcal{Y}} dx_1 dy_1 \dots dx_n dy_n$$

Chern-Bott connection (in a holomorphic frame)

$$\nabla = d - \partial \mathcal{Y} = \nabla' + \nabla''$$

$$\nabla' = \partial - \partial \mathcal{Y}, \quad \nabla'' = \bar{\partial}$$

$$\nabla = d - \partial \mathcal{Y} = d - \sum_{j=1}^{n} \bar{z}_j dz_j$$

curvature: $\Omega = d(-\partial \mathcal{Y}) = -2i \tilde{\omega} = -2i \, d\tilde{\theta}$

$$\tilde{\theta} = -\frac{i}{2} \sum_{j=1}^{n} \bar{z}_j dz_j \qquad \text{symplectic potential}$$

work in $\hat{\mathcal{H}}$ (actually, suitable domains therein)

$$A_j := \nabla_{\frac{\partial}{\partial \bar z_j}} = \frac{\partial}{\partial \bar z_j} \qquad A_j^{+} := \left(\nabla_{\frac{\partial}{\partial \bar z_j}}\right)^{+} = -\frac{\partial}{\partial z_j} + \frac{\partial y}{\partial z_j} = -\frac{\partial}{\partial z_j} + \bar z_j$$

annihilation & creation operators

CCR

$$[A_j, A_k] = [A_j^{+}, A_k^{+}] = 0$$

$$[A_j, A_k^{+}] = I \cdot \delta_{jk}$$

$$\langle \xi, R\eta \rangle_{\mathcal{H}} = \langle R^{+}\xi, \eta \rangle_{\mathcal{H}}$$

identification crucial in theta function theory (both classical and non commutative) (M.S 1986 M.S. 2025 A.Schwarz 2000)

$$\boxed{\text{multiplicity} = \text{dimension of the common kernel of the } A_j \text{ (von Neumann uniqueness theorem 1931)}}$$

$$\mathcal{H} = \{ f \in \hat{\mathcal{H}},\ f \text{ holomorphic} \}$$

Bargmann-Fock space

already a Hilbert space

$\bar{\mathcal{H}}$ conjugate space

$$\hat{\mathcal{H}} \cong \mathcal{H} \otimes \bar{\mathcal{H}} \qquad \{ z_i^{h} \bar z_j^{k} \}$$

orthonormal basis for $\hat{\mathcal{H}}$ up to constants $i,j = 1 \ldots n$, $h \geq 0$ $k \geq 0$

prequantum Hamiltonian

$$Q(\hat h) := -i \nabla_{X_{\hat h}} + \hat\theta(X_{\hat h})$$

$$= -i X_{\hat h} = \sum_{j=1}^{n} \left(z_j \frac{\partial}{\partial z_j} - \bar z_j \frac{\partial}{\partial \bar z_j} \right)$$

Hamiltonian vector field pertaining to $\hat h$

contraction $\rightsquigarrow$ $i_{X_{\hat h}} \hat\omega + d\hat h = 0$

restrict to $\mathcal{H}$:

$$Q(\tilde{u})\big|_{\mathcal{H}} = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}$$

Euler operator (number)

zero point energy missing

Set :

$$a_j := A_j\big|_{\bar{\mathcal{H}}} = \frac{\partial}{\partial \bar{z}_j}$$

$$a_j^{\dagger} := A_j^{\dagger}\big|_{\bar{\mathcal{H}}} = \bar{z}_j$$

CCR :

$$[a_i, a_j] = [a_i^{\dagger}, a_j^{\dagger}] = 0$$

$$[a_i, a_j^{\dagger}] = \delta_{ij} I$$

Similarly one has CCR on $\mathcal{H}$:

$$b_j = \frac{\partial}{\partial z_j} \qquad b_j^{\dagger} = z_j$$

★ upshot: $\hat{\mathcal{H}} \equiv \mathcal{H}_A$ carries a representation π_A of the CCR defined via the Chern-Bott connection. π_A is reducible, with infinite multiplicity given by $\mathcal{H}$. The latter carries a CCR representation π_b, which is in turn irreducible. Similarly one gets π_a on $\bar{\mathcal{H}}$

Bargmann-Fock

$$\pi_A = \pi_b \otimes \pi_a \,, \text{ acting on } \mathcal{H} \otimes \bar{\mathcal{H}} = \mathcal{H}_b \otimes \mathcal{H}_a = \mathcal{H}_A = \tilde{\mathcal{H}}$$

$$z_i^{m_1} \bar{z}_j^{m_2} \longleftrightarrow (b_i^{\dagger})^{m_1} (a_j^{\dagger})^{m_2} |0\rangle$$

$a_i |0\rangle = b_i |0\rangle = 0$

4 Charged particle in a constant magnetic field (on a plane) classical theory

1st approach

$$M = T^*\mathbb{R}^2 \cong \mathbb{R}^4 \qquad \text{phase space}$$

$$\omega = dp_x \wedge dx + dp_y \wedge dy \qquad \text{symplectic form}$$

$$h = \frac{1}{2}\left[(p_x - y)^2 + (p_y + x)^2\right] \qquad \text{Hamiltonian}$$

$$\ell = x p_y - y p_x \qquad \text{z-component of angular momentum}$$

canonical transformation:

$$\begin{cases} P_1 = \frac{1}{\sqrt{2}}(x + p_y) \\ Q_1 = \frac{1}{\sqrt{2}}(y - p_x) \\ P_2 = \frac{1}{\sqrt{2}}(y + p_x) \\ Q_2 = \frac{1}{\sqrt{2}}(x - p_y) \end{cases} \qquad h_j := \frac{1}{2}(P_j^2 + Q_j^2)$$

One gets:

$$h = h_1 = \frac{1}{2}[P_1^2 + Q_1^2] \qquad \ell = h_1 - h_2$$

$$\{h, \ell\} := \omega(X_h, X_\ell) = 0$$

h, ℓ: complete set of first integrals

Also set, for future use

$$\xi := \frac{1}{\sqrt{2}}[P_1 + iQ_1] \qquad z := \frac{1}{\sqrt{2}}[Q_2 - iP_2]$$

$$\Rightarrow \quad h_1 = \xi\bar{\xi}, \quad h_2 = z\bar{z}, \quad \ell = \xi\bar{\xi} - z\bar{z}$$

we have a **completely integrable system** (≈ two harmonic oscillators)

2d - Liouville tori parametrized by (h_1, h_2) or (h_1, ℓ)

$$\omega = dh_1 \wedge d\varphi_1 + dh_2 \wedge d\varphi_2 \qquad (h, \varphi) \text{ action angle variables}$$

2nd approach

equip $M = T^*\mathbb{R}^2$ with a new symplectic form

physical constants reinserted

$$\underset{dd}{\omega'} = \omega + \underbrace{\frac{eB}{c}\, dx \wedge dy}_{\text{magnetic term}} \qquad eB > 0$$

New Hamiltonian : $h' = \frac{1}{2m}(p_x^2 + p_y^2)$

(gauge invariant formulation)

New angular momentum $\ell' = \ell + \frac{eB}{2c}(x^2+y^2)$

Again $\{h', \ell'\} = 0$

we have $X_{h'} = \frac{1}{m}\left(p_x \partial_x + p_y \partial_y - \frac{eB}{c} p_x \partial_{p_y} + \frac{eB}{c} p_y \partial_{p_x}\right)$

$$X_{\ell'} = -y\,\partial_x + x\,\partial_y - p_y\,\partial_{p_x} + p_x\,\partial_{p_y}$$

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$\ast$ Translations

Action on (M, ω')

$$G = (\mathbb{R}^2, +)$$

$$(x, y, p_x, p_y) \longmapsto (x + \underset{\in \mathbb{R}}{a}, y + \underset{\in \mathbb{R}}{b}, p_x, p_y)$$

$\mathfrak{g} = \mathbb{R}^2$ acting on M in a Hamiltonian fashion

$$\partial_x \equiv X_{t_x} \leftrightarrow t_x := p_x - \frac{eB}{c} y \qquad + \text{const}$$

$$\partial_y \equiv X_{t_y} \leftrightarrow t_y := p_y + \frac{eB}{c} x \qquad + \text{const}$$

$$\longrightarrow \quad \{t_x, h'\} = \{t_y, h'\} = 0$$

dynamical symmetry group

$\ast$ Rotations $\qquad S^1 \cong U(1)$ acts on M as

$$\begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix} \longmapsto \left(\begin{array}{cc|cc} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ \hline 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{array}\right) \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix}$$

$\mathfrak{u}(1) \cong \mathbb{R}$ acts via

$$\left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \longmapsto -y\,\partial_x + x\,\partial_y - p_y \partial_{p_x} + p_x \partial_{p_y} = X_{\ell'}$$

$$\{\ell', h'\} = 0$$

From the general formula $\quad [X_f, X_g] = X_{\{f,g\}}$

one finds that $\quad [X_{t_x}, X_{t_y}] = X_{\{t_x, t_y\}} = X_{\tilde{c}} = 0$

"magnetic central extension

$\rightsquigarrow$ **Heisenberg group**

symmetry group of the classical system

the (magnetic central extension of) the translation group yields a symmetry of the **classical** system but it will **break down** at the **quantum** level

Nevertheless, rotational symmetry will survive quantization

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QUANTIZATION

* Geometric quantization (I)

Perform holomorphic geometric quantization on the first ("unprimed") system, get BF:

$$\mathcal{H} = \left\{ \mathcal{F} = \mathcal{F}(\mathbf{z}) \,/\, \mathcal{F} \text{ is holomorphic and } \int |\mathcal{F}(\mathbf{z})|^2 e^{-\mathbf{z}\bar{\mathbf{z}}} d\xi\, d\eta\, dx\, dy < \infty \right\} \qquad \mathbf{z} = (\xi, z), \quad \xi = \xi + i\eta, \quad z = x + iy$$

$$\mathbf{z}\bar{\mathbf{z}} := \xi\bar{\xi} + z\bar{z}$$

$$\mathcal{H} \cong \overset{\text{BF}}{\mathcal{H}_1} \otimes \mathcal{H}_2$$

h_1 and h_2 are directly (holomorphically) geometrically quantizable (their flows preserve the holomorphic polarization)

$$\hat{h}_j \propto \text{ quantum harmonic oscillator}$$

One has an obvious action of $S^1 \times S^1$ on $\mathbb{C}^2$

* orthonormal basis (up to constants)

$$\boxed{\xi^{m_1} z^{m_2} \leftrightarrow (a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2} |0\rangle \qquad m_i \geq 0}$$

$$a_1 = \frac{d}{d\xi} \qquad a_1^\dagger = \xi \qquad a_1 a_1^\dagger - a_1^\dagger a_1 = I \qquad a_1|0\rangle = 0$$

acting on $\mathcal{H}_1$, and similar expressions for $a_2, a_2^\dagger$ acting on $\mathcal{H}_2$. Obviously: $[a_1^\#, a_2^\#] = 0$

$\#$ creation or annihilation operator

- I -

One also has

$$\hat{l} = \hat{h}_1 - \hat{h}_2 = a_1^+ a_1 - a_2^+ a_2$$

quantized angular momentum (2-component)

Remark: The above picture is compatible with **Bohr-Sommerfeld**

M is foliated by Liouville tori $\Lambda \cong S^1 \times S^1$

cohomological condition: $\left[\frac{\Theta}{2\pi}\right] \in H^1(\Lambda, \mathbb{Z})$

Θ: symplectic potential

$$\Theta = m_1 \, d\varphi_1 + m_2 \, d\varphi_2$$

$$m_j \in \mathbb{N}^*$$

$\rightsquigarrow$ BS-tori (+ circles & a point)

+ covariantly constant sections $\nabla s = 0$

"WKB wave functions"

$$\Psi_{m_1, m_2} = e^{i(m_1\varphi_1 + m_2\varphi_2)}$$

$\leftarrow \zeta^{m_1} z^{m_2}$

★ Geometric quantization (II)

use the vertical polarization

$$P_m := T_m Q \subset T_m M \qquad Q = \mathbb{R}^2 \qquad \frac{\partial \psi}{\partial p} = 0$$

P-wave functions

$$\mathcal{H}_P := \left\{ s : \langle s, s \rangle = \int_Q (s,s)\, dx\, dy < +\infty \right\}$$

(s,s) hermitian structure on $L = M \times \mathbb{C}$

obtained via $\Theta = p_x dx + p_y dy + \underset{\substack{\shortparallel \\ eB/2c\hbar}}{B}\left[x\,dy - y\,dx + \frac{i}{4\pi} d(x^2+y^2)\right]$

$$d(s,s) = 2\pi i\,(\Theta - \bar{\Theta})(s,s) = -B\, d(x^2+y^2)\,(s,s)$$

in the trivialization $s_0 \equiv 1$ $\quad s \sim \psi \quad s' \sim \phi$ $\quad$ we get

$$(\psi, \phi) = \bar{\psi}\phi\, e^{-B(x^2+y^2)}$$

Quantizable classical observables (their flow preserves P)

$$f = \underset{\substack{\text{v. field} \\ \text{on } Q}}{v^i(q)}\, p_i + \underset{\substack{\text{smooth} \\ \text{function on } Q}}{u(q)} \qquad \text{(linear in momentum)}$$

★ The Hamiltonian is not quantizable

so quantize $\underset{\text{Heisenberg algebra}}{h(4)} = \operatorname{span}\{1, q_x, q_y, p_x, p_y\}$

and extend it to the inhomogeneous symplectic algebra

$$hsp(4,\mathbb{R}) = \operatorname{span}\{1, q_i, p_j, q_iq_j, q_ip_j, p_ip_j \mid i,j = x,y\}$$

via the squaring von Neumann rule

$$Q(p_j^2) = Q^2(p_j)\ ; \quad Q(q_j^2) = Q^2(q_j)$$

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we get

$$\hat{P}_x \Psi = -i\hbar \partial_x \Psi + \frac{eB}{2c}(y + ix)\Psi$$

$$\hat{P}_y \Psi = -i\hbar \partial_y \Psi - \frac{eB}{2c}(x - iy)\Psi$$

and

$$\hat{x}\Psi = x\Psi$$

$$\hat{y}\Psi = y\Psi$$

Passing to complex coordinates, we get

$$\hat{h} = -\frac{2\hbar^2}{m}\left(\partial_z - \frac{B}{2}\right)\partial_{\bar{z}} + \frac{\hbar eB}{2mc}$$

which is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^2, \mathbb{C}) \subset L^2(\mathbb{C}, \mu) \cong \mathcal{H}_p$ (Schwartz functions)

via standard arguments use Nelson's analytic vector theorem

introduce

$$\hat{a} = -\frac{i}{\sqrt{B}}\partial_{\bar{z}} \qquad \hat{a}^{+} = -\frac{i}{\sqrt{B}}(\partial_z - B\bar{z})$$

$$\boxed{\hat{h} = \frac{eB\hbar}{mc}\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\right)}$$

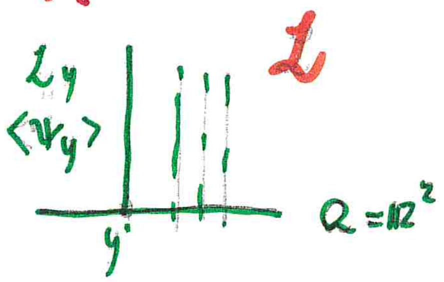
* Translational symmetry breaking

$$\hat{t}_x = -i\hbar(\partial_z + \partial_{\bar{z}}) + i\frac{eB}{4c}(z+\bar{z})$$

$$\hat{t}_y = \hbar(\partial_z - \partial_{\bar{z}}) + \frac{eB}{4c}(z-\bar{z})$$

$$[\hat{t}_{x,y}, \hat{h}] \neq 0$$

* geometric interpretation (Falasso & —, 2016)



$\Psi_y \in$ ground state space of $\hat{h}_y$ (translated Hamiltonian)

$a_y \sim \bar{\partial}_y$ holomorphic structure

$\bar{\partial}_y \Psi_y = 0$

& $b_y \Psi_y = 0$

⟶ get a 1-dimensional space

and an **index bundle** (as in Fourier-Mukai-Nahm theory) carrying a natural connection (Nahm) with non trivial curvature :

Let $\xi \equiv \xi_{\alpha,\beta}(x) = (U(\alpha)V(\beta)\xi_0)(x) = \pi^{-\frac{1}{4}}\exp\left[i\alpha x - \frac{(x-\beta)^2}{2}\right]$

$\xi_0(x) = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}}$

$$[U(\alpha)\phi](x) = e^{i\alpha x}$$

$$[V(\beta)\phi](x) = \phi(x-\beta)$$

$$U(\alpha)V(\beta) = e^{i\alpha\cdot\beta} V(\beta)U(\alpha)$$

CCR in Weyl form

$\xi_{\alpha\beta}$: standard coherent states

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Nahm connection form

$$A = \langle \xi, d\xi \rangle$$

curvature

$$\Omega = dA = d\langle \xi, d\xi \rangle =$$

$$[\langle \partial_\alpha \xi, \partial_\beta \xi \rangle - \langle \partial_\beta \xi, \partial_\alpha \xi \rangle]\, d\alpha \wedge d\beta$$

$$= \quad 2i \, \mathrm{Im} \langle \partial_\alpha \xi, \partial_\beta \xi \rangle \, d\alpha \wedge d\beta$$

A routine computation yields

$$\Omega = -i \, d\alpha \wedge d\beta$$

$\rightsquigarrow$ "translational anomaly"

lack of commutativity with the Hamiltonian detected via the curvature of the coherent state line bundle

In contrast to translations, rotational symmetry survives quantization

$$\hat{\ell} = \hbar (z \partial_z - \bar{z} \partial_{\bar{z}})$$

$$\& \quad [\hat{h}, \hat{\ell}] = 0$$