GEOMETRIC METHODS IN QUANTUM MECHANICS

mauro Spera - UCSC Brescia

Lecture X: Abelian varieties, theta functions & perfect quantization

International Doctoral Program in Science

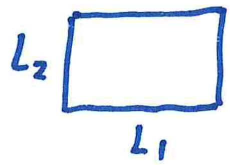
Brescia, Capitolium

Landau levels on a torus & theta functions

free electron on a torus

$$\Sigma_1 = \mathbb{C} / \mathbb{Z} \oplus \tau \mathbb{Z}$$

elliptic curve



$$z = x + iy \qquad \tau = \frac{L_2}{L_1} e^{i\psi} \qquad \mathrm{Im}\,\tau > 0$$

$$\Theta^{k} \to \Sigma_1 \qquad \Theta : \text{theta line bundle}$$

connection

$$A = -\frac{2\pi k}{\mathrm{Im}\,\tau}\, y\, dx$$

curvature

$$F_A = \frac{2\pi k}{\mathrm{Im}\,\tau}\, dx \wedge dy = dA$$

$$c_1(\Theta^k) = \frac{1}{2\pi} \int_{\Sigma_1} F_A = k$$

$$2\pi k = \frac{eB}{hc} L_1^2$$

magnetic field

magnetic flux quantization

raising & lowering operators
a^+ a

$$a = \sqrt{\frac{\mathrm{Im}\,\tau}{\pi k}} \left(\frac{\partial}{\partial \bar z} + i \frac{\pi k}{\mathrm{Im}\,\tau} \mathrm{Im}\, z \right)$$

$$a^+ = -\sqrt{\frac{\mathrm{Im}\,\tau}{\pi k}} \left(\frac{\partial}{\partial z} + i \frac{\pi k}{\mathrm{Im}\,\tau} \mathrm{Im}\, z \right)$$

$$H = \hbar \omega_c \left(a^{+} a + \frac{1}{2} \right)$$

$$\omega_c = \frac{\hbar}{m_e} \frac{2\pi k}{\Im \tau}$$

Hamiltonian

$$[a, a^{+}] = \mathbb{1}$$

Ground state space: holomorphic sections (k-level theta functions)

$$f_\ell(z, \tau) = \vartheta \begin{bmatrix} 0 \\ \ell/k \end{bmatrix} (z, \tau/k)$$

$$\ell = 0, 1, \ldots, k-1$$

★ Theta function & heat equation

$$\vartheta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z}$$

$z \in \mathbb{C}$
$\tau \in \mathbb{H}$

$$\vartheta(z + a\tau + b, \tau) = e^{-\pi i a^2\tau - 2\pi i a z}\,\vartheta(z,\tau)$$

Take $z = x \in \mathbb{R}$ $\quad \tau = it \quad t > 0$

Then $\vartheta(x+1,\tau) = \vartheta(x,\tau)$

$$\boxed{\frac{\partial}{\partial t}\vartheta(x,it) = \frac{1}{4\pi}\frac{\partial^2}{\partial x^2}\vartheta(x,it)}$$ Heat equation

If $t \to 0$, $\vartheta \xrightarrow{\text{distributionally}} \sum_{k\in\mathbb{Z}} \delta_k$

$\vartheta(x,it)$: fundamental solution for $x \in \mathbb{R}/\mathbb{Z}$

sketch of proof (cf. Mumford I)

$f(x) = \sum a_m e^{2\pi i m x}$ test periodic function

$$(\lozenge) := \int_0^1 \vartheta(x,it) f(x)\,dx = \int_0^1 \sum_{n,m} a_m e^{-\pi n^2 t} \cdot e^{2\pi i (n+m)x}\,dx$$

$$= \sum_{n,m} a_m e^{-\pi n^2 t} \underbrace{\int_0^1 e^{2\pi i (m+n)x}\,dx}_{\delta_{m,-n}}$$

$$= \sum_n a_{-n} e^{-\pi n^2 t} \qquad \text{Thus, if } t \to 0^+$$

$$(\lozenge) = \sum_n a_n = f(0) = \delta_0(f)$$ Dirac's delta (counting measure)

prototype of the Coherent state Transform (CST)

$$\Delta^{(-i\Omega)} = -\sum_{i,j=1}^{g} \frac{i}{2\pi} \Omega_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

$M = V/\Lambda$

$\Omega \in \mathbb{H}^g$

$\Omega = \Omega^T$

$\Omega = X + iY$

$Y > 0$

Heat equation:

on $(S^1)^g$
$\cong$
$U(1)^g$

$$2 \frac{\partial u}{\partial t} = \Delta^{(-i\Omega)} u$$

$-i\Omega = -iX + Y$

~~k-level~~ theta functions

$$C_{\frac{1}{k}}^{(-i\Omega)} \left(\theta_\ell^{k} \right) = \vartheta_\ell (z, \Omega) \qquad 0 \leq \ell_\alpha < k$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\sum_n e^{2\pi i k x \cdot n} \qquad\qquad \sum_n e^{\pi i (\ell + kn) \frac{\Omega}{k} (\ell + kn)} e^{2\pi i (\ell + kn) x}$$

$\Updownarrow$ Fourier

$$\delta\left(x - \frac{\ell}{k}\right)$$

$$f(x) = \sum a_n e^{2\pi i x n}$$

$$\left(C_t^{(-i\Omega)} f \right)(x) = \sum a_n e^{it\, n \cdot \Omega n} e^{2\pi i x \cdot n}$$

upshot

$$\text{BS}: \langle \delta_\ell \rangle_{0 \leq \ell < k} \underset{\text{(unitary)}}{\overset{\text{CST}}{\simeq}} \langle \vartheta_\ell \rangle_{0 \leq \ell < k}$$

$\Rightarrow$ perfect quantization:

$$\boxed{\text{Bohr-Sommerfeld GQ} = \text{holomorphic GQ}}$$