GEOMETRIC METHODS IN QUANTUM MECHANICS

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Lecture VIII: Coherent states & Borel-Weil

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Standard **coherent states**

$$H = \frac{1}{2}\{(Q-\alpha)^2 + (P-\beta)^2\}$$

translated ground states of the quantum harmonic oscillator

$$\underset{\displaystyle \frac{Q+iP}{\sqrt{2}}}{\overset{\shortparallel}{a}}\ \psi_{\alpha\beta} = \frac{\alpha+i\beta}{\sqrt{2}}\ \psi_{\alpha\beta}$$

$$\psi_{\alpha\beta} = e^{i(\alpha Q - \beta P)}\ \psi_0 = e^{\frac{i\alpha\beta}{2}}\ U(\alpha)\,V(-\beta)\ \psi_0$$

$$\left(e^A e^B = e^{\frac{1}{2}[A,B]}\ e^{A+B} \quad \text{if} \quad \left[{A \atop B}, [A,B]\right] = 0 \right)$$

$$\psi(\alpha) = \underset{\displaystyle e^{\alpha a^+ - \bar{\alpha}\bar{a}}}{\overset{\shortparallel}{D(\alpha)}}\ \psi_0 \qquad \boxed{D(\alpha)D(\beta) = e^{i\,\mathrm{Im}\,\alpha\bar{\beta}}\ D(\alpha+\beta)}$$

$$D(\alpha)D(\beta)\psi_0 = \boxed{D(\alpha)\,\psi_\beta = e^{i\,\mathrm{Im}\,\alpha\bar{\beta}}\ \psi_{\alpha+\beta}}$$

Heisenberg group coherent states

$$T(g)|\xi\rangle = e^{i\Phi(g,\xi)}\ |g\cdot\xi\rangle$$

$$\boxed{\psi_\alpha = \psi_\alpha(z) = e^{-\frac{|\alpha|^2}{2}}\ e^{\alpha z}}$$

$$\mathbb{C} \longrightarrow P(\mathcal{H}) \qquad \text{embedding}$$
$$\alpha \longmapsto [\psi_\alpha]$$

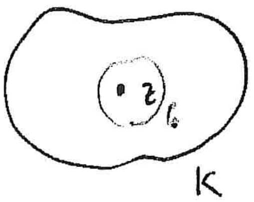
$$f(\alpha) = e^{\frac{|\alpha|^2}{2}} \langle \psi_{\bar{\alpha}}, f\rangle_{\mathcal{H}} \qquad f \in \mathcal{H}$$

$$\int_{\mathbb{C}} |f|^2 e^{-|z|^2}\, dz \wedge d\bar{z} < \infty$$

the **evaluation map is continuous**

($\leadsto$ Rawnsley '77)

Auxiliary computation (Bargmann - Fock)



$$|f(z)| = \left| \frac{1}{2\pi i} \oint_{C} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right|$$

$$\leq \frac{1}{2\pi} \int | f\underbrace{(z + \rho e^{i\vartheta})}_{f_z(\rho e^{i\vartheta})} |\, d\vartheta$$

$$c_1 := \int_0^\infty e^{-\rho^2} \rho\, d\rho = \frac{1}{2} \int_0^\infty e^{-\xi}\, d\xi = \frac{1}{2}$$

$$c_1 |f(z)| \leq \int_0^\infty \int_0^{2\pi} e^{-\rho^2} | f_z(\rho e^{i\vartheta}) |\, \rho\, d\vartheta\, d\rho$$

$$|f(z)| \leq \frac{1}{c_1} \int_{\mathbb{C}} |f_z(\zeta)|\, d\mu \equiv \frac{1}{c_1} \| f_z \|_1$$

$$\leq \frac{1}{c_1} \mu(\mathbb{C})^{\frac{1}{2}} \| f_z \|_2$$

$$\| f \|_K \leq A(K) \| f \|_2$$

$$\| f \|_{\partial K} \leq A(K) \| f \|_2$$

$$\Rightarrow \left(f_n \xrightarrow{L^2} f \Rightarrow f_n \xrightarrow{K} f \right) \Rightarrow f \text{ holomorphic}$$

★ Borel-Weil & coherent states

(see Pressley-Segal '86 Ch 2; Perelomov '86)

G compact Lie group

G possesses a bi-invariant measure (Haar measure)

Example: $SU(2) \cong S^3$ Haar = standard measure inherited from $\mathbb{R}^4$

Let G act linearly on a f.d. vector space $\bar{V}$

Then, upon picking a G-invariant inner product on V (Weyl trick (averaging))

G becomes a subgroup of some $U(n)$

$\Rightarrow$ $\bar{V}$ = orthogonal direct sum of irreducible G-modules

Let $V = \underset{\mathrm{Lie}(G)}{\mathfrak{g}}$ G acts on $\mathfrak{g}$ via the **adjoint representation** (derivative at $x=1$ of $x \mapsto g x g^{-1}$)

$\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$ G acts irreducibly on $\mathfrak{g}_i$

$\mathfrak{g}_i$: Lie subalgebra

$[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ for $i \neq j$

$G \underset{\text{loc isom}}{\sim} \underbrace{G_1 \times \dots \times G_k}$ corresponding to $\mathfrak{g}_i$ can be taken to be simply connected

G_i do not have a non trivial <u>connected</u> normal subgroup

simple group (except S^1)

G semisimple: no circles in

compact simple groups

$SU(n)$, $SO(n)$, $Sp(n)$, G_2, F_4, E_6, E_7, E_8
unitary orthogonal symplectic
rank

$\mathfrak{su}_n = A_{n-1}$ $\mathfrak{so}_{2n+1} = B_n$ $\mathfrak{sp}_n = C_n$
$\mathfrak{so}_{2n} = D_n$

maximal torus T (maximal connected abelian subgroup of G)
= product of ℓ circles
rank of G

any two maximal tori are conjugate
Example : $U(n) \rightsquigarrow \mathbb{T}$ = Diagonal matrices $\begin{pmatrix} e^{i\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\lambda_n} \end{pmatrix}$

G acts on $\mathfrak{g}$ $\rightsquigarrow$ get a complex linear action
on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$

$\star$ decompose $\mathfrak{g}_{\mathbb{C}}$ via the action of T

$\Rightarrow$ $$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha$$ $\alpha : T \to S^1 \subset \mathbb{C}^\times$
root of G

1-dimensional

roots : finite subset of $\hat{T}$ character group
homomorphisms
$\chi : T \to S^1$

α is determined by $\dot{\alpha} \in \mathfrak{t}^*$
$(\mathfrak{t} \to \mathbb{R}$ s.t. $\alpha(e^{\xi}) = e^{i\dot{\alpha}(\xi)})$

$\hat{T}$: lattice in $\mathfrak{t}^*$
weights of G

notationally we blur the distinction between α and $\dot{\alpha}$

Also: α root $\Rightarrow$ $-\alpha$ root :

$$\mathfrak{g}_{-\alpha} = \overline{\mathfrak{g}}_\alpha$$

VIII-4

Example: $G = U(n) \Rightarrow \mathfrak{g}_{\mathbb{C}} = M(n, \mathbb{C})$

$n \times n$ complex matrices

roots α_{ij} $i \neq j$, $1 \leq i, j \leq n$

$$\mathfrak{g}_{\alpha_{ij}} = \left\{ \begin{array}{c|ccc} & & & j \\ i & \cdots & \cdot & \\ & 0 & & 0 \end{array} \right\} \qquad \alpha_{ij} : \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \mapsto \underbrace{x_i x_j^{-1}}_{\xi_i - \xi_j}$$

$Z(G) \subset T$ & $Z(G) = \bigcap_{\alpha} \ker \alpha$. G semisimple $\Rightarrow$ finite centre $\Rightarrow$ the roots span $\mathfrak{t}^*$

centre

$\dim \mathfrak{g}_\alpha = 1$ Let $e_\alpha \in \mathfrak{g}_\alpha$ and assume $e_{-\alpha} = \bar{e}_\alpha$

Then $h_\alpha = -i[e_\alpha, e_{-\alpha}] \in \mathfrak{t}$

$h_\alpha \neq 0$

$\Rightarrow \langle e_\alpha, e_{-\alpha}, h_\alpha \rangle$ = Lie subalgebra of $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{su}(2)$

Normalize e_α:

$$[h_\alpha, e_\alpha] = 2i\, e_\alpha$$

$$[h_\alpha, e_{-\alpha}] = -2i\, e_{-\alpha}$$

$$[e_\alpha, e_{-\alpha}] = i\, h_\alpha$$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

h_α canonically determined by α, $2\pi h_\alpha \in \ker(\exp : \mathfrak{t} \to T)$

Coroot $\forall\, \beta : \mathfrak{t} \to \mathbb{R}$ root $\beta(h_\alpha) \in \mathbb{Z}$ ($e^{2\pi h_\alpha} = 1$)

$\alpha(h_\alpha) = 2$ $h_\alpha \leadsto \eta_\alpha : S^1 \to T$ ~ lattice $\check{T}$

$\eta_\alpha(e^{i\theta}) = \exp(\theta h_\alpha)$ con. dual to $\hat{T}$

G simply connected $\Rightarrow$ the coroots generate $\check{T}$

$\mathfrak{g}_{\mathbb{C}} = \langle e_\alpha, \mathfrak{t} \rangle$

$$
\begin{aligned}
[e_\alpha, e_\beta] &= n_{\alpha\beta}\, e_{\alpha+\beta} && (\alpha+\beta \text{ root})\\
&= i\, h_\alpha && \alpha+\beta = 0\\
&= 0 && \text{otherwise}
\end{aligned}
$$

We can further adjust e_α in such a way that $0 \neq n_{\alpha\beta} \in \mathbb{Z}$

Weyl group, Weyl chambers, positive roots

W Weyl group $:= N(T)/T$ (normalizer)

automorphisms of T obtained via conjugation through G

For $U(n)$: $W = S_n$: symmetric group (permutes the entries)

W: finite group of isometries of $\mathfrak{t}$: preserves $\check{T}$ and permutes the roots in $\hat{T}$

$\alpha \rightsquigarrow S_\alpha = \exp \frac{1}{2}\pi (e_\alpha + e_{-\alpha}) \in N(T)$

S_α has order 2

S_α: reflection in $H_\alpha \subset \mathfrak{t}$, $H_\alpha = \{\xi : \alpha(\xi) = 0\}$

W is generated by the S_α

$S_\alpha(\xi) = \xi - \alpha(\xi)\, h_\alpha$

regular elements of $\mathfrak{t}$: those not lying on H_α

they give rise to **Weyl chambers**

permuted simply transitively by W

Select one of the chambers C and call it

positive

α positive (not.: α_+)

if α is positive on C

α positive is called simple if H_α is a wall of C

If G is semisimple, of rank ℓ, then

$\exists$ ℓ simple roots and $e_{\alpha_i}, e_{-\alpha_i}, h_{\alpha_i}$ generate

$\mathfrak{g}_{\mathbb{C}}$

$U(n)$: positive roots α_{ij}, $i<j$
simple roots $\alpha_{i,i+1}$ $1 \leq i < n$

Irreducible representations & antidominant weights

G compact $\Rightarrow$ every irreducible representation is f.d.

$\rightsquigarrow$ $\exists$ (ε_i) basis of V diagonalizing the action of

T (maximal torus) $\rightsquigarrow$ $\lambda_i : T \to S^1$ **weight** of ε_i

{weights}: invariant under W & finite subset of $\hat{T}$

λ dominates μ if $\lambda - \mu$ is positive on the positive chamber C

λ: antidominant if it is dominated by $w.\lambda$ $\forall w \in W$, i.e. $\lambda(h_{\alpha_+}) \leq 0$ $\forall \alpha_+$ positive

Hence irreps $\leftrightarrow$ $\hat{T}_-$: antidominant weights ~ lowest weight

$\hat{T}_- \cong \hat{T}/W$ (orbits of W in $\hat{T}$)

explicit realization: **Borel - Weil**

Generalized flag manifolds

B^+ Lie: spanned by $\mathfrak{t}_{\mathbb{C}}$ and e_{α_+}

Borel subgroup $B^+ \cap G = T$

$$G/T \cong G_{\mathbb{C}}/B^+$$

complex cells: orbits of B^+

$W = N(T)/T \subset G/T$

$\exists!$ w in each cell

Bruhat decomposition

$C_w = B^+ w$

$\dim C_w = \text{length of } w = \#\{\alpha_+ \mid w\cdot\alpha_+ < 0\}$

G/T is simply connected

$U(n)/T \cong GL(n,\mathbb{C})/B^+$, $B^+ \equiv$ upper triangular matrices

$Fl(\mathbb{C}^n)$

$$\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$$

in general: flag manifolds (B^+ echelon matrices)

Towards BW

every $\lambda : T \to S^1$ extends uniquely to $\lambda : B^+ \to \mathbb{C}^\times$ (holomorphic) $\rightsquigarrow$ get a homogeneous holomorphic line bundle

$$L_\lambda = G_{\mathbb{C}} \times_{B^+} \mathbb{C}_\lambda \longrightarrow G_{\mathbb{C}}/B^+$$

$$\begin{array}{c} \parallel\!\!\!\wr \\ G_{\mathbb{C}} \times \mathbb{C}/\sim \end{array} \qquad (gb, \xi) \sim (g, \lambda(b)\xi)$$

$G_{\mathbb{C}}$ acts on L_λ and on its holomorphic sections

★★★ Borel - Weil

(i) L_λ has no non-zero holomorphic sections unless λ is antidominant
dominant

(ii) $H^0(L_\lambda)$ realizes an irreducible representation of G with lowest weight λ
highest

Relationship with projective embeddings & coherent states

$f : X \to \mathbb{P}(V)$ holomorphic

$\leadsto$ define $L_f \to X$ holomorphic line bundle

$(L_f)_x :=$ line $f(x)$ in $\bar{V}$ $\quad L_f \subset X \times \bar{V}$

fibre at x

and $\exists\ \bar{\pi} : L_f \to \bar{V}$ linear in each fibre

Thus $\alpha : V \to \mathbb{C}$, composed with π

$L_f \xrightarrow{\pi} V \xrightarrow{\alpha} \mathbb{C}$ yields $s \in \Gamma(L_f^*)$
dual line bundle

$\leadsto$ get a linear map $V^* \to \Gamma(L_f^*)$

Conversely, from $L \to X$, assume that $\forall x \in X\ \exists$ a section not vanishing at x. Then $\exists$

$f_L : X \to \mathbb{P}(\Gamma^*)$ $\quad f_L(x) =$ evaluation of f_L at x
space of section

(choose an identification with $\mathbb{C}$ $\sim$ indeterminacy up to a phase) $\leadsto$ Rawnsley's coherent states

Again on BW

irreducible representation of $G_{\mathbb{C}}$ on V

$\exists$ a ray Ω in V^* stable under B^+

Then take the orbit $G_{\mathbb{C}}\,\Omega$ & get an equivariant map

$$f : G_{\mathbb{C}}/B^+ \to \mathbb{P}(V^*)$$

and consequently L_f & a map $V \to \Gamma(L_f^*)$

A vector of highest weight in V^* yields a ray stable under B^+

Important example

the determinant line bundle Det

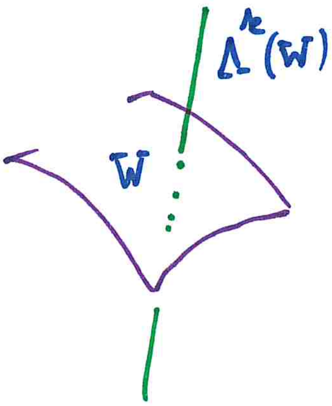
Let $U(n)$ act on $\Lambda^k(\mathbb{C}^n)$ — kth exterior power

$Gr_k(V)$: grassmannian $= \{ k\text{-dimensional subspaces of } V \}$

$\dim_{\mathbb{C}} V = n$ — $\dim = 1$

$Det_{\bar{W}}$: fibre at $\bar{W} \in Gr_k(V) = \Lambda^k(W)$ — top exterior power

$(w_1, \dots w_k) \longmapsto w_1 \wedge w_2 \wedge \dots w_k$



Det does not have holomorphic sections, Det^* (take dual lines throughout) does have sections

Roughly: $\Lambda^k(V^*) \to \Gamma(Det^*)$ $(\star)$

$\alpha_1 \wedge \dots \alpha_k \longmapsto (\lambda\, w_1 \wedge \dots w_k \mapsto \langle \alpha_1 \wedge \dots \alpha_k, w_1 \wedge \dots w_k \rangle$

$\qquad \qquad\quad \cong$

$\det\left((\alpha_i, w_j) \right)$

$(\star)$ is injective and surjective (Hartogs)

Far reaching generalizations...

Kählerian coherent states

(M, ω) compact prequantizable Kähler manifold

$(L, \nabla, (\ ))$ — ∇: connection, $(\ )$: metric — hermitian holomorphic prequantum bundle

$L \to M$

unique up to equivalence if M is simply connected

∇ : Chern-Bott connection

unique connection compatible with $(\,,)$ and the holomorphic structure, with curvature

$$\Omega = -2\pi i \omega$$

$$\nabla^{0,1} = \bar\partial \quad \text{in a holomorphic frame}$$

L^2 : L^2-sections of $L \to L$
(use Liouville measure $d m$)

H : quantum Hilbert space $\equiv H^0(L)$

finite dimensional by compactness

holomorphic sections of $L \to M$

$$H = \ker \Delta \qquad \Delta = (\nabla^{0,1})^\dagger \nabla^{0,1}$$

elliptic

$\#$

$\{0\}$ if L is sufficiently positive

$\leadsto \ h^0(L) = \dim H$ topological invariant

(by Riemann-Roch & Kodaira vanishing)

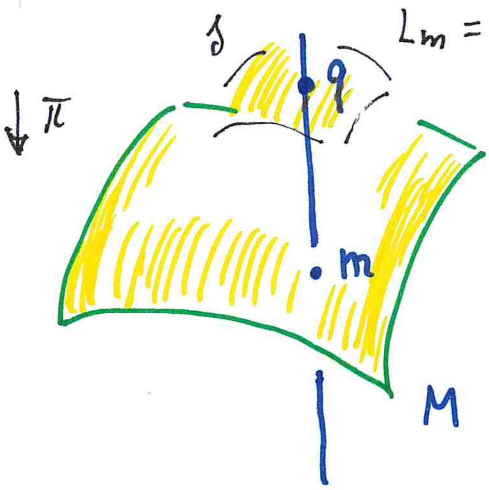
normalization

$$\underbrace{vol(M)}_{\text{phase space volume of the classical system}} := \int_M dm = \underbrace{h^0(L)}_{\dim H}$$

$\dim H$: dimension of the quantum Hilbert space

Bohr correspondence principle

★ Coherent states (Rawnsley '77)



$ev_m : H \longrightarrow L_m$

$ev_m(s) := s(m)$

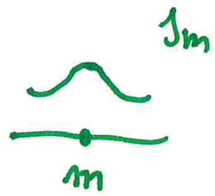
the evaluation map is continuous

$$\Rightarrow \quad s(m) = \langle \underbrace{e_q}_{\in H}, s\rangle \cdot q \qquad \langle\cdot,\cdot\rangle = \int_M (\cdot,\cdot)\, dm$$

e_q: coherent state vector

$$e_{cq} = \bar{c}^{-1} e_q, \quad c \in \mathbb{C}^*$$

equivalent definition (—, 2000)



s_m, $m \in M$ coherent state vector **maximises** $s \longmapsto (s,s)(m) = \langle s, A_m s\rangle$

$\langle s, s\rangle$

$s_m = c\, e_q$

& if $\|e_q\| = 1$, $(s_m, s_m)(m) = (q,q)$

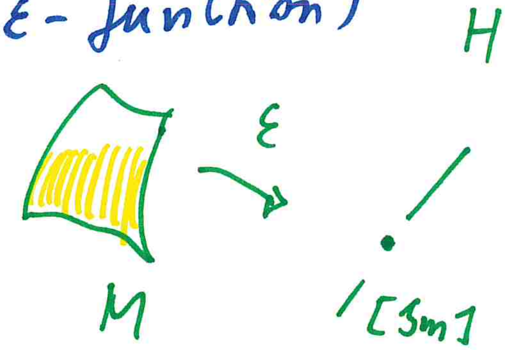
$\mathbb{P}(H)$: projective space pertaining to H

$$[\;\;] : H \setminus \{0\} \longrightarrow \mathbb{P}(H)$$ canonical map

Define ("Rawnsley's ε-function")

$$\varepsilon : M \longrightarrow \mathbb{P}(H)$$

$$\varepsilon(m) = [\, s_m \,]$$



Assumptions

A0: Kodaira vanishing

A1 : ε **well-defined** (absence of base points)

A2 : ε **injective**

A1 & A2 $\Rightarrow$ $V_{\lambda_m} \subset H$ 1-dimensional

λ_m : largest eigenvalue of A_m

A3 $\quad m \mapsto \Lambda(m) = \lambda_m$ constant

Consequence: ε is a symplectic embedding, ε_* is injective & $\varepsilon^*(\Omega) = \omega$

Ω : Fubini-Study

& $\quad \varepsilon^*(\mathcal{O}(1), \nabla_{can}) = (L, \nabla)$

$\mathcal{O}(1)$: hyperplane bundle; ∇_{can} : canonical connection

$$\nabla_{can} = - \frac{\langle v \,|\, dv \rangle}{\|v\|^2}$$

M homogeneous & simply connected $\Rightarrow$ ε constant (Rawnsley '77, Loi '2000)

Warning: Kodaira does not specify curvature consequences of AO – 3

- $(s_{m'}, s_{m'})(m) < (s_m, s_m)(m) \quad (=1)$

$\triangle$

- $\langle s_m, s \rangle \overset{H}{=} (s_m, s)(m)$ $\qquad s_m = 1 = \Lambda$

reproducing property

- $\langle s_1, s_2 \rangle = \int_M \langle s_1, s_m \rangle \langle s_m, s_2 \rangle \, dm$

i.e $\quad \int_M |s_m\rangle\langle s_m| = I$

generalised resolution of the identity

- $(s_m, s_m)(m') = (s_{m'}, s_{m'})(m)$

reciprocity law (local)

- $(s_m, s_m)(m') = e^{-D(m,m')}$

D : Calabi's diastasis function

$$D(m,m') = f(m,m) + f(m',m') - f(m',m) - f(m,m')$$

f any local kähler potential (extended sesquiholomorphically

$$\frac{i}{2} \partial\bar\partial f = \omega$$

Aside

$\underline{P} \rightsquigarrow \Pi(m_1, m_2) := \langle s_{m_1}, s_{m_2} \rangle$ reproducing kernel

projection onto $\mathcal{H}$

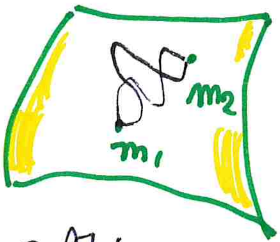
$$\| $$

$$\int_{\mathcal{D}(m_1,m_2)} \underbrace{e^{i\int_\gamma \vartheta^0}}_{T_\nabla(\gamma)} \, \mathcal{D}(\gamma)$$

path integral measure

holonomy of ∇

Klauder formula



paths connecting m_1 and m_2

"topological dynamics" (cf Odjiewicz '92)

f observable, $f^\#$ associated vector field $df = \omega(f^\#, \cdot)$

F prequantum operator

$$F s = -i \nabla_{f^\#} s + f s$$

$Q(f)$ quantum operator $= PFP : H \circlearrowleft$ (Toeplitz)

f quantizable if $Q(f) = F|_H$

For such f's:

$$\langle s_m, Q(f) s_m \rangle = f(m)$$

Tangent space at a highest weight vector $|\lambda\rangle$ $\quad O_\lambda = G|\lambda\rangle \simeq G/T$

orbit

$T_{[|\lambda\rangle]} O_\lambda$: generated by $\begin{cases} r_\alpha, s_\alpha & \alpha \text{ positive root} \\ r_\alpha = \frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha}) \end{cases}$

$\overset{\shortparallel}{[v]}$

$$r_\alpha^\# \Big|_{[v]} = |v\rangle\langle r_\alpha v| + |r_\alpha v\rangle\langle v|$$

$s_\alpha = \frac{i}{\sqrt{2}}(E_\alpha + E_{-\alpha})$

(orhonormal basis in $\mathfrak{g}$ w.r. to the negative Killing form

$$s_\alpha^\# \Big|_{[v]} = |v\rangle\langle s_\alpha v| + |s_\alpha v\rangle\langle v|$$

$$\boxed{s_\alpha^\# \Big|_{[v]} = -J_{[v]}\, r_\alpha^\# \Big|_{[v]}}$$

complex structure

For $SU(2)$, r_α & s_α $\rightsquigarrow$ J_x, J_y components of the spin operator

raising & lowering operators

$J_\pm = J_1 \pm i J_2$

$J_0 = J_3$

$$J_+|j,\mu\rangle = \sqrt{(j-\mu)(j+\mu+1)}\,|j,\mu+1\rangle$$

vectors: $|j,\mu\rangle$ $\quad \dim = 2j+1$

$$J_-|j,\mu\rangle = \sqrt{(j+\mu)(j-\mu+1)}\,|j,\mu-1\rangle$$

$j \in \mathbb{Z}/2$

$$J_-|j,-j\rangle = 0$$

lowest weight

$J_0|j,\mu\rangle = \mu|j,\mu\rangle$

$[J_0, J_\pm] = \pm J_\pm$

$[J_-, J_+] = -2J_0$

$$|j,\mu\rangle = \sqrt{\frac{(j-\mu)!}{(j+\mu)!\,(2j)!}}\; J_+^{\,j+\mu}\,|j,-j\rangle$$

J_+

$|j,-j\rangle$

physicists' notation for spin theory

unitary irreps of $SU(2)$

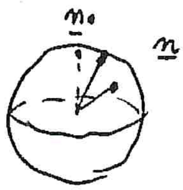
★ Coherent states for S^2

(cf. Perelomov '86)
Onofri '75

SU(2)

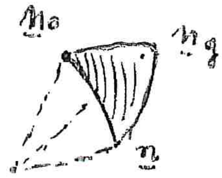
spin

$$T^j(g)\,|\underline{n}\rangle = e^{i\Phi(\underline{n},g)}\,|\underline{n}_g\rangle$$



$$\Phi(n,g) = j\,A(\underline{n}_0,\underline{n},\underline{n}_g)$$

area of a geodesic triangle



infinitesimally

$$T^j(1+\delta g)\,|\zeta\rangle = e^{i\delta\Phi}\,|\zeta+\delta\zeta\rangle$$

$$i\,\delta\Phi = j\left(\frac{\partial F}{\partial\zeta}\,\delta\zeta - \frac{\partial F}{\partial\bar\zeta}\,\delta\bar\zeta\right) \qquad F = \log(1+|\zeta|^2)$$

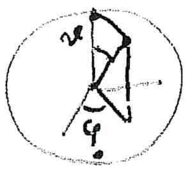
connection form $\qquad$ $\omega = d\Phi$ area form

Borel-Weil

$$X = G/H = G^{\mathbb{C}}/B$$

$$|\underline{n}\rangle = |\zeta\rangle = (1+|\zeta|^2)^{-j}\,e^{\zeta J_+}\,|j,-j\rangle$$

$$\zeta = -\tan\frac{\vartheta}{2}\,e^{-i\varphi} \qquad |j,\mu\rangle$$



$$\underline{n} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\vartheta)$$

stereographic projection

$$D(\underline{n}) = T^j(g_n) = \exp(i\,\theta\,(m\cdot J))$$

$$J_\pm = J_1 \pm iJ_2$$
$$J_0 = J_3$$
$$[J_0, J_\pm] = \pm J_\pm$$
$$[J_+, J_-] = 2J_0$$

$$\boxed{D(\underline{n}_1)\,D(\underline{n}_2) = D(\underline{n}_3)\,\exp(i\,\Phi(n_1,n_2)\,J_0)}$$

max. nilpotent $\qquad$ max solvable $\qquad$ subgroups

$$\begin{pmatrix}\alpha & \beta\\ -\bar\beta & \bar\alpha\end{pmatrix} \qquad \underset{G^{\mathbb{C}}}{g \in} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = z_+\,h\,z_- = z_+\,b_-$$

$$\begin{pmatrix}1 & \beta\delta^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\delta^{-1} & 0\\ 0 & \delta\end{pmatrix}\begin{pmatrix}1 & 0\\ \gamma\delta^{-1} & 1\end{pmatrix}$$

Gauss decomposition

$$T^j(g) f(z) = e^{i S(g,z)} f(z_g)$$

$$z_g = \frac{\alpha z - \bar\beta}{\beta z + \bar\alpha}$$ möbius transformation

$$S(g,z) = \int_0^z (\theta - g\cdot\theta) + S(g,0)$$

$$\theta = \frac{\partial F}{\partial z} dz = \partial F \qquad F = j \log(1 + z\bar z)$$

$$\boxed{\langle z, \zeta \rangle = \left[(1+|\zeta|^2)(1+|z|^2)\right]^{-j} (1 + \bar z \zeta)^{2j}}$$

$$e^{-\frac{1}{2} f(z,\bar z)} \, e^{-\frac{1}{2} f(\zeta,\bar\zeta)} \, e^{f(z,\zeta)}$$

general expression

sesquiholomorphic extension of the Kähler potential